Dynamics of Nonholonomic Systems

by
Ju. I. Neĭmark
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PREFACE

The idea of writing the present book arose during A. A. Andronov's seminar in 1949/50 in connection with the discussions on the formulation of the equations of motion of various technical systems. Apart from furthering scientific aims, the seminar was intended to serve a pedagogical purpose, as was emphasized more than once by Professor Andronov. Among the subjects discussed were the concepts of directed couplings and servo constraints, methods of formulating the equations of electrical circuits, tensor forms of equations of motion, equations of motion of mechanical systems, variational principles of field theory and electrodynamics, questions of the derivation of the equations of motion of electrical machines, and many other problems. Papers were read at the seminar by N. A. Železcov, M. L. Levin, A. V. Gaponov, Ju. I. Neĭmark, N. A. Fufaev, and others. It was during this seminar that certain inaccuracies in a number of works on the mechanics of nonholonomic systems came to light and a connection was revealed between the theory of electrical machines and the mechanics of nonholonomic systems.

Work on a book on the dynamics of nonholonomic systems begun in 1951 by Neĭmark had to be discontinued for a number of reasons. Much later, the authors were again attracted to problems of the mechanics of nonholonomic systems and they published a number of papers concerning the derivation of equations of motion, the applicability of mathematical models in the mechanics of nonholonomic systems, the stability of nonholonomic systems, and the directional stability of rolling systems. Not surprisingly, the idea of writing a book was revived and this time bore fruit.

Between the conception and completion of the book, its plan has undergone certain changes, but the general idea, namely, to bring together and unite the abstract theory of analytic mechanics and the applied problems of stability of rolling systems and the theory of electrical machines, has been preserved. The desire to present the concrete applications as fully as possible has also been preserved and is reflected in the selection of a large number of examples.

Inevitably, the attention devoted to the various aspects of the subject has been determined to a certain extent by the personal scientific

interest of the authors. As a result, many questions either have remained untouched or have not been treated as comprehensively as they warrant.

The same applies to the bibliography, which is extensive, but can in no way pretend to be exhaustive.

The book gives the first comprehensive and systematic exposition of the mechanics of nonholonomic systems, including the kinematics and dynamics of nonholonomic systems with classical nonholonomic constraints, the theory of the stability of nonholonomic systems, technical problems of the directional stability of rolling systems, and the general theory of electrical machines.

The first chapter consists of an exposition of the kinematics of nonholonomic systems. The fundamental concepts are introduced and the criterion of holonomy of kinematic constraints is established. The chapter is concluded with the theory of kinematic integrating mechanisms.

The second chapter is devoted to the study of the motion of nonholonomic systems on the basis of the general laws of dynamics. Čaplygin's theorem on conservation of angular momentum is generalized and classical problems of rolling of rigid bodies on a surface are studied.

The third chapter is devoted to the analytical mechanics of nonholonomic systems. Different forms of the equations of motion of nonholonomic systems are investigated and the use of transpositional relations is elucidated. Impulsive motions of nonholonomic systems are studied; conditions for the existence of first integrals are derived and an exposition given of the theory of Čaplygin's last multiplier.

The problem of the applicability of mathematical models in the mechanics of nonholonomic systems is studied in the fourth chapter. The influence of small parameters on the dynamics of nonholonomic systems is studied in a number of specific examples.

The fifth chapter is devoted to the stability and small oscillations of nonholonomic systems about equilibrium positions and also about steady motions.

In the sixth chapter the general equations of motion of a vehicle on wheels with pneumatic tires are derived. A number of specific technical problems of the directional stability of rolling systems are considered. These include the bicycle, motorcycle, automobile, the landing gear of an aeroplane, and a railroad truck.

In the seventh chapter, the dynamics of nonholonomic systems is studied from the point of view of the general theory of electrical machines. The Lagrange-Maxwell equations for electromechanical systems with

closed and open currents are derived on the basis of a discrete description of the electromagnetic processes in the quasi-steady approximation. Examples are given of electromechanical systems with nonholonomic constraints resulting from sliding contacts. The general equations of electrical machines are derived.

The book contains 172 figures and 5 tables, and 515 references are included in the bibliography.

The table of contents will give the reader a fairly complete idea of the contents of the book. We should like to point out that readers who are only interested in certain aspects of dynamics of nonholonomic systems can turn straight to the relevant chapters and use the remaining chapters for reference purposes.

Ju. I. Neĭmark, N. A. Fufaev

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INTRODUCTION

The birth of the theory of dynamics of nonholonomic systems occurred at the time when the universal and brilliant analytical formalism created by Euler and Lagrange was found, to general amazement, to be inapplicable to the very simple mechanical problems of rigid bodies rolling without slipping on a plane. Lindelöf's error, detected by Čaplygin, became famous and rolling systems attracted the attention of many eminent scientists of the time (Appell, Bobylev, Čaplygin, Cenov, Hamel, Hertz, Maggi, Voronec, Žukovskiĭ, and others). Earlier works of Ferrers, Korteweg, and C. Neumann were not immediately noted. The interest stimulated by the development of the analytic mechanics of nonholonomic systems has been maintained in some form or another to the present day, as may be seen from a glance at the bibliography given at the end of the book.[1)]

The development of many aspects of the analytic mechanics of nonholonomic systems is intimately linked with the analogous problems of the mechanics of holonomic systems, the theory of differential equations, tensor calculus, and differential geometry. The general geometrical treatment of problems of mechanics and its extension to nonholonomic systems have led to the creation of a new branch of differential geometry, the geometry of nonholonomic manifolds (by Schouten, Vagner, Vranceanu, and others).

At the same time, certain aspects of nonholonomic systems have required the study of a number of specific problems. In this connection, one may mention the discussion concerning the realization of nonholonomic constraints, the role of transpositional relations, and the investigation of the stability of nonholonomic systems. Contributions to the discussions on the realization of nonlinear nonholonomic constraints have been made by Appell, Delassus, and Beghin. The mechanics of

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[1)] For example, Agostinelli [1-6], Arrighi [27-29], Aržanyh [30-48], Castoldi [95-100], Cenov [102-109], Četaev [110-114], Dobronravov [131-143], Eden [150, 151], Efimov [152, 153], Fradlin [159-161], Johnsen [215-220], Kane [224-226], Kirgetov [232, 233], Kuhtenko [257-259], Lampariello [263-265], Lobas [276-279], Metelicyn [301-305], Muštari [316], Novoselov [345-356], Pignedoli [366-369], Pogosov [370-375], Rumjancev [394, 395], Šul'gin [425-442].

systems with nonlinear nonholonomic constraints was developed by Appell, Četaev, Hamel, Johnsen, Novoselov, and others.

Problems of the stability of nonholonomic systems have only been studied quite recently. After the work of Whittaker, one may mention the papers of Bottema, Aĭzerman, Gantmaher, Obmoršev, Knjazev, and others.

If nonholonomic systems in mechanics are a comparatively recent discovery, in electrodynamics the analogous discovery was made in the very recent past, although general comments and statements on the analogy between electromechanical systems with sliding contacts and rolling mechanical systems have been made more than once (by Carvallo, Appell, Liénard, Maxwell, Lorentz, and others). The so-called Barlow's wheel has been discussed in particular detail in this connection.

The ideas of analytical mechanics have been consistently extended to electromechanical systems by a natural study of rolling mechanical systems and electromechanical systems with sliding contacts from a common point of view. This has made it possible to include the general analytic theory of electrical machines in the dynamics of nonholonomic systems.

A field of great practical importance which has not yet been included in the general theory of nonholonomic systems are the various systems in which the rolling constraints are not classical. Such systems include the automobile, the landing gear of an aeroplane, the motorcycle, and railroad rolling stock. It was noted long ago that the kinematic constraints that arise during the rolling of an elastic pneumatic tire or a railroad wheel are very different from the classical idealization of the rolling of a rigid body without sliding. However, these new types of constraint were not studied in connection with the mechanics of nonholonomic systems but under the stimulation of technically important problems of directional stability. (Here, one should mention the investigations of Carter, Rocard, Greidanus, Keldyš, and others.)

In the present monograph, an attempt is made to discuss all the aforementioned problems from a common point of view.

CHAPTER I

KINEMATICS OF NONHOLONOMIC SYSTEMS

§1. Holonomic and nonholonomic discrete mechanical systems

In considering many problems of motion and equilibrium of mechanical systems, one may make use of a discrete idealization, i.e., one in which the mechanical state of a given system is defined by a finite number of parameters. In such a case, the system is said to be discrete. A good example is a system consisting of a finite number of material points and absolutely rigid bodies.

In many cases, the very construction of the system precludes an arbitrary motion of its various parts; their motions and positions are interrelated to a certain extent and must satisfy a number of conditions and restrictions. In mechanics, it is customary to say that constraints are imposed on the system. The actual form of these constraints may be very varied. For example, one may have two interlocking pinions, or two separated bodies may be joined by a rod of fixed length. For the present, we shall only be concerned with the restrictions on the geometrical position and motion of the individual parts of the system which result from these constraints. The constraints may impose restrictions on the possible geometrical positions of the individual parts of a system. They are then called geometric constraints. But if they restrict the kinematically possible motions of the system, i.e., the possible values of the velocities of the individual parts, such constraints are said to be kinematic. It is clear that every geometric constraint must, at the same time, give rise to a certain kinematic constraint. However, as we shall see, the converse need not hold, i.e., a constraint on the possible velocities of individual parts of a system need not lead to restrictions on their possible positions. In order to justify this assertion, let us consider a circular disc with a sharp edge that rolls without sliding on a plane.

Consider a disc rolling on a plane π, on which we fix a coordinate system Oxy. The position of the disc is given by the coordinates x and y of the point of contact M, the angle ψ measured from a chosen point of the rim to the point of contact M, the angle φ between the tangent to the disc at the point M and the Ox axis, and the angle of inclination

ϑ between the plane of the disc and π (Figure 1.1). The condition that the disc rolls without sliding on the plane π means that the instantaneous velocity of the point of contact of the disc is equal to zero at all times. An arbitrary small displacement of the disc can be characterized by the variations of the coordinates x, y, ψ, φ, and ϑ, which we shall denote by dx, dy, $d\psi$, $d\varphi$, and $d\vartheta$. Since the disc, of radius R, say, rolls without sliding, the variations of the five coordinates that determine the position of the disc cannot take completely arbitrary values, but must always satisfy two conditions:

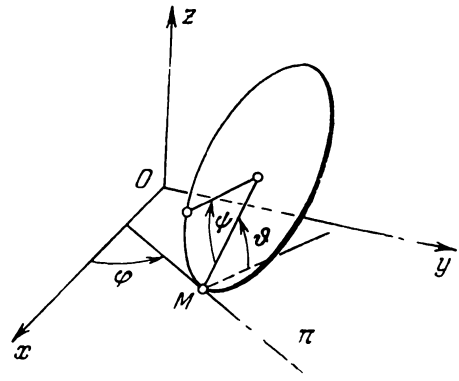

FIGURE 1.1

$$dx = R\cos\varphi\, d\psi, \qquad dy = R\sin\varphi\, d\psi. \tag{1.1}$$

Thus, the condition of rolling without sliding is a kinematic constraint expressed by equations (1.1), or the equations

$$\dot{x} = R\dot{\psi}\cos\varphi, \qquad \dot{y} = R\dot{\psi}\sin\varphi, \tag{1.2}$$

obtained from (1.1) by division by dt. However, although these conditions must be satisfied, the five coordinates x, y, ψ, φ, and ϑ may take all sets of values as the disc rolls on the plane, i.e., the disc can take up any position relative to the plane. To justify this assertion, we now show how the disc can be brought from a given position x_0, y_0, ψ_0, φ_0, ϑ_0 to any other prescribed position x_1, y_1, ψ_1, φ_1, ϑ_1. We first roll the disc from the point $M_0(x_0, y_0)$ of the plane π to the point $M_1(x_1, y_1)$ along any curve of length $R(\psi_1 - \psi_0 + 2\pi k)$, where k is an arbitrary integral number. We then rotate the disc around the perpendicular to the plane π at the point M_1 until the angle φ takes the desired value φ_1. Finally, we tilt the disc into the position $\vartheta = \vartheta_1$.

Thus the kinematic constraints (1.2) do not impose any restrictions on the possible values of the coordinates x, y, ψ, φ, and ϑ.

The existence of kinematic constraints that impose no restrictions on the possible configurations of a mechanical system was recognized in comparatively recent times. Even Lagrange in his celebrated *Mécanique Analytique* did not suspect their existence. Thus he assumed that inde-

pendent coordinates with independent variations could be chosen for any mechanical system once allowance had been made for the conditions imposed by the nature of the system. This oversight was only discovered after a considerable period of time in connection with the study of various cases of motion of rigid bodies constrained to roll without sliding on a plane or a more complicated surface. It was only in 1894 that Hertz introduced the distinction between holonomic and nonholonomic constraints and mechanical systems.

Let us suppose that the existing constraints on a system have been lifted and that its geometrical position is then determined by n quantities $q_1, q_2, \cdots, q_n$, known as generalized coordinates. Then, to an arbitrary variation of these generalized coordinates in time there corresponds a certain motion of the unconstrained system. If the constraints are now reimposed, there will be variations of the generalized coordinates q_1, $q_2, \cdots, q_n$ that no longer correspond to motions of the system. The variations of the generalized coordinates and their values must now satisfy a number of conditions, which follow from the constraints imposed. These conditions may, for example, be expressed by a system of inequalities of the form

$$f_\alpha (q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) \geqslant 0 \quad (\alpha = 1, 2, \ldots, m). \tag{1.3}$$

The corresponding constraints are then said to be one-sided and nonrestrictive or nonlimiting. Alternatively, we may have equations of the form

$$f_\alpha (q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) = 0 \quad (\alpha = 1, 2, \ldots, m), \tag{1.4}$$

and the corresponding constraints are then said to be two-sided and restrictive or limiting. In their turn, restrictive constraints are classified as geometric or kinematic and time-dependent or time-independent, depending on whether or not they contain the time explicitly. Constraints are said to be geometric if they are expressed by equations of the form

$$f_\alpha (q_1, q_2, \ldots, q_n; t) = 0 \quad (\alpha = 1, 2, \ldots, m), \tag{1.5}$$

and kinematic if the corresponding equations have the form

$$f_\alpha (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; q_1, q_2, \ldots, q_n; t) = 0 \quad (\alpha = 1, 2, \ldots, m). \tag{1.6}$$

This definition is completely in accord with what we said about geometric and kinematic constraints at the beginning of this section.

The kinematic constraints (1.6) are integrable if the corresponding

system of differential equations (1.6) is integrable; in the opposite case, they are nonintegrable. Integrable kinematic constraints are, essentially, geometric constraints. In contrast, nonintegrable constraints are not generally equivalent to geometric constraints and cannot be reduced to them. We say "generally" because one could conceive of cases when the system (1.6) is completely nonintegrable, i.e. is not equivalent to any system of equations of the form

$$dF_\alpha(q_1, q_2, \cdots, q_n; t) = 0$$

but nevertheless admits a singular integral which reduces the kinematic constraints to geometric constraints.[1)]

A mechanical system with nonintegrable kinematic constraints that cannot be reduced to geometric constraints is called a nonholonomic system. Thus, the distinguishing property of a nonholonomic system is that not all variations of its generalized coordinates, irrespective of the way the latter are chosen, correspond to a motion of the system which does not violate its constraints. We must emphasize that, according to this definition, the existence of a single nonintegrable constraint does not necessarily mean a system is nonholonomic, since this constraint may prove to be integrable by virtue of the remaining constraint equations. For example, each of the constraints

$$(x^2 + y^2)\, dx + xz\, dz = 0$$

and

$$(y^2 + x^2)\, dy + yz\, dz = 0$$

is nonintegrable but, taken together, they are integrable, since, after simple transformations, they can be reduced to the form

$$(x^2 + y^2)\, d\,(x^2 + y^2 + z^2) = 0, \qquad d \ln \frac{x}{y} = 0.$$

The nonholonomic constraints usually encountered in mechanics, i.e., nonintegrable kinematic constraints, can be expressed in a form which is linear in the generalized velocities:[2)]

$$A_i\,(q_1, q_2, \ldots, q_n; t)\, \dot{q}_i + B\,(q_1, q_2, \ldots, q_n; t) = 0. \tag{1.7}$$

Such constraints are said to be linear nonintegrable kinematic constraints.

[1)] The authors do not know of any concrete examples of systems of this kind.

[2)] If no confusion is possible, the summation sign will on occasion be omitted. Summation is then to be understood over an index that appears twice in a product.

If $B \equiv 0$, they are said to be homogeneous and, in addition, independent of the time if the coefficients A_i and B do not contain the latter explicitly.

Our example of a circular disc with a sharp rim constrained to roll without sliding on a plane is an example of a nonholonomic system with linear homogeneous time-independent nonintegrable kinematic constraints (1.2).

§2. Configuration space

The smallest number of variables needed to define the position of a system at a given moment of time is called the number of its generalized coordinates. If one regards the positions of a system as points of some space, in which proximity is defined naturally in terms of proximity of the corresponding positions of the system, the dimension of the space, which is called the configuration space, is equal to the number of coordinates of the system.

In qualitative dynamics, the topological structure of the configuration space plays an important role. Furthermore, the introduction of a suitable metric in the configuration space enables one to treat the problem of the motion of the system in terms of the motion of a point in a multi-dimensional space. In a number of important cases, this problem reduces to the purely geometrical question of geodesic lines.

We shall now clarify the concept of a configuration space by considering a number of simple examples.

EXAMPLE 1. Consider a system (Figure 1.2) consisting of a block, which can move along the guide rail HH', and a pendulum attached to the block. The position of the system can be specified by the displacement of the block x measured along the guide rail from some initial point O and by the angle φ between the pendulum and some fixed straight line. We shall represent the positions of the system by the points of a plane, establishing a correspondence between the points of the system and a point of the plane with Cartesian coordinates x and φ. Of course, this correspondence between the points of the plane and the configurations of the system is not one-to-one, since two different

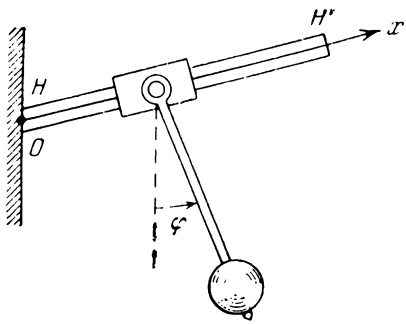

FIGURE 1.2

points of the plane (x, φ) and $(x, \varphi + 2k\pi)$ $(k = 0, \pm 1, \pm 2, \cdots)$ correspond to one and the same position of the block and pendulum. A one-to-one correspondence can be achieved, for example, by distinguishing the strip $0 \leqq \varphi < 2\pi$ on the x, φ plane. However, we should then forfeit the continuity of the correspondence: a continuous displacement of the system would not, in general, correspond to a continuous motion of the representative point in the distinguished strip, since the two neighboring positions of the system $(x, 2\pi - \epsilon)$ and $(x, 0)$ for an arbitrary small $\epsilon > 0$ correspond to widely separated points of the strip. In order to restore the continuity of the correspondence, we must arrange for the points $(x, 2\pi - \epsilon)$ and $(x, 0)$ to be close to each other. To this end, we roll the strip into a cylinder and "glue" together the opposite sides. The geometrical image thus obtained—a cylinder—may be taken as the configuration space of the system, since the correspondence between the positions of the system and the points of the cylinder is one-to-one and continuous.

As the system executes any given continuous motion, the representative point moves along a continuous curve on the cylinder. Thus if the block moves steadily to the right and the pendulum oscillates about some given position, the representative point moves along a sinusoidal curve L on the cylinder (Figure 1.3). A motion of the representative point along the spiral M on the cylinder in the direction indicated by the arrow corresponds to a motion of the system in which the block moves steadily to the right and the pendulum rotates clockwise around its point of suspension.

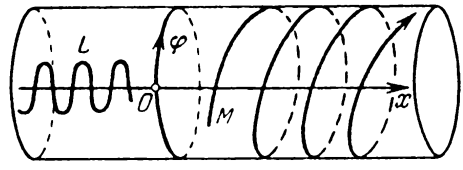

FIGURE 1.3

EXAMPLE 2. We shall now find the configuration space of the so-called double pendulum, which is a system consisting of a pendulum A able to rotate about a fixed axis OO' and a second pendulum B attached to the first in such a way that it can rotate about an axis aa' fixed to the pendulum A (Figure 1.4). As quantities specifying the position of the double pendulum, one can take the angles of rotation φ and ψ of the first and the second pendulum about the axes OO' and aa', respectively. Any position of the double pendulum is then determined

by two angles in the interval between 0 and 2π and may therefore be represented by a point of the square $0 \leqq \varphi \leqq 2\pi$, $0 \leqq \psi < 2\pi$ on the φ, ψ plane. Although the correspondence thus established between the positions of the double pendulum and the points of the square is one-to-one, it is not continuous. In order to make it continuous, we first roll the square into a cylinder, gluing the sides $\varphi = 0$ and $\varphi = 2\pi$, and then bend the cylinder and glue the opposite ends. As a result, we obtain a torus, the configuration space of the double pendulum (Figure 1.5). Actually, there is no need to bend the cylinder into a torus. Instead, we may agree to identify the opposite points of the sides of the square $0 \leqq \varphi < 2\pi$, $0 \leqq \psi < 2\pi$. It is clear that such a square with identified sides is also a configuration space for the double pendulum.

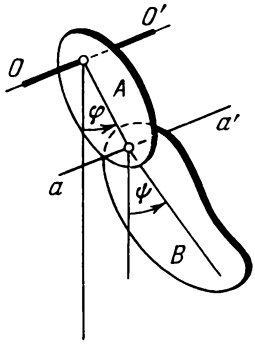

FIGURE 1.4

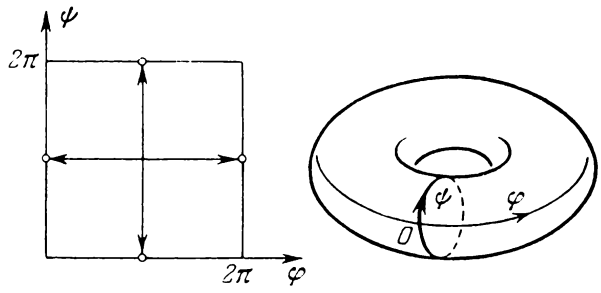

FIGURE 1.5

EXAMPLE 3. Suppose that a body A slides on a fixed plane π, touching the latter at three points. The first two points of contact can slide freely on the plane π in all directions, but the third contact is realized by a knife edge or a small wheel with a sharpened rim which can only move on the plane π along the knife edge or the sharp rim of the wheel (Figure 1.6). The position of the body A on the plane π can be specified by the coordinates x and y of the point of contact M of the knife edge and the angle φ between the direction

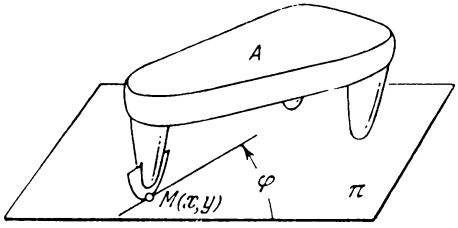

FIGURE 1.6

of the knife edge and some fixed direction on the plane. Although variations of the coordinates x, y, and φ cannot be prescribed arbitrarily, but must satisfy the constraints

$$dy = \tan\varphi dx,$$

the coordinates x, y, and φ themselves are nevertheless independent, since this constraint is nonintegrable. The configuration space of the system is therefore three-dimensional. It can be represented by the layer enclosed between the two planes $\varphi = 0$ and $\varphi = 2\pi$, the opposite points of these planes being identified (Figure 1.7). In contrast to the previous examples, it is no longer possible to visualize a "gluing together" of the identified planes in three-dimensional space.

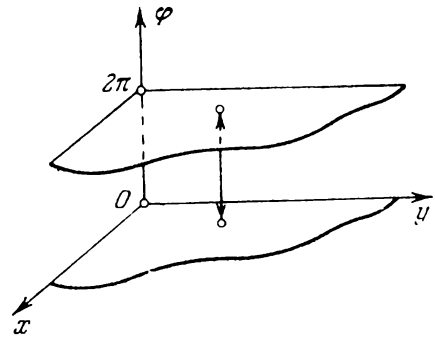

FIGURE 1.7

EXAMPLE 4. *Configuration space of a rigid body with a fixed point.* It is well known that an arbitrary finite displacement of a rigid body with a fixed point can be realized by rotating the body about an axis passing through the fixed point. As usual, we shall represent this rotation by a vector directed along the axis of rotation having a length equal to the angle of rotation. There is no need for the length of the rotation vector to exceed π. Vectors of length π pointing in opposite directions represent the same finite displacement of the body. We take some given position of the body as the reference position and specify any other position by the point at the end of the rotation vector that carries the body from the reference position to the given position. Thus, all possible positions of a rigid body with a fixed point are represented by the points of a sphere of radius π. Identifying diametrically opposed points of the surface of this sphere, we obtain a one-to-one and continuous correspondence between the positions of the body and the points of the sphere, i.e., a sphere

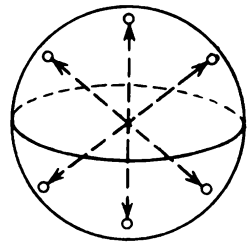

FIGURE 1.8

with the diametrically opposed points of its surface identified[3] is the configuration space for a rigid body with a fixed point (Figure 1.8).

EXAMPLE 5. *The configuration space of a circular disc constrained to roll without sliding on a plane.* We have already seen that the position of the disc can be specified by the five generalized coordinates, x, y, φ, ψ, and θ, and that we cannot dispense with any of them. The configuration space of the disc is therefore five-dimensional. Let us first consider the configuration space of a disc which touches the plane at a given point (x,y). For fixed x and y, the position of the disc is determined by the three angles φ, ψ, and θ, or, equivalently, by a point of a three-dimensional cube. Identifying opposite points of the faces of the cube, we obtain the desired configuration space, which is known as a three-dimensional torus (Figure 1.9). Any position of the disc on the plane can now be specified by two points: a point of the three-dimensional torus and a point of the plane determining the position of the point of contact of the disc on the plane. This correspondence between the positions of the disc and pairs of points of a three-dimensional torus and a plane is one-to-one and continuous. Regarding each pair of points as a point of a new five-dimensional space, in which proximity is defined in terms of proximity of the corresponding points of the three-dimensional torus and the plane,[4] we obtain a five-dimensional configuration space of a disc constrained to roll without sliding on a plane. It should be noted that if the disc moves on the plane but is not constrained to roll without sliding the configuration space is exactly the same, i.e., the imposition of nonholonomic rolling constraints does not affect the configuration space of the disc.

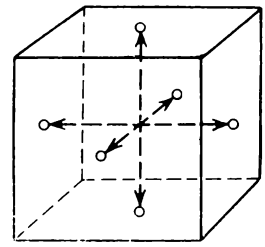

FIGURE 1.9

We shall now consider an example of a system whose constraints, and hence, in general, the totality of its possible positions, depend on the time. In this case, one can only speak of the configuration space of the system at a given instant of time or of a configuration space

[3] Such a topological image is called a three-dimensional projective space.

[4] By this we mean that two points of the five-dimensional space are assumed to be close to each other if the corresponding points of the three-dimensional torus and the plane are simultaneously close to each other. This procedure for obtaining a third topological image from the original two is called topological multiplication.

and the time, i.e., a union of the "set" of spaces corresponding to all instants of time.

EXAMPLE 6. Suppose that a material point M is forced to move around the circumference of a circle of radius $R = R(t)$ lying in a plane, the circle, in its turn, moving in accordance to a prescribed law. For each individual instant of time, the totality of possible positions (and configurations) of the system in space is a circle, but the circle is different at different instants of time. Thus, for each instant of time, there is a corresponding configuration space. The union of the configuration space and the time has the form of a cylinder; for the position of the material point M at the time t may be specified by the angle φ between the radius joining the given point to the center of the circle and some straight line of the moving plane. The set of instants of time t and angles φ can be represented by a point with coordinates t and φ on an infinite strip of width 2π (Figure 1.10). Opposite points of this strip correspond to the same position of the point M at the given instant of time. We must therefore identify these points. As before, the identification can be visualized by rolling the strip into a cylinder (Figure 1.10). The time is plotted along a generator of the cylinder and the angle φ on the circular sections. We emphasize that each of these circular sections of the cylinder is the configuration space of the system considered at the instant of time corresponding to the given section.

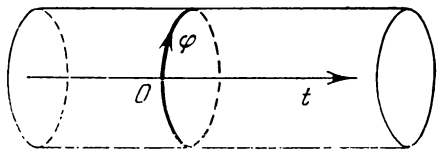

FIGURE 1.10

§3. Virtual displacements. The number of degrees of freedom

An arbitrary motion of the system can be represented in its configuration space and time by a certain curve. If the system is holonomic, the converse is true: any curve in this space, on which, of course, the time must increase, represents a certain motion of the system. However, this is not true for nonholonomic systems. Only certain curves in the configuration space and time correspond to motions of the system compatible with its constraints; for a point of the space representing the position of the system at a given instant of time cannot move in an arbitrary direction. This is because the differentials of the generalized coordinates and the time defining this displacement must now satisfy

a number of nonholonomic constraints

$$\sum_{k=1}^{n} a_{ik}\,dq_k + a_{i0}\,dt = 0 \quad (i = 1, 2, \ldots, m). \tag{3.1}$$

The set of all possible displacements compatible with these conditions forms a hyperplane of dimension $n - m$. Thus, at each point of the space, the possible displacements lie in a hyperplane of the space passing through the given point. Therefore, the curves which represent the kinematically possible motions of the system, and, in particular, its actual motion, are, at each point of the space, tangential to the hyperplane associated with this point. In connection with the problem of eliminating the reactions of ideal constraints—the principal problem in the derivation of the equations of motion of mechanical systems—it is customary to introduce the concept of virtual displacements. Virtual variations of generalized coordinates are variations of the generalized coordinates that satisfy the equations

$$\sum a_{ik}\delta q_k = 0, \tag{3.2}$$

obtained from the constraint equations of the system (3.1) by omitting the terms with dt. Displacements of a system corresponding to virtual variations of its generalized coordinates are called virtual displacements. The virtual displacements of the points of the system can be expressed in terms of the virtual variations of the generalized coordinates by means of the formulas

$$\delta \boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial q_k}\,\delta q_k. \tag{3.3}$$

We must emphasize that we are speaking of virtual displacements of the system at a given instant of time and for a given configuration of the system. According to this definition, the virtual displacements of the system may be regarded as the possible displacements of a system having the same time-independent homogeneous constraints as the original system at the given instant of time.

The number of linearly independent virtual displacements of a system is called the number of its degrees of freedom. For holonomic systems, the number of degrees of freedom is equal to the number of coordinates. This is not the case for a nonholonomic system, for which the number of degrees of freedom is less than the number of generalized coordinates by the number of nonintegrable kinematic constraints.

As a simple example, let us consider a circular disc constrained to roll on a plane. As we have already explained, the position of the disc can be specified by the five quantities x, y, φ, ψ, and θ, and the equations of the rolling constraints have the form

$$\dot{x} = R\dot{\psi}\cos\varphi, \quad \dot{y} = R\dot{\psi}\sin\varphi.$$

The equations of the rolling constraints of the disc do not depend explicitly on the time and are homogeneous. It follows that the equations determining the virtual and possible displacements are the same, and, in the given case, are

$$\delta x - R\cos\varphi\,\delta\psi = 0, \quad \delta y - R\sin\varphi\,\delta\psi = 0. \tag{3.4}$$

They admit three linearly independent solutions, and the disc therefore has three degrees of freedom. As the three linearly independent virtual displacements we may take, for example, the displacements:

1) $\delta x = R\cos\varphi\,\delta\psi$,	$\delta y = R\sin\varphi\,\delta\psi$,	$\delta\psi$,	$\delta\varphi = 0$,	$\delta\theta = 0$;
2) $\delta x = 0$,	$\delta y = 0$,	$\delta\psi = 0$,	$\delta\varphi$,	$\delta\theta = 0$;
3) $\delta x = 0$,	$\delta y = 0$,	$\delta\psi = 0$,	$\delta\varphi = 0$,	$\delta\theta$,

corresponding to virtual rotations of the disc about three straight lines passing through the point of support of the disc. These are the straight line perpendicular to the rim of the disc lying in the plane on which the disc rolls, the normal to the plane, and the tangent to the disc at the point of contact.

At each point of the five-dimensional configuration space of the disc, the set of virtual displacements of the disc (identical with the possible displacements) forms a three-dimensional plane, which is, by virtue of equations (3.4), orthogonal to the vectors

$$\boldsymbol{a}_1(1, 0, -R\cos\varphi, 0, 0) \text{ and } \boldsymbol{a}_2(0, 1, -R\sin\varphi, 0, 0).$$

It is clear that, as the disc rolls from one point to another, the position of this plane changes in accordance with the change of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. We note that if the disc were to roll on a plane that moves in a definite preassigned manner, the virtual displacements of the disc at each given instant would be the same as those of a disc rolling on the fixed plane obtained at each instant of time by arresting the motion of the moving plane.

§4. Phase space

The mechanical state of a system is defined by specifying the positions and velocities of all the material points composing the system.

Let $q_1, q_2, \cdots, q_n$ be the independent generalized coordinates and $\boldsymbol{r}_i$ the radius vector of the ith point

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \ldots, q_n; t). \tag{4.1}$$

Differentiating these equations with respect to the time, we obtain the velocity of the ith point

$$\boldsymbol{v}_i = \frac{d\boldsymbol{r}_i}{dt} = \dot{\boldsymbol{r}}_i = \frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r}_i}{\partial t}. \tag{4.2}$$

Equations (4.1) and (4.2) show that, at a given instant of time, the generalized coordinates $q_1, q_2, \cdots, q_n$ and the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ determine the mechanical state of the system. If the system is subjected to nonholonomic constraints, the generalized velocities are connected by a number of equations, so that the number of generalized velocities necessary to specify the state of the system may be reduced. Thus, if there are m nonholonomic constraints, it is generally sufficient to specify $n - m$ of the generalized velocities and to determine the remaining velocities from the m constraint equations. As quantities defining the velocities of the points of the system at a given instant of time, one is not restricted to taking merely $n - m$ generalized velocities; rather, one can take any $n - m$ independent functions of the generalized velocities, for example, linear functions of the form

$$\omega_i = \alpha_{ij}\dot{q}_j \qquad (i = 1, 2, \cdots, n - m). \tag{4.3}$$

The quantities ω_i are called kinematic characteristics. Thus, the state of a system with n coordinates and $n - m$ degrees of freedom can be specified at a given instant of time t by $2n - m$ quantities, namely by n generalized coordinates $q_1, q_2, \cdots, q_n$ and kinematic characteristics $\omega_1, \omega_2, \cdots, \omega_{n-m}$ as a point in a $(2n - m)$-dimensional space. We shall say that two points of this space are close to each other if the same is true of the corresponding states of the system. Such a space is called the phase space. In the general case, when the constraints of the system depend on the time and the expressions (4.1) of necessity contain the time explicitly, it is only meaningful to speak of the phase space or the state space of the system for a given instant of time. If it is desirable,

in the general case, to geometrize the change of the state of the system in time, one can introduce a state space and time, known as the phase space and time. We shall now consider some simple examples of phase spaces.

EXAMPLE 1. *Phase space of a physical pendulum.* The state of a physical pendulum—a heavy rigid body able to rotate about a horizontal axis—can be specified by the angle of deflection φ of the pendulum from a chosen reference position and the angular velocity $\dot{\varphi}$ of the system. Establishing a correspondence between a state of the physical pendulum defined by the angle φ and the angular velocity $\dot{\varphi}$ and a point of a cylinder with cylindrical coordinates φ and $\dot{\varphi}$ (Figure 1.11), we obtain a one-to-one and continuous correspondence between the points of the cylinder and the mechanical states of the physical pendulum. The phase space of a physical pendulum is therefore a cylinder.

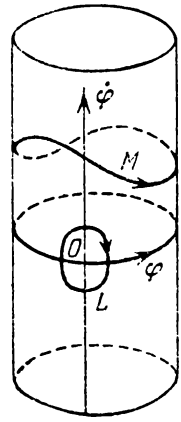

FIGURE 1.11

Let us consider two possible motions of the system. If the physical pendulum executes a periodic oscillatory motion, the representative point on the phase cylinder passes once, for each complete oscillation, around a closed loop (L in Figure 1.11), which does not encircle the cylinder. A periodic rotational motion of the pendulum is represented by a curve that goes right around the cylinder (M in the same figure).

EXAMPLE 2. We now consider the phase space of the nonholonomic system described in Example 3 of §2. The position of this system is determined by not less than three quantities. At the same time, the generalized velocities must satisfy only a single constraint, which, using the generalized coordinates x, y, and φ, we can write in the form

$$\dot{y} = \dot{x} \tan \varphi = 0.$$

Thus, the system has three coordinates and two degrees of freedom, and its phase space must be five-dimensional. The state of Čaplygin's sleigh (this is the name we shall give to this system) can, for example, be specified by the generalized coordinates x, y, and φ and the generalized velocities $\dot{x}$ and $\dot{\varphi}$. Instead of the generalized velocities $\dot{x}$ and $\dot{\varphi}$, one can take any two kinematic characteristics, for example, the velocity of the point of contact of the knife edge v, which is

$$v = \dot{x} \cos \varphi + \dot{y} \sin \varphi,$$

and the angular velocity $\dot{\varphi}$ of the sleigh about the point of contact. In contrast to the preceding example, it is impossible to visualize the five-dimensional phase space of Čaplygin's sleigh. It is however possible to visualize geometrically the forms of the different sections of the space. For example, the sections $\dot{x} = \text{const}$ and $\dot{\varphi} = \text{const}$ form spaces similar to the previously described configuration space of this system. The sections $\dot{x} = \text{const}$, $\dot{y} = \text{const}$, and $\varphi = \text{const}$ are ordinary planes and the sections $x = \text{const}$, $y = \text{const}$, and $\dot{\varphi} = \text{const}$ are cylinders.

EXAMPLE 3. On a fixed plane π, suppose that a material point is constrained to move at all times in the direction of a second point, which moves on the plane in accordance with a prescribed law. We denote by x and y the coordinates of the first point and write the law of motion of the second point with coordinates x_1 and y_1 in the form

$$x_1 = x_1(t), \quad y_1 = y_1(t).$$

The equation of the constraint can now be written in the form

$$\dot{y} = \frac{y - y_1}{x - x_1}\,\dot{x}.$$

This constraint is nonholonomic and depends on the time. The position and the velocity of the first point, i.e., its state at a given instant of time t, can be specified by the two generalized coordinates x and y and the generalized velocity $\dot{x}$ (the second generalized velocity $\dot{y}$ can be found from the equation of the nonholonomic constraint, since t is known). Therefore, the phase space of this system, which again has meaning only when it is associated with a definite instant of time, is a three-dimensional Euclidean space of the points $(x, y, \dot{x})$. The phase space and time of this system, i.e., its state space and time, is four-dimensional and is also a Euclidean space of the points $(x, y, \dot{x}, t)$.

§5. Kinematics of the rolling of one surface on another

Let us suppose that a moving body bounded by a rigid surface S is constrained to touch a given fixed surface S_1 at all times. At each instant of time t, a point O of the surface S touches a point O_1 of the surface S_1. The velocity of the point O of the moving surface at the instant of time at which it is the point of contact lies in the common tangent plane to the moving and fixed surfaces at their point of contact; for suppose the two surfaces touch at the new points O' and O_1' at a subsequent instant of time $t + dt$. During this time, the point O of the surface S moves along the segment OO', whose differential lies in the tangent plane.

The velocity v of the point of contact of the moving surface, which lies, as we have just proved, in the tangent plane, is called the velocity of sliding of the surface S with respect to the surface S_1. If there is no sliding, i.e., $v = 0$, we say that the surface S rolls on the surface S_1.

Apart from translational motion corresponding to a displacement of the point O, the moving body may also rotate about O with a certain angular velocity ω.

Thus, in the general case, the motion of the moving body can be decomposed into a translational motion with velocity of sliding v and a rotation around the point of contact with a certain angular velocity ω.

Let us now consider in more detail the case when the rigid surface S rolls without sliding on the fixed surface S_1, i.e., $v = 0$. Then, at every moment of time, the velocity field of the moving body is the same as if it were rotating with a certain angular velocity ω about some axis passing through the point of contact. Depending on the direction of the instantaneous axis of rotation, we distinguish pure or proper rolling and so-called pivoting. Pure rolling occurs if the instantaneous axis of rotation of the moving surface lies in the tangent plane; pivoting, if the instantaneous axis of rotation is normal to the tangent plane. As an example of pure rolling, we may take the rolling of a cylinder on a plane when the instantaneous axis of rotation is the generator along which the cylinder touches the plane. The spinning of a sphere on a horizontal plane about its vertical diameter is an example of pivoting.

In the general case, the rolling of a surface S on a surface S_1 can be decomposed into pure rolling and pivoting in accordance with the decomposition of the vector ω into a component ω_τ lying in the tangent plane and a component ω_n normal to the surface (Figure 1.12). If the surface S does not slide on S_1, rolling with pivoting can be regarded as pure rolling of a moving axoid Σ on a fixed axoid Σ_1. In this representation, of course, the line of intersection of the moving axoid Σ rolls without sliding along the line of intersection of the fixed surfaces S_1 and Σ_1.

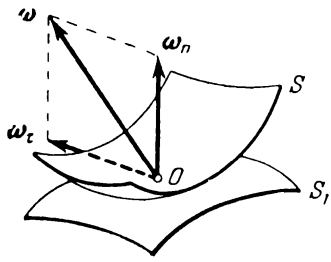

FIGURE 1.12

In order to write down the differential constraint equations of the rolling of one surface on another, one must find the equations that relate the angle of rotation $d\theta$ of the moving surface about its point of

contact with the fixed surface to the element $d\boldsymbol{s}$ described by the point of contact. To this end, we first decompose the angle of rotation $d\boldsymbol{\theta}$ of the moving surface into the angle of rotation associated with pure rolling $d\boldsymbol{\theta}_r$ and that associated with pivoting $d\boldsymbol{\theta}_p$. The element $d\boldsymbol{s}$ does not depend on the angle of rotation associated with pivoting, since the point of contact of the moving surface is not displaced as a result of pivoting. Thus, the displacement of the point of contact actually depends on only the angle of rotation $d\boldsymbol{\theta}_r$ associated with the pure rolling.

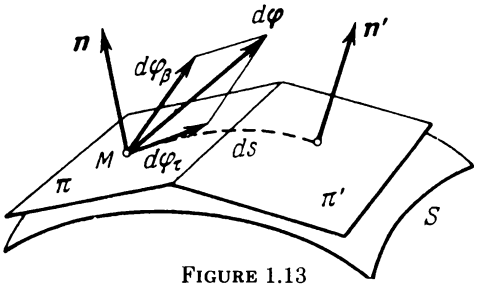

FIGURE 1.13

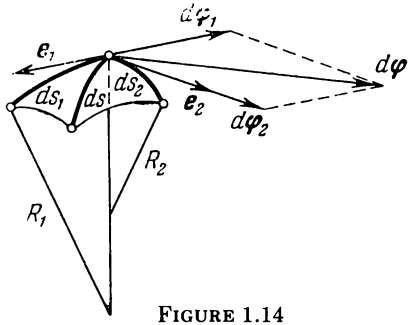

FIGURE 1.14

As the surface S rolls on the surface S_1, the point of contact will describe certain curves L and L_1 on the surfaces. Let $\boldsymbol{n}$ and $\boldsymbol{n}_1$ be the unit normals to S and S_1, their directions being chosen in such a way that they coincide at the point of contact of the surfaces. Then $d\boldsymbol{n}$ and $d\boldsymbol{n}_1$ are the variations of these normals as they are displaced from the point of contact along the curves L and L_1. It is clear from the geometry that the rolling angle $d\boldsymbol{\theta}_r$ is related to the variations of the normals $d\boldsymbol{n}$ and $d\boldsymbol{n}_1$ by the equation

$$d\boldsymbol{\theta}_r = \boldsymbol{n} \times (d\boldsymbol{n} + d\boldsymbol{n}_1) = \boldsymbol{n} \times d\boldsymbol{n} + \boldsymbol{n} \times d\boldsymbol{n}_1. \tag{5.1}$$

Thus, the rolling angle $d\boldsymbol{\theta}_r$ can be expressed directly in terms of the variations of the normals $\boldsymbol{n}$ and $\boldsymbol{n}_1$ as they are displaced by the amount $d\boldsymbol{s}$ along the, at the given instant, fixed curves L and L_1 on the surfaces S and S_1. Therefore, in order to complete our investigation of the relation between $d\boldsymbol{\theta}_r$ and $d\boldsymbol{s}$, it remains to consider how a normal vector varies as it is displaced along a given fixed surface. Let $\boldsymbol{n}$ and π be the unit normal and tangent plane to a surface at the point M of the surface. If the point M is displaced along the surface by $d\boldsymbol{s}$, the tangent plane π will then occupy a new position π', which may be made to coincide with the old position by rotating it through a certain angle $d\boldsymbol{\varphi}$ about the line of intersection of the planes π and π' (Figure 1.13). The rotation

vector $d\boldsymbol{\varphi}$, which is related to the variation of the normal $d\boldsymbol{n}$ by the equations

$$d\boldsymbol{\varphi} = \boldsymbol{n} \times \boldsymbol{n}' = \boldsymbol{n} \times d\boldsymbol{n}, \qquad d\boldsymbol{n} = d\boldsymbol{\varphi} \times \boldsymbol{n}, \tag{5.2}$$

can be decomposed into two components $d\boldsymbol{\varphi}_\tau$ and $d\boldsymbol{\varphi}_\beta$ along the direction of displacement of the point M and along the direction perpendicular to it. The second direction, as also the rotation vector $d\boldsymbol{\varphi}$, lies in the tangent plane. The angle $d\boldsymbol{\varphi}_\beta$ is called the angle of curvature of the surface in the direction $d\boldsymbol{s}$; the angle $d\boldsymbol{\varphi}_\tau$, its angle of torsion in the same direction. If the point M moves along lines of curvature of the surface, there is no torsion and the angle of curvature is

$$d\boldsymbol{\varphi}_\beta = ds/R, \tag{5.3}$$

where R is the corresponding principal radius of curvature of the surface. Regarding the displacement of the point M as made up of two displacements $d\boldsymbol{s}_1$ and $d\boldsymbol{s}_2$ along the lines of curvature, we obtain

$$d\boldsymbol{\varphi} = -\frac{\boldsymbol{e}_1\, ds_2}{R_2} + \frac{\boldsymbol{e}_2\, ds_1}{R_1}, \tag{5.4}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors along the lines of curvature and R_1 and R_2 are the radii of normal curvature of these lines (Figure 1.14). Comparing equations (5.1) and (5.2), we immediately see that the rolling angle $d\boldsymbol{\theta}_r$ of the moving surface S is

$$d\boldsymbol{\theta}_r = d\boldsymbol{\varphi} + d\boldsymbol{\varphi}_1, \tag{5.5}$$

where $d\boldsymbol{\varphi}$ and $d\boldsymbol{\varphi}_1$ are the angles of rotation of the tangent planes to the surfaces S and S_1 corresponding to the displacements of the tangent planes. We now express the angles $d\boldsymbol{\varphi}$ and $d\boldsymbol{\varphi}_1$ by means of equation (5.4), obtaining as a result

$$d\boldsymbol{\theta}_r = -\frac{\boldsymbol{e}_1\, ds_2}{R_2} + \frac{\boldsymbol{e}_2\, ds_1}{R_1} - \left(\frac{\boldsymbol{e}_1\, ds_2}{R_2}\right) + \left(\frac{\boldsymbol{e}_2\, ds_1}{R_1}\right), \tag{5.6}$$

in which the expressions not in parentheses refer to the surface S and the similar expressions in parentheses to the surface S_1.

We now turn to the problem of finding the kinematic constraints imposed on the angle of pivoting $d\boldsymbol{\vartheta}_p$. Let $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_1$ be the unit vectors tangent to the curves L and L_1 and $d\boldsymbol{\tau}$ and $d\boldsymbol{\tau}_1$ be projections of their variations onto the common tangent plane resulting from a displacement by $d\boldsymbol{s}$ along the curves L and L_1, respectively. Geometrically, it is clear that $d\boldsymbol{\tau} - d\boldsymbol{\tau}_1$ is the angle $d\boldsymbol{\theta}_p$ of pivoting of the rolling surface S. There-

fore, the angular velocity associated with the pivoting is directly related to the geodesic curvatures of the curves of pure rolling L and L_1, i.e.,

$$\omega_p = \frac{d\theta_p}{dt} = \frac{d\tau - d\tau_1}{ds} \cdot \frac{ds}{dt} = (K_g - K_{g_1})\, v, \tag{5.7}$$

where v is the velocity of the point of contact of the surfaces and K_g and K_{g_1} are the so-called geodesic curvatures of the curves L and L_1 at the same point. To illustrate the result we have just obtained, let us consider the motion of a sphere of radius R rolling on a plane with a constant angular velocity ω, i.e., we must find the curves which are described by the point of contact on the plane and the surface of the sphere. Since the principal radii of curvature of a plane are equal to infinity and those of the sphere are both equal to R, equation (5.6) leads to the relation

$$\omega_r = \frac{e_1\, ds_2 - e_2\, ds_1}{R},$$

from which it follows directly that the displacement velocity of the point of contact is perpendicular to the angular velocity $d\theta_r/dt$ of the pure rolling of the sphere and that its magnitude is $R\dot{\theta}_r = R\omega_r$. Thus, the point of contact describes a straight line on the plane and, on the surface of the sphere, a curve whose geodesic curvature is constant and, by equation (5.7), equal to

$$K_g = \frac{1}{v}\frac{d\theta_p}{dt} = \frac{\omega_p}{R\omega_r}.$$

A curve of constant geodesic curvature on a sphere is a circle whose radius ρ is related to its geodesic curvature K_g by the equation

$$K_g\rho = \sqrt{1 - \left(\frac{\rho}{R}\right)^2}.$$

To see this, we take a circle of radius ρ on the sphere and project it onto the tangent plane at some point of the circle. The curvature of the resulting curve at the point of contact of the plane and the sphere is, of course, the geodesic curvature K_g of the projected circle. It is clear from geometrical considerations that

$$\frac{1}{K_g} = \frac{\rho}{\cos\theta} \quad \text{and} \quad \rho = R\sin\theta,$$

where θ is the angle shown in Figure 1.15. Eliminating θ, we obtain the above relation between the geodesic curvature K_g and the radius ρ of a circle on a sphere of radius R.

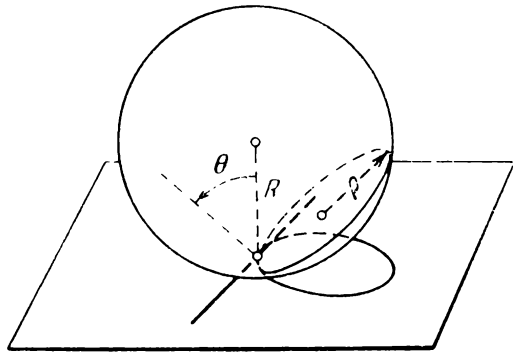

FIGURE 1.15

§6. Kinematic integrating mechanisms

The most important part of the most varied kinds of kinematic integrating mechanisms, such as planimeters, integrimeters, integraphs, or harmonic analyzers, not to mention the integrating device of the most complicated differential analyzers and mammoth mathematical machines capable of solving any differential equation, is a simple, but very accurately made small circular wheel with a sharpened edge, or alternatively, a thin and carefully rounded and occasionally grooved rim. We shall first consider the kinematics of such a wheel and then, taking a few examples, show, without going into constructional details, how the kinematic relationships realized by the wheel can be used for integration tasks.

An integrating wheel with a sharp edge is depicted schematically in Figure 1.16. Being pressed onto a sheet of paper which lies on a plane, it is forced to roll on the paper in the tangential direction without sliding in this or in the direction perpendicular to the tangent.

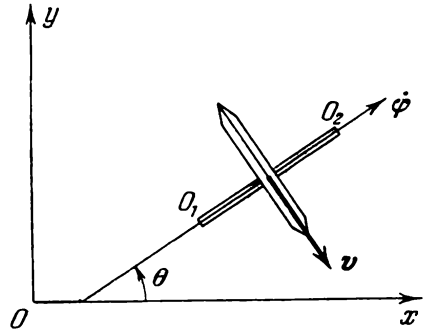

FIGURE 1.16

The position of the wheel can be specified by the angle φ of rotation about its own axis O_1O_2, which is kept in a horizontal position throughout the motion; by the angle θ between the axis of the disc and the Ox

axis of a system of coordinates Oxy set up on the plane on which the wheel rolls; and by the coordinates x and y of the point of contact or, equivalently, the center of the disc.

Thus, the velocity vector of the center of the disc always lies in the plane of the disc and the length of the path traversed by the disc is $R\varphi$, i.e., the constraint equations for the rolling of an integrating wheel with a sharp or, as one occasionally says, cutting edge may be written in the form

$$\dot{x} - \dot{y}\tan\theta = 0, \quad \sqrt{\dot{x}^2 + \dot{y}^2} = R\dot{\varphi}. \tag{6.1}$$

These equations can easily be transformed to a form linear in the generalized velocities by substituting the values first of $\dot{x}$ and then of $\dot{y}$ from the first equation into the second. The result is:

$$\dot{x} = R\,\dot{\varphi}\sin\theta, \quad \dot{y} = R\,\dot{\varphi}\cos\theta. \tag{6.2}$$

Thus, the wheel with a sharp rim is a nonholonomic system with four coordinates and two degrees of freedom, the constraints (6.1) or (6.2) holding when the wheel rolls.

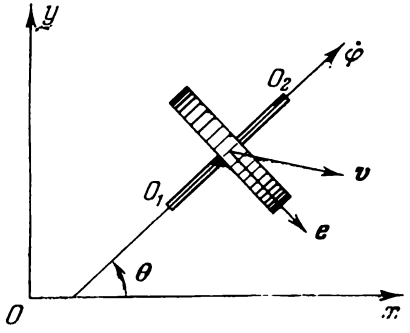

FIGURE 1.17

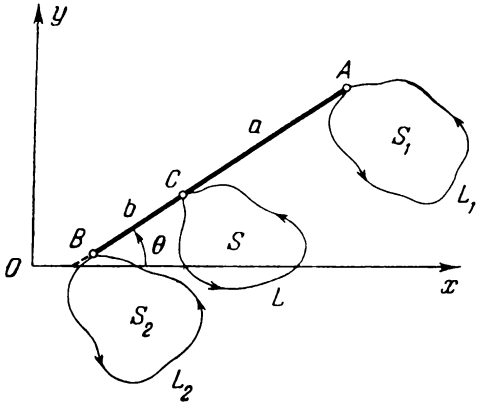

FIGURE 1.18

A wheel with a thin rounded edge or rim with transverse grooves differs from a wheel with a sharp edge in that it can slide in the transverse direction (Figure 1.17). If the wheel is displaced in its plane, it rolls without sliding, but if it moves in the transverse direction, i.e., in the direction of the axis O_1O_2, it slides and does not turn. The four coordinates x, y, θ, and φ, which determine the position of such a wheel, must now satisfy a single constraint, namely,

$$\dot{x}\sin\theta - \dot{y}\cos\theta = R\,\dot{\varphi}, \tag{6.3}$$

where $\dot{x}\sin\theta - \dot{y}\cos\theta$ is the projection of the velocity of the center of the wheel onto its plane.

A planimeter is a device constructed to measure areas enclosed by contours on a plane. To measure the area enclosed by a given contour L, one must guide the pointer of the planimeter around L. The enclosed area can then be read off the drum of the measuring wheel. In order to understand the general principle of a planimeter, we take a segment of a straight line and suppose that it moves on a plane in such a way that its ends A and B describe certain closed curves L_1 and L_2. Taking rectangular coordinate axes Oxy on the plane, we specify the position of the segment AB by the coordinates x and y of some point C on the segment and the angle θ between the segment and the Ox axis (Figure 1.18). The coordinates of the end points A and B of the segments are, respectively,

$$A\ (x + a\cos\theta,\ y + a\sin\theta),\ B\ (x - b\cos\theta,\ y - b\sin\theta), \tag{6.4}$$

where a and b are the lengths of the segments AC and CB.

Using now a well-known formula of integral calculus,[5] we find the areas S_1 and S_2 bounded by the contours L_1 and L_2:

$$\begin{aligned} S_1 &= \frac{1}{2}\oint_L (x + a\cos\theta)(dy + a\cos\theta\, d\theta) - (y + a\sin\theta) \\ &\quad \times (dx - a\sin\theta\, d\theta) = \frac{1}{2}\oint_L (x\,dy - y\,dx) + \frac{a^2}{2}\oint_L d\theta \\ &\quad + \frac{a}{2}\oint_L (\cos\theta\, dy - \sin\theta\, dx) + \frac{a}{2}\oint_L (y\sin\theta + x\cos\theta)\, d\theta \\ &= \frac{1}{2}\oint_L (x\,dy - y\,dx) + \frac{a^2}{2}\oint_L d\theta \\ &\quad + a\oint_L (\cos\theta\, dy - \sin\theta\, dx) + \frac{a}{2}\oint_L d(x\sin\theta - y\cos\theta), \end{aligned} \tag{6.5}$$

$$\begin{aligned} S_2 &= \frac{1}{2}\oint_L (x\,dy - y\,dx) + \frac{b^2}{2}\oint_L d\theta \\ &\quad - b\oint_L (\cos\theta\, dy - \sin\theta\, dx) - \frac{b}{2}\oint_L d(x\sin\theta - y\cos\theta). \end{aligned}$$

[5] The area bounded by a closed contour L is $\frac{1}{2}\oint (x\,dy - y\,dx)$, taken along the closed contour L in the positive direction on the contour.

We now form the difference $S_2 - S_1$, remembering that integrals around a closed contour of the total differentials of single-valued functions vanish:

$$S_2 - S_1 = \frac{b^2 - a^2}{2} \oint d\theta + (a + b) \oint (\sin\theta\, dx - \cos\theta\, dy). \tag{6.6}$$

If, in addition, the segment makes no complete rotations during the motion, then

$$\oint d\theta = 0, \tag{6.7}$$

and the value found for the difference $S_2 - S_1$ can be simplified to the form

$$S_2 - S_1 = l \oint (\sin\theta\, dx - \cos\theta\, dy), \tag{6.8}$$

where l is the length of the segment AB.

The quantity

$$\sin\theta\, dx - \cos\theta\, dy, \tag{6.9}$$

which is equal to the component of the displacement of the point C of the segment perpendicular to the segment, can be transformed by means of the kinematic constraint of rolling on a plane of a wheel with a rounded edge into the angle of rotation of the wheel. To this end, one need only "put" the wheel on the moving segment, attaching it at the point C; for, by the constraint equation of rolling of the wheel (6.3), the quantity

$$\sin\theta\, dx - \cos\theta\, dy = R\, d\varphi \tag{6.10}$$

and, hence,

$$S_2 - S_1 = lR(\varphi_2 - \varphi_1), \tag{6.11}$$

where $\varphi_2 - \varphi_1$ is the number of revolutions made by the wheel as the end points of the segment pass once round the contours L_1 and L_2.

Let us now suppose that one end of the segment describes the contour that encloses the area we wish to determine and the second is constrained to move at all times on a given open curve. The desired area S is then obviously

$$S = lR\, \Delta\varphi. \tag{6.12}$$

If the open curve on which one end of the segment is constrained to move is an arc of a circle, we have a so-called polar planimeter; if it is a straight line, a so-called linear planimeter.

As examples of the use of a wheel with a sharp cutting edge, we shall consider the extraordinarily simple "hatchet" planimeter of A. N. Krylov and the Abdank-Abakanowicz integraph. Krylov's hatchet planimeter is exceptionally simple. It consists of a rod with a cutting wheel attached at one end and a guiding pin at the other. In order to measure a given area with this device, one must start with the guiding pin at the approximate center of the area and then guide it along the contour as shown in Figure 1.19, noting the initial and final positions of the points of contact of the cutting wheel. Suppose that at the completion of the circuit the planimeter must be turned through an angle θ to return it to its original position (Figure 1.19); the desired area is then approximately

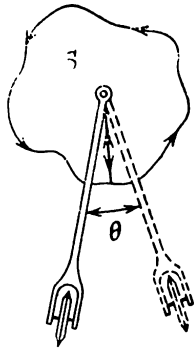

FIGURE 1.19

$$S \approx l^2\theta, \tag{6.13}$$

where l is the distance between the point of the guiding pin and the point of contact of the cutting wheel. For we can neglect the area of the region described by the point of contact of the cutting wheel (it is usually a relatively very small area) and apply equation (6.8), taking the point of support of the cutting wheel as the point C with coordinates x and y. As the pin is guided around the contour of the measured surface, the rolling constraint of the cutting wheel yields

$$\sin\theta\, dx - \cos\theta\, dy = 0, \tag{6.14}$$

and, as the planimeter is returned to the original position,

$$\sin\theta\, dx - \cos\theta\, dy = l\, d\theta. \tag{6.15}$$

Therefore, equation (6.8) gives

$$S \approx l^2\theta. \tag{6.16}$$

We mention that the hatchet planimeter is used not only to measure areas but also for so-called trailing iteration ([**306**], p. 272).

An integraph is a device that traces the graph of the antiderivative function $\int f(x)\,dx$ when the pointer of the device is guided along the graph of the function $f(x)$. The first integraph was constructed in 1882 by Professor Abdank-Abakanowicz at Cracow University (Figure 1.20).

Without going into the details of the construction of the Abdank-Abakanowicz integraph, we mention that the main working part of the device is a cutting wheel, whose displacement along the Ox axis is, by virtue of the construction of the integraph, equal to the corresponding displacement of the pointer. The tangent of the angle ψ between the plane of the cutting wheel and the Ox axis is proportional to the ordinate of the curve $y=f(x)$ along which the pointer moves. The displacement of the center of the wheel along the Oy axis is governed by the rolling constraint. It is clear that, under these conditions, the displacement of the center of the wheel along the Oy axis is

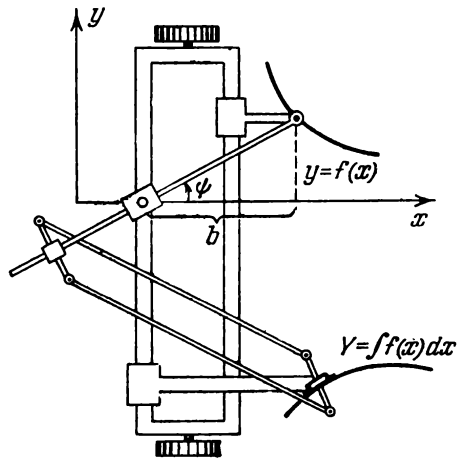

FIGURE 1.20

$$dy = \tan\psi\, dx = f(x)\, dx, \tag{6.17}$$

so that a pencil which moves in the same way as the center of the cutting wheel of the Abdank-Abakanowicz integraph will trace the graph of the antiderivative function

$$y = \int f(x)\, dx. \tag{6.18}$$

as the pointer moves along the graph of the curve $y=f(x)$.

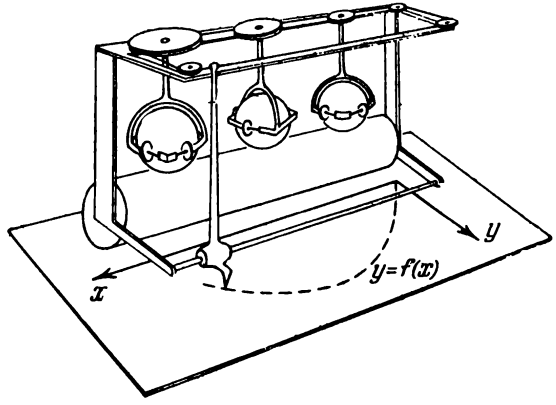

FIGURE 1.21

As our last example, we take the Henrici-Coradi harmonic analyzer (Figure 1.21). The essential part of this instrument is a sphere on which there is a frame carrying two integrating wheels. The sphere is supported on a circular cylinder that rests on a plane, and the frame with the integrating wheels can rotate about the vertical axis. The integrating wheels on the frame, whose position relative to the sphere is evident from Figure 1.22, are pressed lightly onto the sphere the whole time. Using the constraints of rolling of the sphere on the cylinder and of the integrating wheels on the sphere, we express the variations of the angles of rotation α and β of the integrating wheels in terms of the angle of rotation $d\psi$ of the cylinder and the angle of rotation φ of the frame about the vertical axis. If the sphere or the frame rotates about the vertical axis, the integrating wheels do not revolve, sliding over the surface of the sphere. However, if the sphere rotates about the horizontal axis parallel to the axis of rotation of the cylinder, the integrating wheels roll on the surface of the sphere. Let ρ, R, and r be the radii of the cylinder, sphere, and integrating wheels, respectively, and φ the angle of rotation of the frame, measured in the sense shown in Figure 1.22. It is then immediately clear that the angles of rotation $d\alpha$ and $d\beta$ of the integrating wheels resulting from a rotation of the cylinder through an angle $d\psi$ are

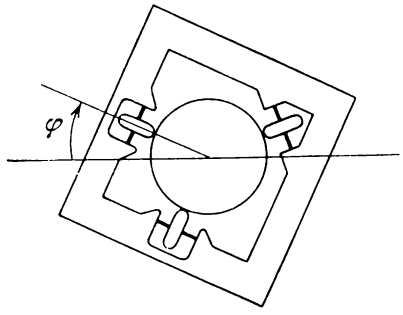

FIGURE 1.22

$$d\alpha = -R\cos\varphi\,\frac{\rho\,d\psi}{Rr} = -\frac{\rho}{r}\cos\varphi\,d\psi, \quad d\beta = \frac{\rho}{r}\sin\varphi\,d\psi. \tag{6.19}$$

The device is so constructed that the relations $\rho d\psi = dy$ and $\varphi = 2\pi n x/l$, where $n = 1, 2, \cdots$, and l is the period of the function $f(x)$, are satisfied as the pointer moves along the graph of the function $y = f(x)$. After the pin has traversed a period of the curve $y = f(x)$, the integrating wheels have turned through angles α and β, equal to

$$\alpha = -\frac{1}{r}\int \cos\frac{2\pi n}{l}x\,dy, \quad \beta = \frac{1}{r}\int \sin\frac{2\pi n}{l}x\,dy. \tag{6.20}$$

Integrating by parts, we can transform these expressions for α and β to the form

$$\alpha = \frac{2\pi n}{rl}\int y\sin\frac{2\pi n}{l}x\,dx, \quad \beta = -\frac{2\pi n}{rl}\int y\cos\frac{2\pi n}{l}x\,dx, \tag{6.21}$$

which show that α and β are proportional to the Fourier coefficients of the function $f(x)$.

§7. Holonomy criteria for a system with linear kinematic constraints

In connection with our introduction of the concepts of holonomic and nonholonomic systems and the numbers of coordinates and degrees of freedom, we are led naturally to the mathematical problem of finding methods of determining the number of coordinates and degrees of freedom of a system from its given constraint equations. Above all, the problem arises of finding criteria which permit one to establish whether a given system is holonomic or nonholonomic. In the case of a system with linear kinematic constraints, the latter problem (except in certain special cases) consists of finding necessary and sufficient conditions for the complete integrability of the system of equations

$$\sum_{k=0}^{n} a_{ik}\, dq_k = 0 \quad (i = 0,1,2,\ldots,m), \tag{7.1}$$

where the time t is denoted by q_0 for the sake of uniformity of the notation. Geometrically, the conditions (7.1) mean that a point of an $(n+1)$-dimensional space $R(q_0, q_1, \cdots, q_n)$ cannot be displaced arbitrarily, but that it must move along a curve that touches, at each of its points, a hyperplane L_{n-m} of dimension $n-m$, which contains all vectors $dq_0, dq_1, \cdots, dq_n$ satisfying the system of equations (7.1) at the given point. A system of Pfaffian equations is completely integrable if all such admissible curves emanating from any point of the space lie on a surface of dimension $n-m$ passing through the point. If, however, the system is nonintegrable, the representative point can reach any point of the configuration space, despite the restrictions imposed on its possible displacements at any given instant.

We shall solve this problem first for the very simple case when the system is simply a material point and the possible displacements or, equivalently, the possible variations of the Cartesian coordinates of the points are restricted by the constraint

$$P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0. \tag{7.2}$$

Geometrically, this constraint means that the point cannot move along any curve, but only along curves that, at each of their points, are perpendicular to the vector (P, Q, R) corresponding to the given point.

If all the admissible curves emanating from the point (x_0, y_0, z_0) at

which the material point is situated lie on some surface $F(x, y, z) = 0$, the system is holonomic. Let us assume that this is the case; then, at every point of the surface, the vector with components P, Q, and R is normal to the surface and therefore the curvilinear integral

$$\int P\,dx + Q\,dy + R\,dz, \tag{7.3}$$

taken along any curve lying on the surface, vanishes. The converse is also true, i.e., if the integral (7.3) taken along any curve lying on the surface $F = 0$ vanishes, the vector (P, Q, R) is normal to the surface at all of its points. Now suppose, in particular, that the integral (7.3) is taken along a closed contour. Then

$$\oint P\,dx + Q\,dy + R\,dz = \iint \operatorname{curl}\,(P,\ Q,\ R)\,d\sigma$$

and, hence, on the surface $F = 0$ we have

$$\operatorname{curl}(P, Q, R)\,d\sigma = 0$$

or

$$(P, Q, R)\operatorname{curl}(P, Q, R) = 0. \tag{7.4}$$

If we require that the admissible curves emanating from each point of a region of the space form a surface, the relation (7.4) must be satisfied in the whole of this region D.

The condition (7.4) is also sufficient for the Pfaffian form

$$P\,dx + Q\,dy + R\,dz$$

to be integrable, i.e., if (7.4) is satisfied, then for any point (x_0, y_0, z_0) of the region D there exists a surface $F = 0$ passing through the point and orthogonal at all of its points to the vector (P, Q, R). The proof is based on the fact that, by virtue of (7.4), the vector $\operatorname{curl}(P, Q, R)$ is tangential to the surface $F = 0$ (if it exists) and, hence, the surface $F = 0$ may be taken to consist of the vortex lines.

Let Γ be any possible curve passing through the point M_0 which is not a vortex line. Through the points of this curve we describe the vortex lines. These vortex lines form the desired surface orthogonal to the vector field (P, Q, R); for the integral (7.3) taken along any curve lying on this surface can be replaced by an integral taken along a closed contour composed of

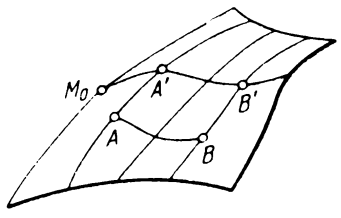

FIGURE 1.23

segments of vortex lines and the curve Γ, as shown in Figure 1.23. This is because the integral vanishes along the sections of the curves AA', $A'B'$, and $B'B$, i.e.,

$$\int_A^B P\,dx + Q\,dy + R\,dz = \int_A^B + \int_B^{B'} + \int_{B'}^{A'} + \int_{A'}^{A} = \oint_{(ABB'A'A)}.$$

However, by virtue of (7.4), the integral vanishes along any closed contour. Hence, the integral also vanishes along any curve Γ, and this proves the assertion.

If in a certain region

$$(P, Q, R)\operatorname{curl}(P, Q, R) \neq 0,$$

then, as was shown by Carathéodory, any two points of the region may be joined by an admissible curve, i.e., despite the constraint (7.2), the position of the point $M(x, y, z)$ in this region is arbitrary.

We now turn to the problem of finding a holonomy criterion for kinematic constraints in the general case. Following Lopšic,[6] we introduce a linear transformation M at each point of the space $R(q_0, q_1, \cdots, q_n)$. This transformation maps all vectors lying in M_{n-m} to zero and maps all vectors perpendicular to L_{n-m} onto themselves. Now suppose that the surface of dimension $n - m$, defined by a parametric equation in a vector form

$$\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, \ldots, u_{n-m}),$$

is integral. Then, by definition,

$$M\frac{\partial \boldsymbol{r}}{\partial u_i} = 0,$$

so that, differentiating with respect to u_j and introducing the notation $\boldsymbol{r}_i = \partial \boldsymbol{r}/\partial u_i$, we find

$$M\frac{\partial^2 \boldsymbol{r}}{\partial u_i\,\partial u_j} + \frac{\partial M}{\partial u_j}\boldsymbol{r}_i = 0. \tag{7.5}$$

Similarly, swapping the positions of the subscripts i and j, we obtain the equation

$$M\frac{\partial^2 \boldsymbol{r}}{\partial u_j\,\partial u_i} + \frac{\partial M}{\partial u_i}\boldsymbol{r}_j = 0. \tag{7.6}$$

---

[6] The very simple and geometrically transparent proof of the necessity and sufficiency of the conditions (7.8) is taken from the original paper of Lopšic [281].

Subtracting (7.6) from (7.5), we obtain

$$\frac{\partial M}{\partial u_j}\boldsymbol{r}_i - \frac{\partial M}{\partial u_i}\boldsymbol{r}_j = 0. \tag{7.7}$$

The relations (7.7) mean that for any two directions $d\boldsymbol{x}$ and $\delta\boldsymbol{x}$ tangent to the surface $\boldsymbol{r}(u_1, u_2, \cdots, u_{n-m})$ or, equivalently, lying in L_{n-m}, the following equation holds:

$$M_{\delta x}d\boldsymbol{x} - M_{dx}\delta\boldsymbol{x} = 0, \tag{7.8}$$

where the quantities $M_{\delta x}$ and M_{dx} are defined by the equations

$$M_{\delta x} = \frac{\partial M}{\partial x_0}\delta x_0 + \ldots + \frac{\partial M}{\partial x_n}\delta x_n, \quad M_{dx} = \frac{\partial M}{\partial x_0}dx_0 + \ldots + \frac{\partial M}{\partial x_n}dx_n.$$

We shall now show that a sufficient condition for the system (7.1) to be completely integrable is that the conditions (7.8) are satisfied for any two admissible displacements $\delta\boldsymbol{x}$ and $d\boldsymbol{x}$. Let $\boldsymbol{\tau}(u_1, u_2, \cdots, u_{n-m})$ be the vector emanating from an arbitrary point $q_0^0, \cdots, q_n^0$ which sweeps out the hyperplane L_{n-m}^0 corresponding to the point $q_0^0, q_1^0, \cdots, q_n^0$ as the parameters $u_1, u_2, \cdots, u_{n-m}$ are varied. From the point $q_0^0, q_1^0, \cdots, q_n^0$, we describe all admissible curves lying in the hyperplanes of dimension m passing through the vector $\boldsymbol{\tau}(u_1, u_2, \cdots, u_{n-m})$ and the vectors perpendicular to L_{n-m}^0. In a sufficiently small neighborhood of the point $q_0^0, q_1^0, \cdots, q_n^0$, each of these admissible curves is determined uniquely by the vector $\boldsymbol{\tau}(u_1, u_2, \cdots, u_{n-m})$ and, taken together, they form a certain surface

$$\boldsymbol{r} = \boldsymbol{r}\,(u_1,\ u_2,\ldots,\ u_{n-m}\ ;\ \sigma),$$

where a monotonic increase in the value of the parameter σ from zero carries us from the point $q_0^0, q_1^0, \cdots, q_n^0$ along the admissible curve which is tangent to the vector $\boldsymbol{\tau}(u_1, u_2, \cdots, u_{n-m})$. By construction,

$$M\frac{\partial \boldsymbol{r}}{\partial \sigma} = M\boldsymbol{r}_\sigma = 0. \tag{7.9}$$

We show that, in addition,

$$M\frac{\partial \boldsymbol{r}}{\partial u_i} = M\boldsymbol{r}_i = 0,$$

i.e., we show that the constructed surface is an integral surface. To this end, we shall find a differential equation satisfied by the vector

$$\boldsymbol{a} = M\boldsymbol{r}_i.$$

We differentiate this equation with respect to σ:

$$\boldsymbol{a}_\sigma = M\boldsymbol{r}_{i\sigma} + \frac{\partial M}{\partial \sigma}\boldsymbol{r}_i$$

$$= M\boldsymbol{r}_{i\sigma} + \left(\frac{\partial M}{\partial x_0}\frac{\partial x_0}{\partial \sigma} + \ldots + \frac{\partial M}{\partial x_n}\frac{\partial x_n}{\partial \sigma}\right)\boldsymbol{r}_i = M\boldsymbol{r}_{i\sigma} + M_{\boldsymbol{r}_\sigma}\boldsymbol{r}_i$$

and equation (7.9) with respect to u_i:

$$M\boldsymbol{r}_{\sigma i} + \frac{\partial M}{\partial u_i}\boldsymbol{r}_\sigma = M\boldsymbol{r}_{\sigma i} + M_{\boldsymbol{r}_i}\boldsymbol{r}_\sigma = 0.$$

Subtracting the one equation from the other, we obtain

$$\boldsymbol{a}_\sigma = M_{\boldsymbol{r}_\sigma}\boldsymbol{r}_i - M_{\boldsymbol{r}_i}\boldsymbol{r}_\sigma. \tag{7.10}$$

We decompose the vector $\boldsymbol{r}_i$ into a sum of the vectors $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$, where $\boldsymbol{a}_i$ is perpendicular to the hyperplane L_{n-m} at the corresponding point and $\boldsymbol{b}_i$ lies in the hyperplane. Clearly,

$$\boldsymbol{a}_i = M\boldsymbol{r}_i = \boldsymbol{a}.$$

Now, replacing $\boldsymbol{r}_i$ in the relation (7.10) by $\boldsymbol{a} + \boldsymbol{b}_i$ and noting that

$$M_{\boldsymbol{r}_i} = M_{\boldsymbol{a}} + M_{\boldsymbol{b}_i},$$

we obtain

$$\boldsymbol{a}_\sigma = M_{\boldsymbol{r}_\sigma}\boldsymbol{a} - M_{\boldsymbol{a}}\boldsymbol{r}_\sigma + M_{\boldsymbol{r}_\sigma}\boldsymbol{b}_i - M_{\boldsymbol{b}_i}\boldsymbol{r}_\sigma,$$

or, since the vectors $\boldsymbol{r}_\sigma$ and $\boldsymbol{b}_i$ lie in the hyperplane L_{n-m}, by virtue of equation (7.8),

$$M_{\boldsymbol{r}_\sigma}\boldsymbol{b}_i - M_{\boldsymbol{b}_i}\boldsymbol{r}_\sigma = 0,$$

and, hence, the vector $\boldsymbol{a}$ satisfies the linear homogeneous differential equation

$$\boldsymbol{a}_\sigma = M_{\boldsymbol{r}_\sigma}\boldsymbol{a} - M_{\boldsymbol{a}}\boldsymbol{r}_\sigma,$$

from which it follows that $\boldsymbol{a} = 0$ since the vector $\boldsymbol{a} = M\boldsymbol{r}_i$ is the null vector for $\sigma = 0$. This completes the proof of the assertion.

From the necessary and sufficient condition for integrability (7.8), we now wish to obtain explicitly similar conditions for the coefficients a_{ki} of the constraints (7.1). To this end, we first find the form of the transformation M, which can be written in components as follows:

$$dp_j = m_{jk}\,dq_k \quad (j = 0,\ 1,\ 2,\ \ldots,\ m). \tag{7.11}$$

The transformation M must map every vector $(dq_0, dq_1, \cdots, dq_n)$ satisfying the conditions (7.1) to the null vector. Hence,

$$m_{jk}\,dq_k = \lambda_{ji}a_{ik}\,dq_k,$$

where λ_{ji} are $(m+1)^2$ as yet undetermined factors. Vectors perpendicular to L_{n-m} and, in particular, the $m+1$ vectors $(a_{i0}, a_{i1}, \cdots, a_{in})$ are mapped by M onto themselves. Therefore, we have the equations

$$\lambda_{ji}a_{ik}a_{sk} = a_{sj}. \tag{7.12}$$

The conditions (7.12) determine the factors λ_{ji} uniquely. For our purposes, it is only important to note that equations (7.12) admit a solution for which the determinant of nth order composed of the factors λ_{ji} does not vanish. To see this, we augment the matrix of coefficients a_{ik}, assumed to have rank $m+1$, making it into a square matrix in such a way that its determinant does not vanish. Then the factors λ_{ji} can be found uniquely and are the elements of the inverse of the augmented matrix $\|a_{ik}\|$. By what we have proved, a necessary and sufficient condition for the system of equations (7.1) to be completely integrable is that at any point of the space $R(q_0, q_1, \cdots, q_n)$ and for all displacements $\delta\boldsymbol{q}(\delta q_0, \cdots, \delta q_n)$ and $d\boldsymbol{q}(dq_0, \cdots, dq_n)$ lying in L_{n-m} the following vector equation is satisfied:

$$\delta M\,d\boldsymbol{q} - dM\,\delta\boldsymbol{q} = \left(\frac{\partial M}{\partial q_0}\,\delta q_0 + \cdots + \frac{\partial M}{\partial q_n}\,\delta q_n\right)d\boldsymbol{q} - \left(\frac{\partial M}{\partial q_0}\,dq_0 + \cdots + \frac{\partial M}{\partial q_n}\,dq_n\right)\delta\boldsymbol{q} = 0.$$

Using equations (7.11), we can represent this vector equation as a system of equations in component form:

$$\delta\,(\lambda_{ji}a_{ih})\,dq_h - d\,(\lambda_{jh}a_{ih})\,\delta q_h = 0$$

or in the form of the system

$$\lambda_{jk}(\delta a_{ik}\,dq_k - da_{ik}\delta q_k) = 0.$$

Since the determinant $|\lambda_{ji}|$ may be assumed nonzero, the last equations yield the conditions

$$\delta a_{ik}\,dq_k - da_{ik}\delta q_k = 0, \tag{7.13}$$

which express the assertion of the well-known theorem of Frobenius, namely, that a necessary and sufficient condition for a system of Pfaffian forms $d\omega_0, d\omega_1, \cdots, d\omega_m$ to be completely integrable is the vanishing of all the so-called bilinear covariants $d\delta\omega_i - \delta d\omega_i$ whenever the conditions

$$d\omega_0 = d\omega_1 = \ldots = d\omega_m = 0,$$
$$\delta\omega_0 = \delta\omega_2 = \ldots = \delta\omega_m = 0$$

are satisfied.

As a simple example, let us take the Pfaffian equation

$$d\omega = a_1 dq_1 + a_2 dq_2 + a_3 dq_3 = 0, \tag{7.14}$$

defined in a three-dimensional space $R(q_1, q_2, q_3)$. As we have just seen, the condition for this equation to be completely integrable can be expressed by the equation

$$\delta a_1 dq_1 + \delta a_2 dq_2 + \delta a_3 dq_3 - da_1 \delta q_1 - da_2 \delta q_2 - da_3 \delta q_3 = 0$$

or, equivalently, the equation

$$\left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1}\right)(\delta q_2\, dq_1 - \delta q_1\, dq_2) + \left(\frac{\partial a_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1}\right)(\delta q_3\, dq_1 - \delta q_1\, dq_3) + \left(\frac{\partial a_2}{\partial q_3} - \frac{\partial a_3}{\partial q_2}\right)(\delta q_3\, dq_2 - \delta q_2\, dq_3) = 0 \tag{7.15}$$

for all $\delta q_1, \delta q_2, \delta q_3$ and dq_1, dq_2, dq_3 that make the Pfaffian form $d\omega$ vanish. It is readily seen that equation (7.15) can be represented in the very compact form

$$\operatorname{curl} A \cdot \delta q \times dq = 0, \tag{7.16}$$

where A is the vector with the components a_1, a_2, a_3. Since the displacement vectors δq and dq satisfy the conditions

$$A\,\delta q = 0, \quad A\,dq = 0,$$

i.e., they are perpendicular to A, the vector $\delta q \times dq$ has the same direction as the vector A. Hence, the condition (7.16) may be written in the form

$$A \operatorname{curl} A = 0, \tag{7.17}$$

which is identical with the previously found condition (7.4).

Thus, a necessary and sufficient condition for the equation (7.14) to be integrable is that equation (7.17) holds.

As a second, simple example, we shall determine whether the conditions for complete integrability for the constraint equations of a rolling disc are satisfied, i.e., for the constraints

$$d\omega_1 = dx - R\cos\varphi\, d\psi = 0,$$
$$d\omega_2 = dy - R\sin\varphi\, d\psi = 0.$$

We form the bilinear covariants

$$\delta\, d\omega_1 - d\,\delta\omega_1 = R\sin\varphi\, d\psi\,\delta\varphi - R\sin\varphi\, d\varphi\,\delta\psi,$$
$$\delta\, d\omega_2 - d\,\delta\omega_2 = -R\cos\varphi\, d\psi\,\delta\varphi + R\cos\varphi\, d\varphi\,\delta\psi.$$

Since the differentials $d\varphi$, $\delta\varphi$, $d\psi$, and $\delta\psi$ may take any values, the bilinear covariants could only vanish if

$$\sin\varphi = 0, \quad \cos\varphi = 0.$$

Thus, we see once again that the constraint equations for the rolling of a disc on a plane are nonintegrable.

We now turn to the more general problem of determining the number of degrees of freedom and the number of coordinates of a system from its constraint equations (7.1). Suppose we are given the system of equations (7.1). We shall say that a function $f(q_0, q_1, \cdots, q_n)$ is an integral of the system if for all $dq_0, dq_1, \cdots, dq_n$ satisfying the system we have

$$df = \frac{\partial f}{\partial q_0}\, dq_0 + \cdots + \frac{\partial f}{\partial q_n}\, dq_n = 0,$$

i.e., if the equation $df = 0$ is a consequence of the constraint equations. It is clear that every integral of the system of equations (7.1) permits one to reduce by one the number of coordinates determining the position of the systems. Conversely, if the number of independent coordinates can be reduced because of the existence of a functional connection $f(q_0, q_1, \cdots, q_n) = 0$ between the coordinates $q_1, q_2, \cdots, q_n$ and the time q_0, the function f is an integral of the system of equations (7.1). By virtue of this result, the problem of determining the number of coordinates and the number of constraints of a system reduces to the problem of finding all functionally independent integrals of the system of equations (7.1) or, rather, to the problem of determining their number. To this end, we shall first reduce the problem of finding integrals of the system of Pfaffian equations (7.1) to that of finding the integrals of a system of linear homogeneous partial differential equations.

Let $e_1, e_2, \cdots, e_{n-m}$ be $n - m$ unit orthogonal basis vectors of the hyperplane L_{n-m} of admissible directions of the system of equations (7.1). Any admissible displacement dq may be decomposed with respect to its basis vectors $e_1, e_2, \cdots, e_{n-m}$ at every point of the space $R(q_0, q_1, \cdots, q_n)$. The surface $f = 0$ is an integral surface if for every admissible displacement dq we have

$$\nabla f \cdot dq = 0, \tag{7.18}$$

where ∇f denotes the vector with components $\partial f/\partial q_0, \partial f/\partial q_1, \cdots, \partial f/\partial q_n$. It follows, in particular, that the equations

$$\nabla f \cdot e_1 = 0,\ \nabla f \cdot e_2 = 0,\ \ldots,\ \nabla f \cdot e_{n-m} = 0 \tag{7.19}$$

are satisfied for an integral surface. If the equations (7.19) hold for some function f, it is clear that the function f satisfies equation (7.18) for any admissible direction dq, i.e., f is an integral of the Pfaffian system (7.1). By virtue of this result, the determination of the integrals of a Pfaffian system is equivalent to the determination of the integrals of the system of partial differential equations (7.19). Denoting by $e_{i0}, e_{i1}, \cdots, e_{in}$ the components of the vector e_i, we write the system of equations (7.19) in the form

$$A_i f = e_{i0}\frac{\partial f}{\partial q_0} + e_{i1}\frac{\partial f}{\partial q_1} + \cdots + e_{in}\frac{\partial f}{\partial q_n} = 0 \quad (i = 1, 2, \ldots, n-m). \tag{7.20}$$

We now consider the integration of the system (7.20). If f is an integral of this system, then, as well as the equations

$$A_i f = 0, \quad A_j f = 0$$

the further equations

$$A_i A_j f - A_j A_i f = 0$$

hold for all i and j. Therefore, if we augment the system of equations (7.20) by equations of the same form constructed by means of the so-called Poisson brackets

$$[A_i A_j] = A_i A_j - A_j A_i$$

of the operators $A_1, A_2, \cdots, A_{n-m}$ occurring in the system, the resultant system will have the same integrals as the original system. As a result of such an extension of the system, the number of linearly independent equations is generally increased by virtue of the fact that the new system contains more equations than the original system. Repeating this process of extending the system, we arrive, after not more than $m + 1$ steps, at the so-called complete system, i.e., a system which cannot be extended by the addition of equations constructed by means of Poisson brackets, since such equations are now consequences of the already existing system of equations. An extension of a system of partial differential equations means, however, a decrease in the number of equations of the corresponding Pfaffian system. Thus, if the augmented system corresponding to the system (7.20) has $n - \bar{m}$ linearly independent

equations, where $\bar{m} \leqq m + 1$, the corresponding Pfaffian system contains $\bar{m} + 1$ equations. An important fact is that a system of Pfaffian equations corresponding to a complete system of partial differential equations is completely integrable.[7] This means that the equations (7.1) have common integrals with the reduced, completely integrable Pfaffian system. A completely integrable system of $\bar{m}$ Pfaffian equations has exactly $\bar{m}$ integrals $f_1, f_2, \cdots, f_{\bar{m}}$. Combining the $\bar{m}$ equations $df_1 = 0, \cdots, df_{\bar{m}} = 0$ with the linearly independent equations of the original system (7.1), we reduce the system (7.1) to the form

$$df_1 = 0, \ldots, \quad df_{\bar{m}} = 0,$$
$$d\omega_{\bar{m}+1} = 0, \ldots, \quad d\omega_m = 0.$$

Thus, the system of constraint equations (7.1) is equivalent to $\bar{m}$ (but not more) holonomic constraints

$$f_1 = C_1, \quad f_2 = C_2, \ldots, \quad f_{\bar{m}} = C_{\bar{m}}$$

and $m - \bar{m}$ nonholonomic, completely nonintegrable constraints

$$\omega_{\bar{m}+1} = 0, \ \ldots, \ \omega_m = 0.$$

This means that the system has $n - \bar{m}$ generalized coordinates and $n - m + \bar{m}$ degrees of freedom.

§8. Quasi-coordinates. Transpositional relations

The velocities of the points of a system in a given configuration and at a given instant of time may be specified either by means of the generalized velocities or, more generally, by means of the so-called kinematic characteristics, which are certain linear combinations of the generalized velocities. As an example of widely used kinematic characteristics, we may mention the projections of the angular velocity of a rigid body onto its principal axes of inertia. These projections can be expressed in terms of the Eulerian angles and their time derivatives by the well-known formulas

$$\begin{aligned} p &= \dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi, \\ q &= \dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi, \\ r &= \dot{\psi} \cos\theta + \dot{\varphi}. \end{aligned} \tag{8.1}$$

The quantities p, q, and r differ from the generalized velocities $\dot{\varphi}$, $\dot{\psi}$, and $\dot{\theta}$ by the fact that they are not total derivatives with respect to the time

[7] This is proved, for example, in Raševskiĭ's book [386], pp. 101-107.

of any of the generalized coordinates. This difference has no significance for the specification of the velocities of the points of a system, but it may be important from the analytical point of view. For, if the kinematic characteristics

$$\omega_i = \alpha_{ik}\dot{q}_k + \alpha_{i0} \tag{8.2}$$

are integrable linear forms, i.e., if for all values of the subscripts i, k, and s the equations

$$\frac{\partial \alpha_{ih}}{\partial q_s} - \frac{\partial \alpha_{is}}{\partial q_h} = 0 \tag{8.3}$$

are satisfied, the generalized coordinates determining the position of the system may be found by simple integration of the kinematic characteristics ω_i with respect to the time. On the other hand, if the kinematic characteristics are nonintegrable linear combinations of the generalized velocities, the determination of the generalized coordinates from kinematic characteristics given as functions of the time necessitates the integration of a system of linear differential equations, a difficult problem in general. However, in this case, we can, in analogy with the foregoing example, introduce the quantities

$$\pi_i = \int_0^t \omega_i \, dt, \tag{8.4}$$

which have a certain resemblance to coordinates but are no longer functions of the position of the system. In contrast to the ordinary generalized coordinates, these quantities π_i are called quasi-coordinates.

Before we turn to a systematic investigation of various questions connected with quasi-coordinates, let us consider in more detail the example quoted above of nonintegrable kinematic characteristics.

Suppose that a moving body has a fixed point at the common origin O of a fixed coordinate system $Oxyz$ and a system $O\xi\eta\zeta$ moving with the body (Figure 1.24). The position of the moving body is determined by its Eulerian angles φ, ψ, and θ, and p, q, and r are the projections of the angular velocity ω of the body onto the axes of the moving system of coordinates. An arbitrary virtual displacement of the rigid body can be decomposed into three rotations $\delta\varphi$, $\delta\psi$, and $\delta\theta$ about the Oz axis, the $O\zeta$ axis, and

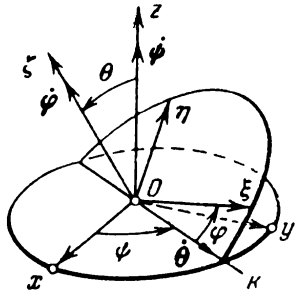

FIGURE 1.24

the line of nodes OK, respectively, or into three rotations δP, δQ, and δR about the moving axes $O\xi$, $O\eta$, and $O\zeta$. The real displacements can be decomposed in the same way, the total angles of rotation corresponding to the first method of decomposition being the Eulerian angles φ, ψ, and θ, and those corresponding to the second method, the quasi-coordinates P, Q, and R, given by

$$P = \int_0^t p\,dt, \quad Q = \int_0^t q\,dt, \quad R = \int_0^t r\,dt. \tag{8.5}$$

Thus, the only difference between the quasi-coordinates P, Q, and R and the Eulerian angles φ, ψ, and θ is that the former are total angles of rotation corresponding to a different decomposition of the elementary displacements of the rigid body from that adopted for the Eulerian angles.

In order to bring home the situation more clearly, let us imagine that the moving body carries three counters indicating the angles P, Q, and R through which the body has turned about the axes $O\xi$, $O\eta$, and $O\zeta$ during its motion, and that there are a further three counters that measure the angles of rotation of the moving body about the Oz, $O\zeta$, and OK axes. The first angles of rotation are quasi-coordinates but the second are coordinates. What then is the difference between them? In what way do the readings of the first three counters differ from those of the second set? Let us represent the variations of the Eulerian angles φ, ψ, and θ and the quasi-coordinates P, Q, and R by the curves described by the points with coordinates φ, ψ, and θ and P, Q, and R, putting time marks on the curves. Both curves describe the motion completely, i.e., the motion can be reproduced entirely if the curves are known. In this respect, there is again no difference between the coordinates φ, ψ, and θ and the quasi-coordinates P, Q, and R. However, whereas every point of the space of φ, ψ, and θ corresponds to a quite definite position of the body, a point in the space of P, Q, and R may, by itself, correspond to any position of the body. Of course, the points of a curve Γ describing the motion of the body in the space of P, Q, and R represent quite definite positions of the body, but individual points of this space do not correspond to any definite position of the body. To justify this statement, we point out that a correspondence between each point of the space of quasi-coordinates and a certain position of the body could be established if and only if any closed contour in the space of quasi-coordinates corresponded to a motion of the body in which it returns to its original position. However, this does not happen in the present case.

For example, passing from a point A of the space of quasi-coordinates to a point C along different paths ABC and $AB'C$ we arrive at different positions of the body. Hence, if the representative point passes round the closed contour $CBAB'C$, the body does not return to the original position. This circumstance, which prevents one from regarding the space of quasi-coordinates P, Q, and R as a coordinate space, is intimately related to the fact that rotations around the axes of a moving system of coordinates are noncommutative. In contrast, the rotations $\Delta\varphi$, $\Delta\psi$, and $\Delta\theta$ are commutative, two successively realized rotations $\Delta\varphi$, $\Delta\psi$, $\Delta\theta$ and $\Delta\varphi_1$, $\Delta\psi_1$, $\Delta\theta_1$ being equivalent to the rotation $\Delta\varphi + \Delta\varphi_1$, $\Delta\psi + \Delta\psi_1$, $\Delta\theta + \Delta\theta_1$.

Let us consider this question in more detail. Suppose that φ, ψ, and θ vary in such a way that the representative point describes a closed curve Γ in the configuration space. Let us find the variations ΔP, ΔQ, and ΔR of the quasi-coordinates P, Q, and R corresponding to the passage of the representative point around the closed contour. Using the kinematic Eulerian formulas (8.1), we immediately find that the variations of the quasi-coordinates are given by the expressions

$$\begin{aligned} \Delta P &= \oint_\Gamma \sin\theta \sin\varphi\, d\psi + \cos\varphi\, d\theta, \\ \Delta Q &= \oint_\Gamma \sin\theta \cos\varphi\, d\psi - \sin\varphi\, d\theta, \\ \Delta R &= \oint_\Gamma \cos\theta\, d\psi + d\varphi. \end{aligned} \tag{8.6}$$

These expressions show clearly that a closed contour Γ in the configuration space corresponds to an open contour in the quasi-coordinate space because the kinematic characteristics are nonintegrable. Taking an arbitrarily small contour in the configuration space consisting of the successive displacements $(\delta\varphi, \delta\psi, \delta\theta)$, $(d\varphi, d\psi, d\theta)$, $(-\delta\varphi, -\delta\psi, -\delta\theta)$ and $(-d\varphi, -d\psi, -d\theta)$, we can readily show that the corresponding variations of the quasi-coordinates P, Q, and R are

$$\begin{aligned} \Delta P &= (\delta d - d\delta) P = (\delta\varphi\, d\psi - \delta\psi\, d\varphi) \sin\theta \cos\varphi + \\ &\quad + (\delta\theta\, d\psi - d\theta\, \delta\psi) \cos\theta \sin\varphi + (\delta\theta\, d\varphi - d\theta\, \delta\varphi) \sin\varphi, \\ \Delta Q &= (\delta d - d\delta) Q = (\delta\psi\ d\varphi - d\psi\, \delta\varphi) \sin\theta \sin\varphi + \\ &\quad + (\delta\theta\, d\psi - d\theta\, \delta\psi) \cos\theta \cos\varphi + (\delta\theta\, d\varphi - d\theta\, \delta\varphi) \cos\varphi, \\ \Delta R &= (\delta d - d\delta) R = (\delta\psi\, d\theta - d\psi\, \delta\theta) \sin\theta. \end{aligned} \tag{8.7}$$

These expressions for the variations of the quasi-coordinates corresponding to the motion of the representative point around the chosen infinitely small contour in the configuration space are called transpositional relations. They are given this name because they tell us the difference resulting from the reversal of the operations of variation δ and d; for a displacement along the curve ABC corresponds to two successive rotations through angles ΔP and ΔQ about the axes $O\xi$ and $O\eta$, while a displacement along the curve $AB'C$ corresponds to the same rotations, but executed in the reverse order, i.e., first around $O\eta$ and then around $O\xi$.

The question posed earlier about the difference between the readings of the first and second set of counters can now be answered easily. The difference is that, whereas an individual reading of the counters of the Eulerian angles φ, ψ, and θ determines the position of the rigid body, an individual reading of the counters of the quasi-coordinates P, Q, and R does not tell us anything about the position of the body. The points of the space of the quasi-coordinates P, Q, and R do not correspond to any definite positions of a rigid body with a fixed point, but this does not prevent us from representing a given motion of the rigid body in the quasi-coordinate space, just as we can in the configuration space. For example, regular precession defined by the equations

$$\varphi = mt + \varphi_0, \quad \psi = nt + \psi_0, \quad \theta = \theta_0 \tag{8.8}$$

is represented by a straight line in the plane $\theta = \theta_0$ in the configuration space φ, ψ, θ, whereas it is represented by a spiral on a circular cylinder in the space of quasi-coordinates.[8] The equation of the spiral has the form

$$\begin{aligned} P &= \int_0^t p\,dt = -\frac{n}{m}\sin\theta_0\,[\cos(mt + \varphi_0) - \cos\varphi_0], \\ Q &= \int_0^t q\,dt = +\frac{n}{m}\sin\theta_0\,[\sin(mt + \varphi_0) - \sin\varphi_0], \\ R &= \int_0^t r\,dt = (n\cos\theta_0 + m)\,t, \end{aligned} \tag{8.9}$$

[8] This example illustrates graphically how different points in the space of quasi-coordinates can correspond to the same position of the body.

where the displacement velocity of the representative point along the spiral has a very simple intuitive meaning. It is the instantaneous angular velocity of rotation of the rigid body, i.e., the velocity components of the representative point along the P, Q, and R axes are the components of the angular velocity vector of the body along the moving axes. If we are mainly interested in representing the motion of the vector ω in the body, the second method of representation of the motion of a rigid body is actually preferable.

CHAPTER II

STUDY OF THE MOTIONS OF NONHOLONOMIC SYSTEMS ON THE BASIS OF THE GENERAL LAWS OF DYNAMICS. CLASSICAL PROBLEMS OF RIGID BODIES ROLLING ON A SURFACE

By the general laws of dynamics we understand the laws governing the changes in the momentum, angular momentum, and kinetic energy, and also the various conditions under which integrals of motion may be obtained from these laws. Despite the considerable successes of analytical mechanics, the general laws of dynamics and the integrals of motion obtained from them continue to play a very important role. In his investigations, Žukovskiĭ made extensive use of the general laws of dynamics. In 1893, the solution was given to the difficult problem of the motion without sliding on a horizontal plane of a hollow sphere containing a gyroscope. In 1897, Čaplygin found a number of new conditions under which integrals of motion exist, these integrals being generalizations of the well-known integrals corresponding to the conservation of momentum and angular momentum. He illustrated their use in the study of various systems consisting of several rigid spheres rolling and sliding on each other. In 1903, Čaplygin gave a brilliant solution to the general problem of the rolling of a symmetric sphere on a horizontal plane. This solution was based on his generalization of the law of conservation of angular momentum (the theorem of areas).

In the present chapter, we shall study the general laws of dynamics and apply them systematically to the study of mechanical systems with nonholonomic constraints.

Using these general laws, we shall set up and, to a greater or lesser extent, integrate the equations of motion of various specific systems. We shall assume that the nonholonomic constraints of these systems are ideal, i.e., they are such that the reaction forces of the constraints do no work under any virtual displacement of the system. In the case of rolling, which we shall always assume occurs without sliding, this means the absence of the so-called frictional forces of rolling and pivoting.

In writing down the equations of motion, we shall generally make use of moving axes, invoking the fundamental formula[1)]

$$\left(\frac{dA}{dt}\right)_K = \left(\frac{dA}{dt}\right)_M + \Omega \times A, \tag{0.1}$$

which relates the derivatives of a vector A with respect to the time t in a system K and a system M rotating with respect to K with an angular velocity Ω. It should be noted that successful use of moving axes depends largely on skillful choice of the moving axes and also the variables describing the motion.

§1. General laws of dynamics. Generalization of the angular momentum integral

Let us consider a system consisting of N material points with masses m_i $(i = 1, 2, \cdots, N)$. Suppose the system admits a virtual rotation about some axis L – a fixed straight line or straight line of fixed direction passing through the center of mass of the system. Since the center of mass is generally in motion, a straight line of fixed direction through the center of mass will also be in motion in space. If the external forces do not exert a moment about this axis, the law of conservation of angular momentum of the system about this axis is, of course, satisfied. Čaplygin drew attention to the fact that an integral of the motion can also be obtained in the more general case when the axis L moves in such a way that the coordinates of the center of mass r_C and the coordinates r_A of some point A of this axis are related at all times by the equations

$$x_C = \lambda x_A + x_0, \; y_C = \lambda y_A + y_0, \; z_C = \lambda z_A + z_0, \tag{1.1}$$

where λ, x_0, y_0, and z_0 are arbitrary constants. The case of a fixed straight line corresponds to the special case $x_A = \text{const}$, $y_A = \text{const}$, and $z_A = \text{const}$, and the case of a straight line passing through the center of mass of the system corresponds to $\lambda = 1$, $x_0 = y_0 = z_0 = 0$. To see this, we note that for constant values of x_A, y_A, and z_A the straight line L passing through the point A is a fixed axis, and, in the second case ($\lambda = 1$, $x_0 = y_0 = z_0 = 0$), we arrive at the conservation law for the case when the system rotates about an axis of fixed direction passing through the center of mass.

[1)] Henceforth, a derivative in a moving system of coordinates will be denoted by a tilde above the derivative, i.e., $(dA/dt)_M \equiv \tilde{dA}/dt$.

These conditions of Čaplygin can, in fact, be generalized slightly further. We shall show that the only condition necessary for the existence of Čaplygin's integrals is that the projections of the velocity v_C of the center of mass and the velocity of some point A of the axis L onto the plane perpendicular to L are always parallel.

We shall imagine that the system is freed from the constraints, which we shall replace by the, in general, unknown reaction forces $\boldsymbol{R}_i$ of these constraints. We shall assume, at the same time, that the system admits an arbitrary virtual displacement as a fixed rigid system.

We now write down the d'Alembert-Lagrange equation

$$\sum_{i=1}^{N} (m_i \boldsymbol{w}_i - \boldsymbol{F}_i - \boldsymbol{R}_i)\, \delta \boldsymbol{r}_i = 0, \tag{1.2}$$

where w_i is the acceleration of the material point with mass m_i and $\boldsymbol{F}_i$ is the resultant of the external forces acting on this point. As virtual displacements $\delta \boldsymbol{r}_i$, we shall only consider displacements under which the system moves as an absolutely rigid body. In this case, an arbitrary displacement $\delta \boldsymbol{r}_i$ can, of course, be represented in the form

$$\delta \boldsymbol{r}_i = \delta \boldsymbol{r}_A + \delta \boldsymbol{\varphi} \times \boldsymbol{r}'_i, \tag{1.3}$$

where $\boldsymbol{r}_A$ is the radius vector from the fixed center O to some, in general, moving point A and $\delta \boldsymbol{\varphi}$ is the angle through which the system turns as a result of the virtual rotation about the axis L passing through the point A. Substituting the expression (1.3) for the virtual variations $\delta \boldsymbol{r}_i$ into (1.2), we obtain the equation

$$m_i \ddot{\boldsymbol{r}}_i\, \delta \boldsymbol{r}_A + m_i \ddot{\boldsymbol{r}}_i \cdot \delta \boldsymbol{\varphi} \times \boldsymbol{r}'_i = (\boldsymbol{F}_i + \boldsymbol{R}_i)\, \delta \boldsymbol{r}_A + (\boldsymbol{F}_i + \boldsymbol{R}_i)\, \delta \boldsymbol{\varphi} \times \boldsymbol{r}'_i. \tag{1.4}$$

The terms on the right-hand side of this equation represent the virtual work of the forces $\boldsymbol{F}_i$ and the reaction forces $\boldsymbol{R}_i$; it may be written in the form

$$\delta A_{\boldsymbol{F}} + \delta A_{\boldsymbol{R}} = (\boldsymbol{F} + \boldsymbol{R})\, \delta \boldsymbol{r}_A + (\boldsymbol{M}'_{\boldsymbol{F}} + \boldsymbol{M}'_{\boldsymbol{R}})\, \delta \boldsymbol{\varphi},$$

where $\boldsymbol{F}$ and $\boldsymbol{R}$ are the resultants of the forces $\boldsymbol{F}_i$ and $\boldsymbol{R}_i$, respectively, and which, in fact, reduce to the sum of the external forces; $\boldsymbol{M}'_{\boldsymbol{F}}$ and $\boldsymbol{M}'_{\boldsymbol{R}}$ are the moments of these forces about the point A, which also reduce to the resulting moment of the external forces alone. Denoting by $\boldsymbol{Q} = m_i \boldsymbol{v}_i$ the momentum vector of the system, we have

$$m_i \ddot{\boldsymbol{r}}_i \delta \boldsymbol{r}_A = \frac{d\boldsymbol{Q}}{dt}\, \delta \boldsymbol{r}_A.$$

Similarly, introducing the vector $G' = r_i' \times m_i v_i$, the angular momentum of the system about the moving point A, and using the vector equation $r_i = r_A + r_i'$, we obtain

$$m_i \ddot{r}_i \cdot \delta\varphi \times r_i' = \delta\varphi \cdot r_i' \times m_i \ddot{r}_i = \delta\varphi \left(\frac{dG'}{dt} + \dot{r}_A \times Q \right),$$

and equation (1.4) can then be written in the form

$$\frac{dQ}{dt} \delta r_A + \left(\frac{dG'}{dt} + \dot{r}_A \times Q \right) \delta\varphi = \delta A_F + \delta A_R. \tag{1.5}$$

By virtue of the independence of the virtual displacements δr_A and $\delta\varphi$, equation (1.5) yields the well-known law of variation of the momentum

$$\frac{dQ}{dt} = F + R \tag{1.6}$$

and the law of variation of the angular momentum in the form

$$\frac{dG'}{dt} + v_A \times Q = M_F' + M_R'. \tag{1.7}$$

Now suppose $\delta\varphi$ is a virtual rotation of the system with a fixed direction satisfying the condition

$$v_A \times Q \cdot \delta\varphi = v_A \times v_C \cdot m \delta\varphi = 0, \tag{1.8}$$

where m is the mass of the complete system and v_C is the velocity of the center of mass. Assuming that the constraints are ideal, we obtain $\delta A_R = 0$. From the general equation (1.4) or from the equation of moments (1.7), we find that, if the moment of the forces about the axis L of the virtual rotation vanishes, i.e., if $M_F' \delta\varphi = 0$, the projection of the angular momentum G' onto the direction $\delta\varphi$ is conserved, since, in this case,

$$\frac{dG'}{dt} \delta\varphi = \frac{d}{dt} (G'l) \delta\varphi = 0, \tag{1.9}$$

where l is a unit vector along the fixed direction of the axis L passing through the, in general, moving point A. It follows from (1.9) that

$$(G'l) = \text{const}. \tag{1.10}$$

What is the physical meaning of the condition (1.8)? It follows from (1.8) that the vectors v_A, v_C, and l are parallel to one and the same plane. Let the fixed direction of L coincide with the direction of the Oz axis. The condition (1.8) then reduces to the following equation:

$$\frac{\dot{y}_C}{\dot{x}_C} = \frac{\dot{y}_A}{\dot{x}_A}, \tag{1.11}$$

where the quantities $\dot{z}_C$ and $\dot{z}_A$ may be arbitrary. Thus, for a generalized integral of angular momentum to hold, it is necessary and sufficient that the projections of the velocity of the center of mass and the velocity of some point A of the axis L onto the plane perpendicular to this axis are always parallel. This is the desired generalization of Čaplygin's conditions (1.1). If the conditions (1.1), which can be written in the vector form

$$\boldsymbol{r}_C = \lambda \boldsymbol{r}_A + \boldsymbol{r}_0, \tag{1.12}$$

hold, the condition (1.8) is satisfied identically, since, in such a case,

$$\boldsymbol{v}_A \times \boldsymbol{v}_C = \lambda \boldsymbol{v}_A \times \boldsymbol{v}_A \equiv 0.$$

We shall now give some examples of the use of the generalized angular momentum integral.

EXAMPLE 1. *Inertial rolling of a dynamically symmetric ball on the surface of a sphere.* A ball that touches a sphere admits an arbitrary virtual rotation about the point of contact. The moment of the external forces about the point of contact vanishes and the velocity of the point of contact is always parallel to the velocity of the center of the ball, which coincides with its center of mass. It follows that the angular momentum of the ball about the moving point of contact of the ball and the sphere is conserved, i.e., $G' = G'_0$.

In the case of a homogeneous sphere, we have

$$\boldsymbol{G}' = (J + 1)\,\boldsymbol{\omega} - (\boldsymbol{\omega}\boldsymbol{n})\,\boldsymbol{n}, \tag{1.13}$$

where J is the moment of inertia of the ball about any axis passing through its center; $\boldsymbol{\omega}$ is the instantaneous angular velocity of rotation; $\boldsymbol{n}$ is the unit normal vector to the sphere at the point of contact of the ball; and the mass and radius of the ball are taken as the unit mass and unit length, respectively. Substituting the value $J = 2/5$ into the expression (1.13), we obtain the generalized angular momentum integral

$$7\boldsymbol{\omega} - 5\,(\boldsymbol{\omega}\boldsymbol{n})\,\boldsymbol{n} = \text{const}.$$

This relation shows that the angular velocity $\boldsymbol{\omega}$ of rotation of the ball as it rolls on the sphere depends only on the direction of the normal $\boldsymbol{n}$. Therefore, in the special case when the ball rolls on a plane, the vector

ω is constant, and, hence, the ball rolls on the plane with constant angular velocities of pure rolling and pivoting. In this case, the point of contact of the ball and the plane moves along a straight line with the constant velocity $v = \omega \times n$. We have already considered the kinematics of the rolling of a ball on the plane in §5 of Chapter I.

We mention that, apart from the generalized angular momentum integral, there is also an integral corresponding to the conservation of the kinetic energy in the general case of inertial rolling of a homogeneous ball on a sphere:

$$2T = \omega G' = h = \text{const}, \tag{1.14}$$

where h is a constant determined by the initial conditions.

Let us decompose the instantaneous angular velocity ω of the ball into the angular velocities of pure rolling ω_r and pivoting ω_p. In accordance with (1.13) and (1.14), we have

$$G' = (J + 1)\,\omega_r + J\omega_p = G_0', \tag{1.15}$$

$$2T = (\omega_r + \omega_p)\,G' = (J + 1)\,\omega_r^2 + J\omega_p^2 = h. \tag{1.16}$$

Comparing equation (1.16) with the relation

$$(J + 1)^2\,\omega_r^2 + J^2\omega_p^2 = |\,G_0'\,|^2,$$

obtained by squaring (1.15), we find that the magnitudes of the angular velocities of rolling ω_r and pivoting ω_p of the ball are constant. Thus, the ball rolls on the sphere with constant angular velocities of rolling and pivoting. Projecting equation (1.15) onto the normal to the surface at the point of contact of the ball and the sphere, we obtain the relation

$$G_0' n = J\omega_p = \text{const},$$

which means that the projection of the constant vector G_0' onto the normal to the sphere always remains constant during the motion of the ball on the sphere. It also follows from (1.15) that the direction of the vector ω_r coincides with the direction of the projection of the vector G_0' onto the tangent plane to the sphere at the point of contact of the ball. Taken together, these results show that the point of contact A of a ball rolling on a sphere describes a circle, which is determined by the intersection of the sphere and the plane passing through the point A perpendicular to the constant vector G_0' of the angular momentum of the ball. The point of contact moves around this circle with a constant angular velocity.

EXAMPLE 2. *Heavy symmetric ball on an inclined plane.* We introduce a moving system of coordinates $Ax'y'z'$ with origin at the point of contact of the ball and the inclined plane (α is the angle of slope of the plane) and the Ay' axis always horizontal. Then, in accordance with Figure 2.1, the moment of the gravitational forces is always directed along Ay' and, hence, $M_x' = M_z' = 0$. We obtain two generalized angular momentum integrals

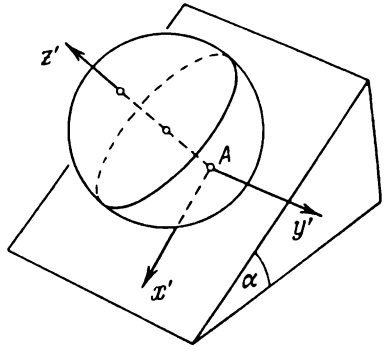

FIGURE 2.1

$$G'\boldsymbol{i} = \text{const}, \quad G'\boldsymbol{k} = \text{const}, \tag{1.17}$$

where $\boldsymbol{i}$ and $\boldsymbol{k}$ are the unit vectors of the axes Ax' and Az', respectively. Assuming that the ball has unit radius and unit mass, we obtain $G'\{(J+1)\omega_x, (J+1)\omega_y, J\omega_z\}$, where J is the moment of inertia of the ball about a straight line passing through its center; ω_x, ω_y, and ω_z are the projections of the instantaneous angular velocity of the ball. The integrals (1.17) yield $\omega_x = \Omega_1 = \text{const}$ and $\omega_z = \Omega_2 = \text{const}$. In addition, we also have an energy integral in the problem. Denoting by x, y, and 1 the coordinates of the center of mass of the ball, which coincides with its geometrical center, we obtain

$$\frac{1}{2}[(J+1)\Omega_1^2 + (J+1)\omega_y^2 + J\Omega_2^2] - xg\sin\alpha = h, \tag{1.18}$$

where h is an arbitrary constant. In order to obtain the equation of motion of the ball on the inclined plane, we use the equations

$$\ddot{x} - \omega_y = 0, \quad \ddot{y} + \omega_x = 0, \tag{1.19}$$

which can be obtained from the condition of rolling without sliding. The second of equations (1.19) yields

$$y = -\Omega_1 t + y_0. \tag{1.20}$$

Substituting the value of ω_y from (1.19) into the expression (1.18), we obtain an equation of the form

$$\left(\frac{dx}{dt}\right)^2 = ax + b, \tag{1.21}$$

where a and b are constant quantities, expressed in terms of the previously introduced constants by the relations

$$a = \frac{2g \sin \alpha}{J+1}, \qquad b = \frac{2h - (J+1)\Omega_1^2 - J\Omega_2^2}{J+1}.$$

Integrating equation (1.21), we obtain

$$x = \frac{at^2}{4} - \frac{b}{a},$$

from which, together with (1.20), it follows that the point of contact of the ball on the inclined plane moves along a parabola.

EXAMPLE 3. *Heavy ball inside a sphere.* If a heavy homogeneous ball rolls on the interior surface of a sphere, we have, apart from the energy integral, only a single generalized angular momentum integral: constancy of the projection of the vector G' onto the vertical axis passing through the point of contact of the ball and the sphere.

Apart from the theory we have just expounded, which leads to the generalized angular momentum integral, Čaplygin established certain other conditions under which new integrals of motion can be obtained. For example, suppose a system can be divided into two parts Σ_1 and Σ_2 for each of which the equations (1.7) relative to parallel and, in general, moving axes L_1 and L_2 can be written in the form

$$\frac{d\boldsymbol{G}_1'}{dt} = \boldsymbol{M}_1, \qquad \frac{d\boldsymbol{G}_2'}{dt} = \boldsymbol{M}_2,$$

where the moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of the reaction of the parts Σ_1 and Σ_2 of the system on each other are proportional, so that

$$\boldsymbol{M}_1 + \mu \boldsymbol{M}_2 = 0,$$

where μ is a constant number. Then, as is immediately obvious, we have the integral

$$G_1' + \mu G_2' = \text{const}, \tag{1.22}$$

which characterizes the transfer of angular momentum from one part of the system to the other.

Now suppose that the equations for the individual parts Σ_1 and Σ_2 of the system, expressing the laws of variation of the momentum and angular momentum, can be written in the form

$$\frac{dG_{1z}'}{dt} = M_{1z}, \qquad \frac{dQ_{2x}}{dt} = F_{2x}, \qquad \frac{dQ_{2y}}{dt} = F_{2y},$$

and that the force of reaction between the parts of the system is such that the relation

$$M_{1z} = \mu F_{2x} + \nu F_{2y}$$

holds, where μ and ν are constant numbers. Then, as above, we have the integral

$$G'_{1z} + \mu Q_{2x} + \nu Q_{2y} = \text{const}, \tag{1.23}$$

which characterizes the exchange between the angular momentum of one part of the system and the momentum of the other part of the system.

As illustrations, we now take some of Čaplygin's examples of systems for which conservation laws of the type (1.22) and (1.23) are satisfied.

EXAMPLE 4. *Ball inside a sphere on an inclined plane.* Suppose that a ball, dynamically symmetric about its center, is placed inside a dynamically symmetric hollow sphere that is constrained to roll without sliding on an inclined plane forming an angle φ with the horizontal. The ball rolls in the hollow sphere without sliding. We introduce a fixed system of coordinate axes $Oxyz$. We take the axes Ox and Oy in the inclined plane with Oy horizontal and Ox along the line of greatest slope, and the Oz axis upward, perpendicular to the plane Oxy. Let A_1 and A_2 be the points of contact of the hollow sphere with the plane and the internal ball with the hollow sphere, respectively, and let B_1 be the point of intersection of the straight line A_1A_2 with the straight line through the center of the ball parallel to the Oz axis (Figure 2.2). Let m_1 and m_2 be the masses, r_1 and r_2 the radii, and C_1 and C_2 the centers of the sphere and ball, respectively. We divide the system into two parts: the hollow sphere (Σ_1) and the internal ball (Σ_2). The only reaction forces between the systems are the reaction force R applied at the point A_2 of the internal ball and the reaction force $-R$ applied at the point A_2 of the hollow sphere. The velocity of the point of contact A_1 of the hollow sphere and the inclined plane coincides with the velocity of the center of mass of the hollow sphere C_1. Similarly, the velocity of the point B_1 is always equal to the velocity of the center of the internal ball, since the segment joining them has constant length and a fixed direction. In addition, the moments of the forces acting on the hollow sphere and

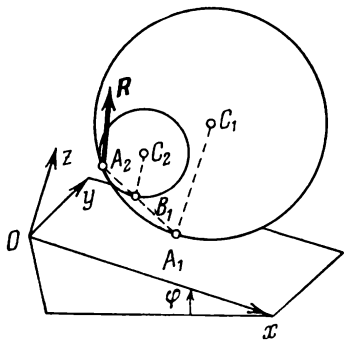

FIGURE 2.2

on the internal ball bear a constant ratio, since the lengths of the segments A_1A_2 and B_1A_2 bear a constant ratio, namely:

$$\frac{A_1A_2}{B_1A_2} = \frac{C_1A_2}{C_2A_2} = \frac{r_1}{r_2} = \mu = \text{const.}$$

We denote by G_1' and G_2' the angular momenta of the hollow sphere and the ball about the points A_1 and B_1, respectively. By what we have proved, these angular momenta satisfy the equations

$$\frac{d\boldsymbol{G}_1'}{dt} = \boldsymbol{M}_{1g} + \boldsymbol{M}_{1R}, \quad \frac{d\boldsymbol{G}_2'}{dt} = \boldsymbol{M}_{2g} + \boldsymbol{M}_{2R}, \tag{1.24}$$

where $\boldsymbol{M}_{1g}$ and $\boldsymbol{M}_{2g}$ are the constant moments of the forces of gravity and $\boldsymbol{M}_{1R}$ and $\boldsymbol{M}_{2R}$ are the moments of the reaction forces at the point of contact of the ball and the sphere. The latter moments bear a constant ratio:

$$\boldsymbol{M}_{1R} + \frac{r_1}{r_2}\boldsymbol{M}_{2R} = 0.$$

Hence and by equations (1.24) we immediately obtain

$$\frac{d\boldsymbol{G}_1'}{dt} + \mu\frac{d\boldsymbol{G}_2'}{dt} = \boldsymbol{M}_{1g} + \mu\boldsymbol{M}_{2g},$$

or, after integration,

$$\boldsymbol{G}_1' + \mu\boldsymbol{G}_2' = (\boldsymbol{M}_{1g} + \mu\boldsymbol{M}_{2g})\,t + \boldsymbol{c}, \tag{1.25}$$

where c is a constant vector. Projecting the integral (1.25), written in the vector form, onto the axes of the fixed system of coordinates $Oxyz$, we obtain two integrals of the form (1.22)

$$G_{1x}' + \mu G_{2x}' = c_x, \quad G_{1z}' + \mu G_{2z}' = c_z$$

and a further integral, which contains the time explicitly:

$$G_{1y}' + \mu G_{2y}' = gt\,r_1\,(m_1 + m_2)\sin\varphi + c_y.$$

In the case of a horizontal plane ($\varphi = 0$), the second integral ceases to depend on the time t explicitly.

EXAMPLE 5. *Absolutely smooth body inside a hollow sphere.* Let us imagine a hollow sphere with a mass distribution symmetric about its center C that rests at the point A on an inclined plane. In the cavity of the sphere, we imagine a rigid body bounded by an arbitrary smooth convex surface. Let M and m be the masses of the hollow sphere and the internal body; x', y', and z', the coordinates of the center of gravity

C' of the body with respect to moving axes (Figure 2.3); x, y, and a, the coordinates of the point C with respect to the fixed axes. Let N be the force of the normal pressure of the body on the sphere at their point of contact; ω the instantaneous angular velocity of the sphere; v_A, the velocity of the point A of the sphere as it slides on the inclined plane; and J_x, J_y, and J_z, the moments of inertia of the sphere about the moving coordinate axes. The motion of the point C' is determined by the equations

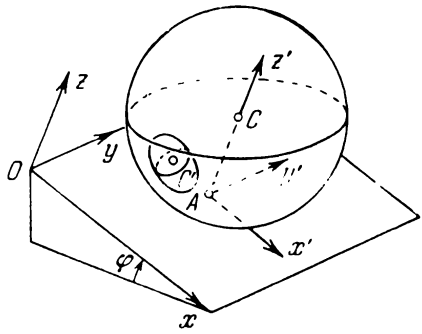

FIGURE 2.3

$$\left.\begin{aligned} m(\ddot{x}+\ddot{x}') &= -N_x + mg\sin\varphi, \\ m(\ddot{y}+\ddot{y}') &= -N_y, \\ m\ddot{z} &= -N_z - mg\cos\varphi. \end{aligned}\right\} \tag{1.26}$$

In addition, the law of variation of the angular momentum of the sphere about the axes Ax' and Ay' yields

$$\frac{d}{dt}(J_x\omega_x - aMv_{Ay}) = -aN_y, \quad \frac{d}{dt}(J_y\omega_y + aMv_{Ax}) = aN_x,$$

where v_A is the velocity of sliding of the point A on the plane Oxy. Eliminating the unknown reaction forces N_x and N_y from these equations by means of (1.26), we obtain two equations, which, after integration, yield the desired integrals

$$\left.\begin{aligned} am\left(\frac{dy}{dt}+\frac{dy'}{dt}\right) - J_x\omega_x + aMv_{Ay} &= \text{const}, \\ am\left(\frac{dx}{dt}+\frac{dx'}{dt}\right) + J_y\omega_y + aMv_{Ax} &= mgat\sin\varphi + \text{const}. \end{aligned}\right\} \tag{1.27}$$

Apart from these conservation laws, the usual integral of dynamics holds in this problem:

$$J_z\omega_z = \text{const},$$

since the moment of the forces acting on the sphere about the Az' axis always vanishes.

§2. Rolling of a disc and a torus on a horizontal plane

When speaking of a circular disc or simply a disc in this section, we shall mean a rigid body which is dynamically symmetric about a certain axis and is bounded by a sharp edge having the form of a circle. The plane in which this circle lies is perpendicular to the dynamical symmetry axis and the center of the circle coincides with the center of mass, which lies on the dynamical axis. We shall assume that the radius of the disc, i.e., the radius of its bounding circle, has unit length and that the disc also has unit mass. We shall denote the principal central moments of inertia of the disc by A and C, where A is the equatorial moment and C the polar moment of inertia.

A good example of the motion which we now intend to study theoretically is that of a coin set in motion on a table or a circular hoop rolling on a horizontal plane. We know from experience that, as long as they roll rapidly, such objects exhibit a remarkable stability that is quite foreign to them in a state of rest. One of the tasks of our theoretical investigation must therefore be the study of the stability of the rolling of a disc and the dependence of the stability on the parameters of the disc. Thus, the problem reduces to studying the dynamics of rolling of a disc on a plane. In order to eliminate the reaction forces of the supporting plane from the start, we shall, when writing down the equations of motion of the disc, consider the variation of the angular momentum of the disc about its point of support. The disc has three degrees of freedom, so that the law of variation of the angular momentum together with the equations of the kinematic constraints gives a complete system of dynamical equations. The position of the disc on the plane may be specified as in §1 of Chapter I by the five generalized coordinates x, y, φ, ψ, and θ. Let $Oxyz$ be a fixed orthogonal system of coordinates with Ox and Oy in the horizontal plane on which the disc rolls and Oz directed vertically upward. Then x and y are the coordinates of the point A of contact of the disc and the plane. In addition, we introduce a moving system of coordinates $A\xi\eta\zeta$ with origin at A. We take the axis $A\xi$ along the tangent

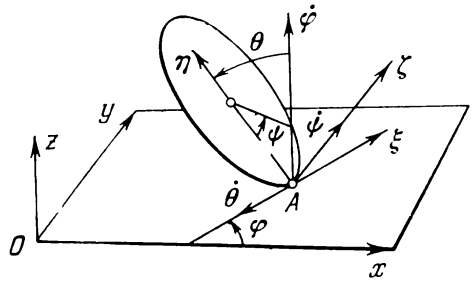

FIGURE 2.4

to the sharp edge of the disc, $A\eta$ along its radius, and $A\zeta$ perpendicular to the plane of the disc, denoting the unit vectors of the moving system of coordinates by ξ^0, η^0, and ζ^0. The remaining generalized coordinates are then: φ, the angle between Ox and $A\xi$; θ, the angle between $A\zeta$ and the horizontal plane xOy; and ψ, the angle measured round the disc from the point A to some fixed point on the disc (Figure 2.4). In accordance with (1.7), the equation determining the variation of the angular momentum of the disc about the point A can be written in the form

$$\frac{\tilde{d}\boldsymbol{G}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{G} + \boldsymbol{v}_A \times \boldsymbol{v}_C = \boldsymbol{M}_g, \tag{2.1}$$

where Ω is the angular velocity of the system of coordinates $A\xi\eta\zeta$; v_A and v_C are the velocities of the point of contact and the center of the disc; and M_g is the moment of the forces of gravity, given by the equation

$$\boldsymbol{M}_g = \eta^0 \times [-g(\eta^0 \cos\theta + \zeta^0 \sin\theta)] = -g\xi^0 \sin\theta. \tag{2.2}$$

Expressing Ω in terms of the generalized velocities, we find

$$\boldsymbol{\Omega} = -\dot{\theta}\xi^0 + \dot{\varphi}\cos\theta\eta^0 + \dot{\varphi}\sin\theta\zeta^0. \tag{2.3}$$

With respect to the moving system of coordinates $A\xi\eta\zeta$, the disc rotates with an angular velocity $\dot{\psi}$ about $A\zeta$; therefore, its instantaneous angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\psi}\zeta^0. \tag{2.4}$$

Since the disc rolls without sliding, the velocity v_A of the point of contact of the disc and the plane is

$$\boldsymbol{v}_A = -\dot{\psi}\xi^0. \tag{2.5}$$

Since the coordinates axes $A\xi$, $A\eta$, and $A\zeta$ are the principal axes of the ellipsoid of inertia of the disc about the point A, the moments of inertia about these axes are, respectively, $A + 1$, A, and $C + 1$. Denoting by p, q, and r the projections of the instantaneous angular velocity of the disc onto the axes of the moving system of coordinates, we obtain

$$\boldsymbol{G} = (A + 1)p\xi^0 + Aq\eta^0 + (C + 1)r\zeta^0. \tag{2.6}$$

Using the kinematic relationships (2.3) – (2.5), the expressions (2.2) and (2.6), and also the fact that the velocity of the center of the disc is

$$\boldsymbol{v}_C = \boldsymbol{\omega} \times \eta^0 = p\zeta^0 - r\xi^0,$$

we can express all the quantities occurring in equation (2.1) in terms

of the components q and r of the vector ω and the angle θ between the plane of disc and the vertical. Using the relations

$$p = -\dot{\theta}, \quad \dot{\varphi}\cos\theta = q, \quad \dot{\psi} = r - q\tan\theta,$$

we find that

$$-(A+1)\ddot{\theta}\xi^0 + A\dot{q}\eta^0 + (C+1)\dot{r}\zeta^0 + [(C+1)qr - Aq^2\tan\theta]\xi^0 + [(C+1)r\dot{\theta} - (A+1)q\dot{\theta}\tan\theta]\eta^0 + [(A+1)q\dot{\theta} - Aq\dot{\theta}]\zeta^0 + \dot{\theta}(q\tan\theta - r)\eta^0 = -g\xi^0\sin\theta.$$

Projecting onto the moving coordinate axes, we obtain the following equations of motion of the disc:

$$\left.\begin{aligned} (A+1)\ddot{\theta} + Aq^2\tan\theta - (C+1)qr &= g\sin\theta, \\ A\dot{q} + Cr\dot{\theta} - Aq\dot{\theta}\tan\theta &= 0, \\ (C+1)\dot{r} + q\dot{\theta} &= 0. \end{aligned}\right\} \tag{2.7}$$

Once these equations have been integrated, the remaining generalized coordinates of the disc may be found from the relations

$$\dot{\varphi}\cos\theta = q, \quad \dot{\psi} = r - q\tan\theta, \quad \dot{x} = -\dot{\psi}\cos\varphi, \quad \dot{y} = -\dot{\psi}\sin\varphi. \tag{2.8}$$

We note that, by virtue of equations (2.7), the condition $\theta = \text{const}$ implies that the components q and r of the angular velocity ω are constant. Turning to the investigation of the equations (2.7), we first consider the so-called cyclic motions of the disc, i.e., motions with constant values of θ, q, and r. In this case, as follows directly from equations (2.7), the quantities θ, q, and r satisfy the relation

$$Aq^2\tan\theta - (C+1)qr = q\sin\theta. \tag{2.9}$$

Thus, this type of motion of the disc is possible, in general, for the arbitrary values $\theta = \theta_1$ and $q = q_1$. It follows from equation (2.9) that, for example, $q = 0$ or $r = 0$ when $\theta = 0$. The case $q = 0$ corresponds to uniform rectilinear rolling of the disc along a horizontal plane, and the case $r = 0$, to uniform spinning about a fixed vertical diameter. If $\theta = \theta_0 \neq 0$, we find from equations (2.8) that

$$x = \left(\sin\theta_0 - \frac{r}{q}\cos\theta_0\right)\sin\left(\frac{qt}{\cos\theta_0}\right) + x_0,$$
$$y = -\left(\sin\theta_0 - \frac{r}{q}\cos\theta_0\right)\cos\left(\frac{qt}{\cos\theta_0}\right) + y_0,$$

i.e., the point of contact of the disc describes a circle on the horizontal

plane with constant angular velocity $q/\cos\theta_0$, corresponding to uniform circular rolling of the disc.

We now turn to the question of the stability of rectilinear or circular motion of a disc on a horizontal plane. It is clear that such motions are unstable in the sense of Ljapunov, for a rectilinear motion of the disc can be transformed into a circular motion, albeit with a very large radius, by an arbitrarily small perturbation. The positions of the disc in the perturbed and unperturbed motions are thus separated by an arbitrarily large distance after a sufficiently long interval of time. Similarly, if a circular motion is subjected to an arbitrarily small perturbation, in particular, one for which the motion remains circular but with a different value of the angular velocity $\dot{\varphi}$, the positions of the disc in the perturbed and unperturbed motions may be separated by a distance of the order of the radius of the circle described by the disc after an appreciable length of time. On the other hand, the motion of the disc is stable with respect to a change of the angle θ of inclination of the disc to the horizontal plane. It is precisely this property, i.e., stability with respect to the angle of inclination, that explains the remarkable ability of the disc to roll on a plane without falling.

Usually, stability of a disc is investigated on the basis of the linearized equations of small oscillations about the unperturbed motion, assumed to be rectilinear or circular in the more general case. Since the disc is a conservative system, the roots of the characteristic equation are either real (and the motion of the disc is then unstable) or purely imaginary complex conjugate numbers. In the latter case, the motion is usually assumed to be stable. However, such a conclusion is invalid; for if we study the motion of the disc as the motion of a conservative nonholonomic system, the second alternative is a so-called doubtful Ljapunov case and therefore requires further investigation.

Turning to the study of the stability, we eliminate the time from the last two equations of the system (2.7) and write them in the form

$$\frac{dq}{d\theta} = -\frac{Cr}{A} + q\tan\theta, \quad \frac{dr}{d\theta} = -\frac{q}{C+1}. \tag{2.10}$$

To these equations, which permit one to find q and r as functions of the variable θ, we add the energy integral

$$\boldsymbol{\omega}\cdot\boldsymbol{G} + 2g\cos\theta = h,$$

which can be represented in the form

$$(A + 1)\,\dot{\theta}^2 = h - 2g\cos\theta - (C + 1)\,r^2 - Aq^2. \tag{2.11}$$

Using equations (2.12) and (2.11), one can study the general nature of the motion of the disc and, in particular, give a precise answer to the question of the stability of rectilinear or circular rolling of the disc on a horizontal plane with respect to the angle θ.

Let $q = q(\theta; \theta_0, q_0, r_0)$ and $r = r(\theta; \theta_0, q_0, r_0)$ be the solution of equations (2.10) corresponding to the initial conditions $q = q_0$ and $r = r_0$ with $\theta = \theta_0$. Substituting this solution into the energy integral (2.11), we obtain an equation of the form

$$\dot{\theta}^2 = f(\theta; \theta_0, q_0, r_0, h). \tag{2.12}$$

A solution of equation (2.12) exists only in the region where the right-hand side of the equation is positive. The character of this solution depends essentially on the form of the graph of the function $f(\theta; \theta_0, q_0, r_0, h)$. Since a motion certainly exists for the given initial conditions, we have $f \geqq 0$ for $\theta = \theta_0$.

Let θ_1 and θ_2 be the neighboring zeros of the function f between which the value $\theta = \theta_0$ lies. If the zeros θ_1 and θ_2 are finite and simple, the solution $\theta = \theta(t)$ of equation (2.12) is a periodic function of the time, varying within the limits θ_1 and θ_2. In the special case $\theta_1 = \theta_2 = \theta_0$, this periodic solution goes over into the solution $\theta = \theta_0$. If the interval (θ_1, θ_2) is unbounded, the function $\theta = \theta(t)$ has a different form, increasing or decreasing unboundedly. If one of the roots, say θ_1, is multiple, the function $\theta = \theta(t)$ will tend asymptotically to this value $\theta = \theta_1$.

It follows from what we have said that a necessary and sufficient condition for the stability of cyclic motion of the disc is that the function f has a strict maximum at $\theta = \theta_0$; for only in this case will the perturbed solution $\theta = \theta(t)$ obtained as a result of arbitrary small variations of the initial conditions θ_0, q_0, and r_0 (or, equivalently, the corresponding small variations of the graph of the function $f = f(\theta; \theta_0, q_0, r_0, h)$) remain in an arbitrarily small neighborhood of the unperturbed solution $\theta = \theta_0$.

Thus, if the cyclic motion $\theta = \theta_0$ is stable, the curve $f = f(\theta; \theta_0, q_0, r_0, h)$ touches the axis $f = 0$ at the point $\theta = \theta_0$ and is convex in the neighborhood of this point. It follows that the first derivative of the function $f(\theta)$ at the point $\theta = \theta_0$ must vanish and the second derivative must be negative. Differentiating the right-hand side of equation (2.11) with respect to θ and replacing dq/dt and dr/dt by their expressions (2.10), we obtain

$$
\left.\begin{aligned}
{}^1/_2(A+1)\frac{df}{d\theta} &= g\sin\theta + (C+1)qr - Aq^2\tan\theta,\\
{}^1/_2(A+1)\frac{d^2f}{d\theta^2} &= g\cos\theta - \frac{C(C+1)}{A}r^2 + (3C+1)qr\tan\theta\\
&\quad - q^2\left(1 + 2A\tan^2\theta + \frac{A}{\cos^2\theta}\right).
\end{aligned}\right\} \tag{2.13}
$$

Setting $df/d\theta = 0$, we obtain, as one would expect, the condition (2.9) of cyclic motion of the disc. A necessary and sufficient condition for such a motion to be stable is that $d^2f/d\theta^2$ is negative. For rectilinear rolling ($\theta = 0$, $q = 0$, $r = r_0$), the stability condition takes the form

$${}^1/_2(A+1)\left(\frac{d^2f}{d\theta^2}\right)_0 = g - \frac{C(C+1)}{A}r_0^2 < 0.$$

Thus, rectilinear rolling of a disc is stable with respect to a variation of the angle θ if the velocity of rolling is greater than some critical value, and it is otherwise unstable.

Our stability condition

$$r_0^2 > \frac{gA}{C(C+1)} = r_{cr}^2 \tag{2.14}$$

is identical with the condition usually obtained from the linear approximation. However, the linear theory suggests that the rectilinear motion of a disc is unstable for $r_0 = r_{cr}$. In contrast, the exact treatment shows that this case is stable, since $d^2f/d\theta^2$ vanishes at $r_0 = r_{cr}$. This means that the curve $f = f(\theta; \theta_0, q_0, r_0, h)$ and the abscissa $f = 0$ have an osculation of higher order. Hence, in order to determine the behavior of the curve in the neighborhood of the point $\theta = \theta_0$, we must calculate the higher order derivatives. For rectilinear motion, we have

$$\frac{1}{2}(A+1)\left(\frac{d^3f}{d\theta^3}\right)_0 = -g\sin\theta + \frac{C(3C+1)}{A}r_{cr}^2\tan\ \theta = 0.$$

Evaluating the fourth derivative, we find

$$\frac{1}{2}(A+1)\left(\frac{d^4f}{d\theta^4}\right)_0 = -g - \frac{2C}{A}(2C+1)r^2 < 0,$$

i.e., the motion is also stable for $r = r_{cr}$.

Our investigation of the rolling of a disc leads naturally to the problem of the rolling of a torus on a plane. The study of the motion on a rough plane of a figure of revolution whose equatorial plane is a symmetry plane also reduces to this problem. Let ρ be the radius of curvature of the meridian of the torus at the equator, and let $a + \rho$ be the radius of the equatorial

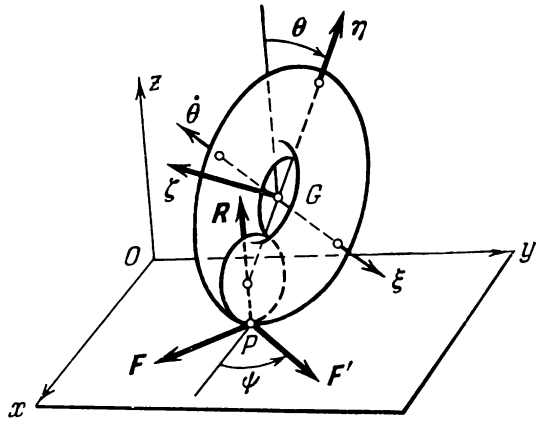

FIGURE 2.5

circle, whose center coincides with the center of mass of the torus. We shall denote the principal central moments of inertia of the torus by A and C, where A is the equatorial moment and C the polar moment of inertia. We shall write down the equations of motion of the torus on the horizontal plane with respect to the moving axes of the system of coordinates $G\xi\eta\zeta$ with origin at the center of gravity of the torus, $G\zeta$ along the symmetry axis of the torus, $G\xi$ horizontal, and $G\eta$ in the vertical plane (Figure 2.5). Denoting by θ the angle between $G\zeta$ and the horizontal plane, we introduce the following vectors and their components with respect to the moving axes: $F\{F', R\cos\theta - F\sin\theta, R\sin\theta + F\cos\theta\}$, the force of friction at the point P of contact of the torus and the horizontal plane; $r_P\{0, -(a+\rho\cos\theta), -\rho\sin\theta\}$, the radius vector through the center of gravity to the point P; $\omega(-\dot\theta, q, r)$, the instantaneous angular velocity of rotation of the torus; $\omega'\{-\dot\theta, q, q\tan\theta\}$, the angular velocity of rotation of the system of coordinates $G\xi\eta\zeta$; and $v(u, v, w)$, the velocity of the center of gravity of the torus.

If ψ is the angle between $G\xi$ and a fixed horizontal direction, then $q = \dot\psi\cos\theta$. The equations of motion of the torus with this notation can be written in the form

$$\left.\begin{aligned} m(\dot u + qw - qv\tan\theta) &= F', \\ m(\dot v + uq\tan\theta + \dot\theta w) &= (R - mg)\cos\theta - F\sin\theta, \\ m(\dot w - v\dot\theta - uq) &= (R - mg)\sin\theta + F\cos\theta, \end{aligned}\right\} \tag{2.15}$$

$$\left.\begin{aligned} -A\ddot\theta + Crq - Aq^2\tan\theta &= -aR\sin\theta - F(a\cos\theta + \rho), \\ A\dot q - A\dot\theta q\tan\theta + Cr\dot\theta &= -F'\rho\sin\theta,\ C\dot r = F(a + \rho\cos\theta), \end{aligned}\right\} \tag{2.16}$$

where m is the mass of the torus. The condition for rolling without sliding yields

$$\left.\begin{aligned} u = -r(a+\rho\cos\theta) + q\rho\sin\theta,\ v = \rho\dot{\theta}\sin\theta, \\ w = -\dot{\theta}(a+\rho\cos\theta). \end{aligned}\right\} \tag{2.17}$$

Equations (2.15) – (2.17) form a closed system for the determination of $u, v, w, F', F, R, \theta, q$, and r. Once these equations have been integrated, the coordinates x and y of the point P of contact of the torus and the plane may be found from the relations

$$\left.\begin{aligned} \dot{x} &= (a+\rho\cos\theta)(q\tan\theta - r)\cos\psi - \rho\dot{\theta}\sin\psi, \\ \dot{y} &= (a+\rho\cos\theta)(q\tan\theta - r)\sin\psi + \rho\dot{\theta}\cos\psi. \end{aligned}\right\} \tag{2.18}$$

The relations (2.18) can be obtained from the condition that the torus rolls without sliding. For the principal radii of curvature of the torus at the point P are, respectively, ρ and $a/\cos\theta + \rho$. The increments of the arc lengths along the principal directions are

$$ds_1 = \rho\dot{\theta}\,dt,\ ds_2 = (a+\rho\cos\theta)(q\sin\theta - r\cos\theta)\frac{dt}{\cos\theta}.$$

Taking Ox along the fixed direction from which ψ is measured, we obtain

$$dx = \cos\psi\,ds_2 - \sin\psi\,ds_1, \qquad dy = \sin\psi\,ds_2 + \cos\psi\,ds_1,$$

from which equations (2.18) follow.

Turning to the investigation of equations (2.15) – (2.17), we consider a motion of the torus for which θ, q and r remain constant. In this case, equations (2.15) and the last two equations of (2.16) yield

$$F' = v = w = 0,\ F\cos\theta = -muq,\ R = mg,$$
$$u = -r(a+\rho\cos\theta) + q\rho\sin\theta,$$

and the first equation of (2.16) yields a relation which θ, q, and r must satisfy during such a motion:

$$\begin{aligned} &Cqr - Aq^2\tan\theta + mga\sin\theta \\ &\qquad + mg\left(a + \frac{\rho}{\cos\theta}\right)[r(a+\rho\cos\theta) + q\rho\sin\theta] = 0. \end{aligned} \tag{2.19}$$

It follows from equations (2.18) that, in such a motion, the point P of contact of the torus and the plane describes a circle. In the special case when $\theta = 0$, the relation (2.19) is satisfied if $q = 0$ or $r = 0$. The case $q = 0$ corresponds to uniform rectilinear rolling of the torus on the plane and $r = 0$ to uniform pivoting about the fixed straight line passing through P and the center of gravity of the torus.

Let us investigate the stability of uniform rectilinear rolling of the torus on the plane. Proceeding as in the investigation of the stability of a disc, we eliminate the time and F' from the first equation of (2.15) and the last two equations of (2.16). We then obtain

$$\left.\begin{aligned} \frac{dq}{d\theta} &= \frac{-Crf_1(\theta) + qf_2(\theta)}{f_3(\theta)}, \\ \frac{dr}{d\theta} &= \frac{-m(a+\rho\cos\theta)[(C-A)r\rho\sin\theta - Aaq]}{f_3(\theta)}, \end{aligned}\right\} \tag{2.20}$$

where

$$f_1(\theta) = C + m(a+\rho\cos\theta)^2 + m\rho\sin^2\theta,$$

$$f_2(\theta) = A[C + m(a+\rho\cos\theta)^2]\tan\theta + Cm\rho\left(a + \rho\frac{\sin^2\theta}{\cos\theta}\right)\sin\theta,$$

$$f_3(\theta) = A[C + m(a+\rho\cos\theta)^2] + Cm\rho^2\sin^2\theta.$$

To these equations we add the energy integral, which can be written in the form

$$\dot\theta^2 = \frac{2[h - mg(\rho + a\cos\theta)] - Aq^2 - m[(a+\rho\cos\theta)r - q\rho\sin\theta]^2}{A + m(a^2 + 2a\rho\cos\theta + \rho^2)}. \tag{2.21}$$

Equations (2.20) and (2.21) enable us to give a precise answer to the question of the stability of steady motion. Let $q = q(\theta; \theta_0, q_0, r_0)$ and $r = r(\theta; \theta_0, q_0, r_0)$ be the solution of equations (2.20) corresponding to the initial conditions $q = q_0$, $r = r_0$, $\theta = \theta_0$. Substituting this solution into the energy integral (2.21), we obtain an equation of the form

$$\dot\theta^2 = f(\theta;\ \theta_0,\ q_0,\ r_0,\ h). \tag{2.22}$$

A solution to equation (2.22) exists only in the region in which the right-hand side of the equation is positive. The nature of this solution depends essentially on the form of the graph of the function $f(\theta; \theta_0, q_0, r_0, h)$. As in the case of a disc, a necessary and sufficient condition for the stability of the motion of the torus is that f, regarded as a function of θ, has a strict maximum at $\theta = 0$. Hence, the first derivative of $f(\theta)$ must vanish at $\theta = 0$ and the second derivative must be negative. Differentiating the right-hand side of (2.21) with respect to θ and replacing $dq/d\theta$ and $dr/d\theta$ by their expressions (2.20), we find $(df/d\theta)_{\theta=0} = 0$. Evaluation of the second derivative gives

$$\left(\frac{d^2f}{d\theta^2}\right)_{\theta=0} = \frac{2\{Amga - C[C + m(a+\rho)^2]r_0^2\}}{A[A + m(a+\rho)^2]},$$

from which we obtain the stability condition

$$r_0^2 > \frac{Amga}{C[C + m(a+\rho)^2]}.$$

§3. The Bobylev-Žukovskiĭ problem of a rolling sphere containing a gyroscope

In 1892 Bobylev published a detailed investigation of the interesting nonholonomic problem of the rolling on a horizontal plane of a hollow sphere containing a gyroscope. He succeeded in expressing all the parameters determining the position of the sphere and the gyroscope in terms of elliptical functions of the time and he investigated the form of the curves which the point of contact of the sphere on the plane describes. Bobylev reproduced these curves experimentally by constraining a gyroscopic sphere which he had constructed to roll on a plane on which lycopodium powder was strewn.

In Bobylev's instrument, the gyroscope occupied a central position within a homogeneous spherical shell and it seems, at the first glance, that this type of shell would correspond to the simplest type of motion of the instrument. However, this is not the case; Žukovskiĭ [**509**] showed that the theoretical investigation of the motion of Bobylev's gyroscopic sphere can be greatly simplified if a ring which lies in the equatorial plane of the gyroscope is added to the spherical shell, the parameters being chosen such that the difference of the polar and equatorial moments of inertia of the sphere is equal to the equatorial moment of inertia of the gyroscope (see Figure 2.6). In 1897, Žukovskiĭ published his "On Bobylev's gyroscopic sphere", a geometrically lucid investigation of the motion of such a sphere containing a gyroscope.

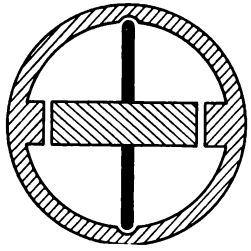

FIGURE 2.6

In 1897 Čaplygin published a paper in which he pointed out that Lindelöf's solution of the problem of the rolling of an arbitrary heavy figure of revolution on a horizontal plane was completely erroneous. He added his own investigation of the question, which he generalized by adding to the rolling body a rotating gyroscope whose axis coincided with the symmetry axis of the body. Čaplygin reduced the question of whether the equations of motions obtained could be solved in quadratures to the same question for a single second-order linear differential equation and found a number of special cases when the integration could be carried out to the end. Considering these cases, Čaplygin solved the problem in quadratures and made a number of comments on the general nature of the motion of the system. Rather

surprisingly, it was found that the addition of the rotating gyroscope in no way affected the problem of integration in quadratures.

Let us assume that the Bobylev-Žukovskiĭ gyroscopic sphere described above is placed on a horizontal plane on which it can roll without sliding. We associate with it an orthogonal system of coordinates $Axyz$ in translational motion with origin at the point of contact of the sphere and the plane and the Az axis vertically upward. We take the vector of the angular velocity of the sphere Ω along the axis of instantaneous rotation, which passes through the point A, since the velocity of the latter vanishes by virtue of the condition of no sliding. Let Ω_n and Ω_e be the projections of the angular velocity Ω onto the axis of dynamical symmetry of the sphere and onto its equatorial plane, and let ω be the angular velocity of rotation of the gyroscope about its own axis, i.e., the projection of the angular velocity of the gyroscope onto its own axis. Further, let A_{sph} and C_{sph} and A_g and C_g, respectively, be the equatorial and polar moments of inertia of the sphere and the gyroscope about their common center.

The kinetic energy, multiplied by two, of the system as it rotates about its center of mass is

$$A_{\text{sph}}\Omega_e^2 + C_{\text{sph}}\Omega_n^2 + A_g\Omega_e^2 + C_g\omega^2$$

or, with allowance for Žukovskiĭ's condition $A_{\text{sph}} + A_g = C_{\text{sph}}$,

$$C_{\text{sph}}(\Omega_e^2 + \Omega_n^2) + C_g\omega^2 = C_{\text{sph}}\Omega^2 + C_g\omega^2.$$

We can now see why Žukovskiĭ's proposal to add a ring to the sphere is, in fact, a simplification; for the kinetic energy of the system due to its rotation about the center of mass then no longer depends on the orientation of the sphere on the plane. We note that ω is constant, since the total moment of the forces about the axis of the gyroscope always vanishes. Thus, the value of the kinetic energy depends only on the magnitude of the vector Ω.

The total kinetic energy, multiplied by two, of the sphere and gyroscope is

$$2T = C_{\text{sph}}\Omega^2 + C_g\omega^2 + Ma^2\Omega_e^2,$$

where a is the radius of the sphere and M is the mass of the complete system. Denoting by p, q, and r the projections of the vector Ω onto the coordinate axes of the system $Axyz$ and introducing the radius of gyration k by the formula $C_{\text{sph}} = Mk^2$, we can rewrite the expression for the kinetic energy (multiplied by two) of the system in the form

$$2T = M[(k^2 + a^2)(p^2 + q^2) + k^2r^2] + C_g\omega^2.$$

As the system moves, the kinetic energy T remains constant, since the force of friction at the point of contact of the sphere and the plane does no work and the height of the center of mass of the system does not change. Hence, since $\omega = \text{const}$, we obtain the integral

$$k^2r^2 + (k^2 + a^2)(p^2 + q^2) = \frac{h}{M}. \tag{3.1}$$

As follows from Čaplygin's generalized angular momentum theorem (see §1 of Chapter II), the vector of the angular momentum of the system about the point A is constant. We can see this directly. Let L be a vector of length $C_g\omega$ directed along the axis of the gyroscope and L_x, L_y, and L_z its projections onto the coordinate axes. Let X and Y be the projections onto Ax and Ay of the force of friction (the reaction of the ideal nonholonomic constraint) developed at the point A of contact of the gyroscopic sphere and the plane. We write the equations of motion of the center of mass and the law of variation of the angular momentum of the system about the center of mass in terms of the projections onto the coordinate axes of the system $Axyz$

$$\frac{d}{dt}(Maq) = X, \quad \frac{d}{dt}(Map) = -Y, \quad \frac{d}{dt}(Mk^2p + L_x) = aY,$$
$$\frac{d}{dt}(Mk^2q + L_y) = -aX, \quad \frac{d}{dt}(Mk^2r + L_z) = 0.$$

Eliminating the forces of reaction X and Y from these equations, we obtain

$$\frac{d}{dt}[M(a^2 + k^2)p + L_x] = 0, \quad \frac{d}{dt}[M(a^2 + k^2)q + L_y] = 0,$$
$$\frac{d}{dt}[Mk^2r + L_z] = 0$$

or, after integration,

$$G_x = \text{const}, \quad G_y = \text{const}, \quad G_z = \text{const},$$

from which it is readily seen that the vector G of the angular momentum of the gyroscopic sphere remains constant as it rolls on the plane.

It follows from the energy integral (3.1) that the vector G varies in such a way that its end, which has the coordinates

$$x = M(a^2 + k^2)p, \; y = M(a^2 + k^2)q, \quad z = Mk^2r, \tag{3.2}$$

always lies on the ellipsoid of revolution

$$\frac{x^2+y^2}{a^2+k^2}+\frac{z^2}{k^2}=Mh. \tag{3.3}$$

In accordance with Figure 2.7, the constant vector G satisfies $G=K+L$, where L is a vector of constant length directed along the axis of the gyroscope. It follows that, as the sphere moves on the plane, the end of the vector K moves along the line of intersection of the ellipsoid of revolution (3.3) and the sphere

$$(K-G)^2=L^2 \tag{3.4}$$

of radius L with center at the point at which the end of the vector G is situated (Figure 2.7).

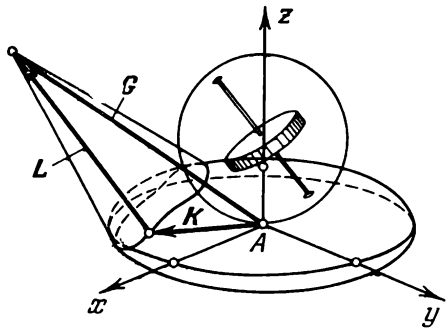

FIGURE 2.7

The curve of the intersection of the ellipsoid (3.3) and the sphere (3.4) defines two cones with vertices at the origin of the coordinates A and at the end of constant vector G. The vector K moves on the first of these cones. If a translational displacement is applied to the second cone in such a way that its vertex is transferred to the center of the sphere, we obtain the cone on which the vector L or, equivalently, the axis of the gyroscope moves.

We now turn to the investigation of the curve of intersection S of the ellipsoid (3.3) and the sphere (3.4). To this end, we find the projections of the curve onto the coordinate planes xAz and yAz.

Arranging the system of coordinates so that the fixed vector G lies in the plane xAz, we denote by θ the angle between G and Ax. The equation of the sphere (3.4) can then be written in the form

$$y^2=L^2-(x-G\cos\theta)^2-(z-G\sin\theta)^2. \tag{3.5}$$

Substituting this value of y into equation (3.3), we obtain

$$x=\alpha-\gamma(z+\beta)^2, \tag{3.6}$$

where

$$\alpha=\frac{G^2k^2\sin^2\theta+a^2[Mh(a^2+k^2)+G^2-L^2]}{2a^2G\cos\theta},$$

$$\beta=\frac{Gk^2}{a^2}\sin\theta,\quad \gamma=\frac{a^2}{2k^2G\cos\theta}.$$

Equation (3.6) shows that the projection of S onto the plane xAz lies on a parabola with apex at the point $(\alpha, -\beta)$ and axis parallel to Ax.

We can find the projection of S onto the coordinate plane yAz by substituting the expression (3.6) for x into equation (3.5). This substitution transforms the right-hand side of (3.5) into a certain polynomial of fourth degree with leading term $-\gamma^2 z^4$. Let z_1, z_2, z_3, and z_4 be the roots of this polynomial; we can then write the equation of the projection of the curve S onto the plane yAz in the form

$$y^2 = -\gamma^2(z - z_1)(z - z_2)(z - z_3)(z - z_4). \tag{3.7}$$

We note that the roots z_1, z_2, z_3, and z_4 are the coordinates z of the points of intersection of the parabola (3.6) with the circle obtained from (3.5) when $y = 0$ and, at the same time, with the meridional section $y = 0$ of the ellipsoid (3.3) (Figure 2.8). Thus, at least two of the roots are real.

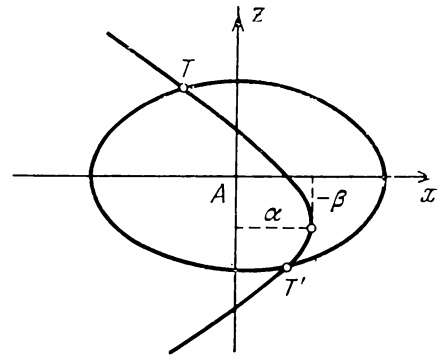

FIGURE 2.8

It is clear that the form of the curve S depends on the nature of the roots z_1, z_2, z_3, and z_4. If all the roots are real, S consists of two separate ovals. If two roots approach and then coincide, two cases can arise. In the first case, the ovals approach each other and then coalesce, forming a figure of eight; in the second case, one of the ovals contracts into a point. If only two of the roots are real and the other two are complex conjugates, S consists of only a single oval.

Our investigation has enabled us to determine the form of the curve described in the moving system of coordinates $Axyz$ by the ends of the vectors $\boldsymbol{K}$ and $\boldsymbol{L}$.

We shall now find the laws governing their motion in time. The Eulerian formula

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\omega} \times \boldsymbol{L}$$

yields the following equation:

$$\frac{dL_z}{dt} = pL_y - qL_x.$$

Using the relation (3.2) and the expressions

$$L_x = -x + G_x = -x + G\sin\theta, \quad L_y = -y,$$
$$L_z = -z + G_z = -z + G\cos\theta,$$

we transform the equation obtained to the form

$$\frac{dz}{dt} = \frac{G\cos\theta}{M(a^2+k^2)}\,y. \tag{3.8}$$

Introducing y from equation (3.7) and then integrating, we find the desired dependence of z on the time

$$t = \pm\frac{2Mk^2(a^2+k^2)}{a^2}\int^{z}\frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}}\,. \tag{3.9}$$

Substituting now $z = z(t)$ into equations (3.6) and (3.7), we obtain the functions $x(t)$ and $y(t)$.

We now turn to the determination of the law of motion and the form of the curve described by the point of contact A as the gyroscopic sphere rolls on a horizontal plane. Let ξ and η be the coordinates of this point with respect to fixed axes $O\xi\eta$ in the plane of rolling parallel to the moving axes Axy. The displacement velocity of the point of contact is equal to the velocity of the center of the gyroscopic sphere; therefore

$$\dot{\xi} = aq, \quad \dot{\eta} = -ap. \tag{3.10}$$

Expressing q and p in these equations in terms of y and x, respectively, by means of (3.2), and then using the relations (3.6) and (3.8), we obtain differential equations which enable us to determine ξ and η in terms of the variable z:

$$\dot{\xi} = \frac{a}{G\cos\theta}\,\dot{z}, \quad \dot{\eta} = -\frac{a}{M(a^2+k^2)}\,[\alpha - \gamma(z+\beta)^2]. \tag{3.11}$$

The first equation of (3.11) can be integrated easily,

$$\xi = \frac{a}{G\cos\theta}\,z + \text{const} \tag{3.12}$$

and shows that ξ, like z, varies periodically in time or, in the special cases when one or more of the roots z_1, z_2, z_3, or z_4 coincide, varies monotonically, tending asymptotically to a certain constant value.

The solution of the second equation of (3.11) in the general case has the form

$$\eta = \mu t + \nu(t),$$

where μ is a constant and $\nu(t)$ is a periodic function of the time with the same period of variation as the variable z. We should mention

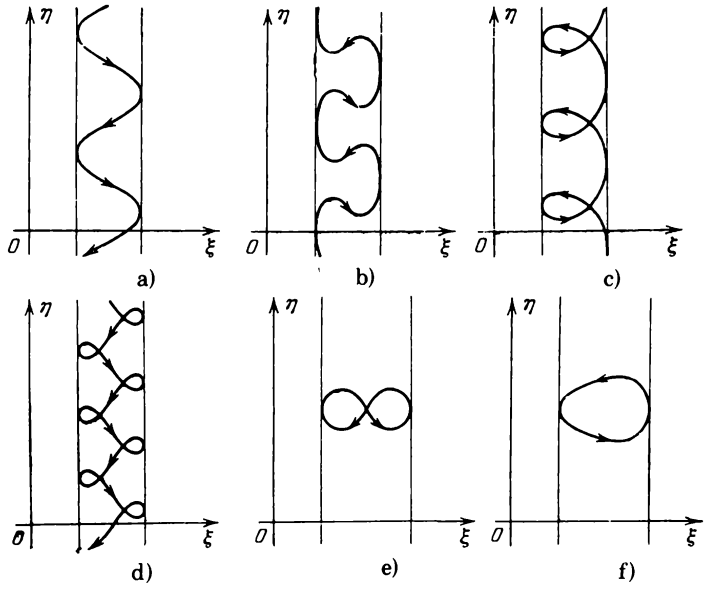

FIGURE 2.9

that, by the second equation of (3.10), the sign of the derivative $\dot{\eta}$ is opposite to that of x. Hence, if the section of the parabola (3.6) contained within the meridional section (Figure 2.8) of the ellipsoid (3.5) intersects the Az axis, the derivative $\dot{\eta}$ changes its sign with the course of time. However, if this section of the parabola (3.6) does not intersect the Az axis, the sign of $\dot{\eta}$ does not change. It follows that the curve described by the point A of contact of the sphere and the horizontal plane can have several qualitatively different forms. The most important forms are shown in Figure 2.9. In Figure 2.9a, we show the first type of curve, which corresponds to the case when the complete section TT' (Figure 2.8) of the parabola (3.6) is contained in the right half of the ellipse. If this is the case, x takes only positive values and, in accordance with the relation

$$\frac{d\eta}{dt} = -\frac{a}{M(a^2 + k^2)}x, \tag{3.13}$$

η decreases monotonically. The point A executes a similar trajectory in the case when the complete section TT' lies in the left half of the ellipse, but, in this case, the point A moves along the trajectory in the opposite direction to that shown by the arrow in Figure 2.9a.

The second type of curve is obtained if the section TT' lies in both halves of the ellipse in such a way that both of its ends are situated in the left half but its middle in the right half. As we move along the parabola in this case from T to T', we first have negative values of x, then positive, and then again negative. It follows from equation (3.13) that η first increases, then decreases, and then increases again. The trajectory of A has the form shown in Figure 2.9b.

The third type of curve arises when the section TT' lies with one of its ends T in the left half of the ellipse and other T' in the right half (this case is shown in Figure 2.8). Moving along the section of the parabola from T to T', we have first negative values of x and then positive values. The coordinate η first increases and then decreases. The trajectory of A is a curve with successive loops (Figure 2.9c). It should be noted that, moving along the curves of Figures 2.9b and 2.9c, the point A will have been displaced in the positive direction of the $O\eta$ axis, after the completion of a full period. However, one can also have trajectories of these types for which the point A has been displaced in the negative direction of the $O\eta$ axis after each complete period. In this case, the curve corresponding to Figure 2.9b takes the form shown in Figure 2.9d. In the limiting case between Figures 2.9b and 2.9d, the curve is closed, having the form of a figure of eight (Figure 2.9e). Similarly, in the limiting case for Figure 2.9c, the curve takes the form of an oval (Figure 2.9f).

§4. Čaplygin's problem of nonholonomic motion on a plane[2)]

Suppose that a rigid body can move on a fixed horizontal plane π, being supported on the latter by three points, of which two slide freely (without friction) on the plane and the third is a knife edge (or the edge of a cutting wheel) that is rigidly fixed to the moving body. It is assumed that the point of contact of the knife edge can move freely on the plane π along the knife edge, but not in the direction perpendicular to it. We use two reference systems: a fixed coordinate system Oxy on the horizontal plane π and a moving system $A\xi\eta$ fixed to the moving

[2)] Čaplygin stated and solved in quadratures the problem studied in this section in 1911 [85]. Without quoting Čaplygin, Carathéodory [88] published a detailed investigation in 1933 of a special case of this problem, calling the system a sleigh. In connection with the nonholonomic geometry, Vagner has considered this problem in the general case.

body. We take the origin of the moving system at the point of support of the knife edge with $A\xi$ along the knife edge and the other axis $A\eta$ perpendicular to it, both axes being parallel to the plane π. The position of the body, or, equivalently, the system of coordinates $A\xi\eta$, is determined by the coordinates x and y of the point A and the angle φ between Ox and $A\xi$.

The investigation of the motion of this system, which we shall call a sleigh, reduces ultimately to finding x, y, and φ as functions of the time. However, rather than set up an equation of motion directly for these quantities, we shall find it more convenient to determine certain intermediate quantities, namely, the projections u and v of the velocity of the point A of the body onto the moving axes $A\xi$ and $A\eta$, and also the angular velocity $\omega = \dot{\varphi}$ of rotation of the body about a vertical axis.

Let m be the mass of the body; a and b, the coordinates of the center of mass in the $A\xi\eta$ system; J_0, the moment of inertia of the sleigh about the vertical axis passing through the center of mass; and R, the horizontal component of the reaction of the point of support of the knife edge, directed, by hypothesis, along $A\eta$. We shall restrict ourselves in this section to a consideration of the special case $b = 0$, when the center of mass lies on $A\xi$ (Figure 2.10).

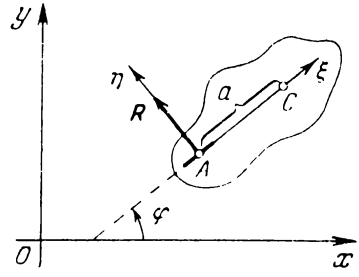

FIGURE 2.10

The equations of motion of the center of mass of the sleigh can be written in the form

$$\left.\begin{aligned} m(\dot{u} - \omega v - \omega^2 a) &= 0, \\ m(\dot{v} + \omega u + \dot{\omega} a) &= R. \end{aligned}\right\} \tag{4.1}$$

We shall also use the law of variation of the angular momentum about the vertical axis passing through the center of mass

$$J_0\dot{\omega} = -Ra. \tag{4.2}$$

The unknown force of reaction R must be such that the knife edge is not displaced in the transverse direction, i.e., by virtue of the constraint imposed on the system, the component v of the velocity of A must vanish. Assuming this is the case, we can find the reaction R from the second equation of (4.1):

$$R = m(\omega u + a\dot{\omega}). \tag{4.3}$$

Introducing the notation

$$k^2 = 1 + \frac{J_0}{ma^2}, \tag{4.4}$$

and eliminating R, we can write the remaining equations of (4.1) and (4.2) in the form

$$\dot{u} = a\omega^2, \quad ak^2 \frac{\dot{\omega}}{\omega} + u = 0. \tag{4.5}$$

With a view to integrating the system of equations (4.5), we eliminate the variable u and find the equation which the angular velocity ω satisfies:

$$ak^2 \frac{d}{dt}\left(\frac{\dot{\omega}}{\omega}\right) = -\dot{u} = -a\omega^2.$$

Multiplying this equation by $\dot{\omega}/\omega$ and then integrating, we arrive at the following first-order equation:

$$k^2\left(\frac{\dot{\omega}}{\omega}\right)^2 = k^2C^2 - \omega^2, \tag{4.6}$$

which contains the arbitrary constant C. Instead of ω, we introduce a new variable ψ by the relation

$$\omega = kC\cos\psi \tag{4.7}$$

in the time interval corresponding to a variation of ψ in the interval between $-\pi/2$ and $+\pi/2$. After this change of variables, equation (4.6) is reduced to the form

$$\dot{\psi}^2 = C^2\cos^2\psi.$$

Since the sign of the constant C in equation (4.6) is still unknown, we choose it in such a way that the equation obtained implies

$$\dot{\psi} = C\cos\psi. \tag{4.8}$$

Equation (4.8) can be integrated easily and, with a suitable choice of the time origin, we obtain

$$Ct = \int_0^{\psi} \frac{d\alpha}{\cos\alpha} = \frac{1}{2}\ln\frac{1+\sin\psi}{1-\sin\psi},$$

and, therefore,

$$\sin\psi = \frac{e^{Ct}-e^{-Ct}}{e^{Ct}+e^{-Ct}} = \tanh Ct, \quad \cos\psi = \frac{2}{e^{Ct}+e^{-Ct}} = \cosh^{-1}Ct.$$

These formulas show that ψ varies from $-\pi/2$ to $+\pi/2$ as t varies from $-\infty$ to $+\infty$. In particular, it follows that the change of variables (4.7) encompasses the complete time interval $-\infty < t < +\infty$.

From equations (4.7) and (4.8), we obtain directly

$$\omega = k\dot{\psi}. \tag{4.9}$$

Integrating this equation and choosing the reference direction for φ suitably, we obtain $\varphi = k\psi$. This shows that the auxiliary variable ψ is proportional to the angle φ of rotation of the sleigh and that the total angle through which the sleigh turns as t varies from $-\infty$ to $+\infty$ is $k\pi$. The time dependence of the angle φ is given by the formula

$$\varphi = k\psi = k\sin^{-1}(\tanh Ct). \tag{4.10}$$

We shall now find the trajectory of the point of support A of the knife edge. The point A moves in the direction of the knife edge with a velocity u in a direction which makes an angle φ with the Ox axis. Therefore,

$$\dot{x} = u\cos\varphi, \quad \dot{y} = u\sin\varphi. \tag{4.11}$$

By (4.5) – (4.7), the velocity u can be expressed in terms of ψ:

$$u = ak^2\frac{\dot{\omega}}{\omega} = Cak^2\sin\psi.$$

Substituting this value of u into (4.11) and noting that, by (4.8), $dt = d\psi/C\cos\psi$, we arrive at the differential equations

$$\frac{dx}{d\psi} = ak^2\tan\psi\cos k\psi, \qquad \frac{dy}{d\psi} = ak^2\tan\psi\sin k\psi,$$

which can be integrated to give the trajectory of the point of support A of the knife edge in a parametric representation (ψ is a running parameter):

$$x = ak^2\int_0^{\psi}\tan\psi\cos k\psi\,d\psi + x_0, \quad y = ak^2\int_0^{\psi}\tan\psi\sin k\psi\,d\psi + y_0 \tag{4.12}$$

Thus, irrespective of the initial conditions, the point of support A of the knife edge always describes the same curve provided the initial kinetic energy of the sleigh is not equal to zero. The velocity u with which the point A passes along this curve at all of its points is proportional to the constant of integration C. The physical meaning of the constant of integration C is directly apparent from the expression for the kinetic energy of the sleigh

$$T = \frac{1}{2}ma^2k^4C^2,$$

which does not vary during the motion of the sleigh.

It only remains for us to determine the form of the trajectory of the point A for different values of the parameter k. (If the quantity a, which also occurs in the parametric equation (4.12), is varied, the trajectory is merely altered by a change of scale.) The simplest type of motion of Čaplygin's sleigh occurs when $a = 0$ (and, hence, $k = \infty$), when the projection of the center of mass onto the plane π coincides with the point of support of the knife edge. In considering this degenerate case, we cannot use the formulas obtained above directly, since the change of variables (4.7) and even the representation of the equations of motion in the form (4.5) are only meaningful if k is different from zero and infinity. Of course, the desired equations for this case could be found by making a passage to the limit. However, it is simpler to solve the problem ab initio, since it is very elementary. Thus, if $a = 0$, the moment of the force of reaction R about the vertical axis passing through the center of mass vanishes and, therefore, $\dot{\varphi} = \text{const}$. Since the kinetic energy is constant, this relation also implies $u = \text{const}$. The differential equations (4.1) and (4.2) lead to the same conclusion. Thus, the point of support A of the knife edge moves with a constant speed in a direction that rotates with a constant angular velocity ω, i.e., the point A describes a circle of radius $r = \omega/u$.

In the general case, when $a \neq 0$, the curve (4.12) has a unique singular point at $\psi = 0$. The nature of this singularity can be determined by expanding the functions x and y in a power series in ψ in the neighborhood of the value $\psi = 0$.

The expansion has the form

$$x - x_0 = \frac{1}{2} ak^2\psi^2 + \cdots, \quad y - y_0 = \frac{1}{3} ak^3\psi^3 + \cdots$$

and shows that the curve (4.12) has a cusp at $\psi = 0$. Since the difference $x - x_0$ does not change when ψ is replaced by $-\psi$ and the difference $y - y_0$ merely changes sign, the branches of the curve (4.12) corresponding to $\psi > 0$ and $\psi < 0$ are situated symmetrically with respect to the straight line $y = y_0$. As $\psi \to +\pi/2$, the angular coefficient of the tangent to the curve (4.12), given by

$$dy/dx = \tan k\psi,$$

tends to $\tan(k\pi/2)$ and $-\tan(k\pi/2)$. Since the difference

$$y - x\tan\frac{k\pi}{2} = ak^2\int_0^{\psi} \tan\psi \sin k\left(\psi - \frac{\pi}{2}\right) d\psi$$

tends to a finite number $\alpha \neq \infty$ as $\psi \to \pi/2$, the curve (4.12) has the asymptotic form

$$y = x \tan \frac{k\pi}{2} + \alpha.$$

In order to represent the curve (4.12) correctly, it should be noted that the tangent to the curve rotates continuously in the same direction

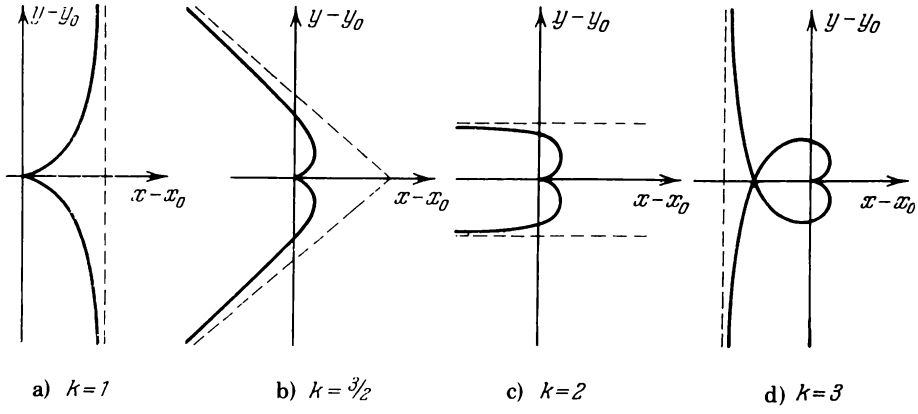

FIGURE 2.11

as ψ varies from 0 to $\pi/2$. The same remark applies to the case when ψ varies from 0 to $-\pi/2$. In Figure 2.11, we show the curves (4.12) for $k = 1$, $3/2$, 2, and 3, respectively.

We mention that the knife edge of the sleigh comes to rest instantaneously at the cusp of these curves, i.e., at this point, the velocity u is equal to zero.

§5. Rolling of a sphere on an absolutely rough surface

In this section, we shall consider the problem of a homogeneous sphere rolling on an absolutely rough surface under the influence of forces whose resultant passes through the center of the sphere O_1 (coinciding with the center of mass of the sphere). We introduce a system $O_1\xi\eta\zeta$ of moving coordinate axes with origin at the point O_1 and axis $O_1\zeta$ passing through the point of contact A of the sphere and the surface (Figure 2.12). We shall project all vector quantities onto the moving axes $O_1\xi\eta\zeta$. Let $\Omega(\theta_1, \theta_2, \theta_3)$ be the instantaneous angular velocity of the axes $O_1\xi\eta\zeta$; $\omega(\omega_1, \omega_2, \omega_3)$, the in-

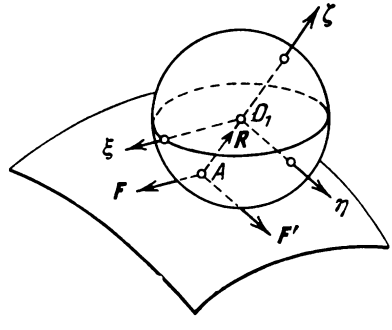

FIGURE 2.12

stantaneous angular velocity of the sphere; $v(u, v, 0)$, the velocity of the center O_1 of the sphere; $R(F, F', R)$, the reaction force at the point of support; and $F(X, Y, Z)$, the resultant of the applied forces. The equations of motion of the sphere can then be written in the form

$$\left.\begin{aligned} \dot{\omega}_1 - \theta_3\omega_2 + \theta_2\omega_3 &= \frac{F'a}{k^2}, \\ \dot{\omega}_2 - \theta_1\omega_3 + \theta_3\omega_1 &= -\frac{Fa}{k^2}, \\ \dot{\omega}_3 - \theta_2\omega_1 + \theta_1\omega_2 &= 0, \end{aligned}\right\} \tag{5.1}$$

$$\left.\begin{aligned} \dot{u} - \theta_3 v &= X + F, \\ \dot{v} + \theta_3 u &= Y + F', \\ -\theta_2 u + \theta_1 v &= Z + R, \end{aligned}\right\} \tag{5.2}$$

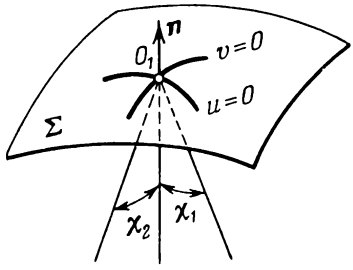

FIGURE 2.13

where a is the radius of the sphere; k is the radius of gyration; and the sphere is assumed to have unit mass. The condition of rolling without sliding reduces to the two equations

$$u - a\omega_2 = 0, \quad v + a\omega_1 = 0. \tag{5.3}$$

Eliminating F, F', ω_1, and ω_2 from these equations, we obtain

$$\left.\begin{aligned} \dot{u} - \theta_3 v &= \frac{a^2}{a^2 + k^2} X + \frac{k^2}{a^2 + k^2} \theta_1 a\omega_3, \\ \dot{v} + \theta_3 u &= \frac{a^2}{a^2 + k^2} X + \frac{k^2}{a^2 + k^2} \theta_2 a\omega_3. \end{aligned}\right\} \tag{5.4}$$

Thus, the center of the sphere, which lies on the surface Σ formed by the ends of all the normals of length equal to the radius of the sphere to the given surface, moves under the influence of the applied forces reduced by the factor $a^2/(a^2 + k^2)$ and also under the influence of additional forces with the components

$$\frac{k^2}{a^2 + k^2} \theta_1 a\omega_3 + \theta_3 v, \quad \frac{k^2}{a^2 + k^2} \theta_2 a\omega_3 - \theta_3 u.$$

We take the $O_1\xi$ and $O_1\eta$ axes along the lines of curvature of the surface Σ. Let ρ_1 and ρ_2 be the radii of curvature of the normal sections of this surface along the directions $O_1\xi$ and $O_1\eta$. Then,

$$\theta_1 = -\frac{v}{\rho_2}, \quad \theta_2 = \frac{u}{\rho_1}. \tag{5.5}$$

Let χ_1 and χ_2 be the angles formed by the radii of curvature of the lines of curvature with the normal to the surface Σ at the point O_1 (Figure 2.13). Using Meusnier's theorem, we arrive at the relation

$$\theta_3 = \frac{u}{\rho_1}\tan\chi + \frac{v}{\rho_2}\tan\chi_2. \tag{5.6}$$

Substituting (5.3) and (5.5) into the third equation of (5.1), we obtain

$$a\dot{\omega}_3 = uv\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right). \tag{5.7}$$

The motion of the sphere may be investigated as follows. Let x, y, and z be the coordinates of the center of the sphere. The quantities u and v can then be determined from the equations of the surface in terms of $\dot{x}$, $\dot{y}$, and $\dot{z}$ by means of a decomposition of the velocity along the directions of the coordinate axes $O_1\xi$ and $O_1\eta$. Eliminating u, v, θ_1, θ_2, and θ_3 by means of (5.4), (5.5), and (5.6), we obtain three equations containing x, y, z, and ω_3 and their derivatives with respect to the time. These equations, together with the equation of the surface, constitute a complete system for the determination of the motion of the sphere.

It is sometimes convenient to take the $O_1\xi$ axis along the tangent to the trajectory of the point O_1. In this case, $v = 0$ and $\omega_1 = 0$. If U is the velocity of the point O_1, then $u = U$. Let R be the radius of torsion of the geodesic that touches the trajectory at the point O_1, and ρ the radius of curvature of the normal section passing through the tangent to the trajectory at the point O_1. Then, $\theta_1 = U/R$ and $\theta_2 = U/\rho$. In these expressions, the quantity R is, as usual, positive, if the torsion in the direction O_1 is from the $O_1\eta$ to the $O_1\zeta$ axis. Denoting by χ the angle between the radius of curvature of the trajectory and the normal to the surface, we obtain $\theta_3 = (U/\rho)\tan\chi$. The equations (5.4) are now replaced by

$$\left.\begin{aligned} \dot{U} &= \frac{a^2}{a^2+k^2}X + \frac{k^2}{a^2+k^2}\frac{U}{R}a\omega_3, \\ \frac{U^2}{\rho}\tan\chi &= \frac{a^2}{a^2+k^2}Y + \frac{k^2}{a^2+k^2}\frac{U}{\rho}a\omega_3, \end{aligned}\right\} \tag{5.8}$$

and, instead of equation (5.7), we obtain

$$a\dot{\omega}_3 = -\frac{U^2}{R}. \tag{5.9}$$

The last of equations (5.2), which determines the magnitude of the pressure of the sphere on the surface, can be written in one of the alternative forms

$$\frac{u^2}{\rho_1} + \frac{v^2}{\rho_2} = -Z - R, \qquad \frac{U^2}{\rho} = -Z - R. \tag{5.10}$$

We note that if the constraint imposed on the rolling sphere is lifted, the sphere leaves the surface when the sign of R changes.

Having formulated the problem of the rolling of a sphere on an arbitrary rough surface in complete generality, we shall now solve the problem in some special cases. We shall consider examples when the surface belongs to one of the known simple types.

EXAMPLE 1. *Rolling of a sphere on a plane.* In this case, $\theta_1 = \theta_2 = 0$, and, assuming that the moving system of coordinates $O_1\xi\eta\zeta$ is chosen in such a way that $\theta_3 = 0$, we can simplify equations (5.4) considerably:

$$\dot{u} = \frac{a^2}{a^2 + k^2} X, \quad \dot{v} = \frac{a^2}{a^2 + k^2} Y.$$

Hence, the motion of the center of the sphere does not depend on the magnitude of the projection ω_3 of the angular velocity, which is a constant in this case.

EXAMPLE 2. *Rolling of a heavy sphere on a spherical surface.* Let the radius of the sphere be $b - a$; then $\rho_1 = \rho_2 = b$ and $\omega_3 = \text{const}$. Since any normal section is a principal section, we choose the direction of the $O_1\xi$ axis along the tangent to the trajectory. Hence, the radius of torsion is infinite, $R = \infty$, and equations (5.8) show that the center of mass moves as if it were a material point subjected to the applied forces reduced by the factor $a^2/(a^2 + k^2)$, together with an additional force of magnitude

$$\frac{k^2}{a^2 + k^2} \frac{a\omega_3}{b} U.$$

in the direction of the $O_1\eta$ axis (i.e., perpendicular to the trajectory of the motion).

Let O be the center of the fixed sphere; θ, the angle between the $O_1\zeta$ axis and the vertical Oz; ψ, the angle between the plane zOO_1 and the fixed plane passing through the vertical Oz. The integral expressing the conservation of mechanical energy then takes the form

$$\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta = F - \frac{2g}{b} \frac{a^2}{a^2 + k^2} \cos\theta, \tag{5.11}$$

where F is a constant determined by the initial conditions. In addition, the angular momentum about the Oz axis is conserved,

$$\dot{\psi} \sin^2\theta = \varepsilon - \frac{k^2}{a^2 + k^2} \frac{a\omega_3}{b} \cos\theta, \tag{5.12}$$

where ϵ is an arbitrary constant, which is also determined by the initial conditions.

The integrals of the motion (5.11) and (5.12) completely determine the quantities $\dot{\theta}$ and $\dot{\psi}$ given the initial conditions. The pressure of the moving sphere on the fixed sphere is given by

$$R = g\frac{3a^2 + k^2}{a^2 + k^2}\cos\theta - bF.$$

Equations (5.11) and (5.12) are similar to the equations of motion of the axis of a heavy spinning top, so that the investigation and results obtained for these equations are the same as those of the theory of motion of a top.

For, eliminating $\dot{\psi}$ from these equations, we obtain

$$\dot{\theta}^2 = F - \alpha\cos\theta - \frac{(\varepsilon - \beta\cos\theta)^2}{\sin^2\theta},$$

where

$$\alpha = \frac{2g}{b}\frac{a^2}{a^2 + k^2} > 0, \quad \beta = \frac{a\omega_3}{b}\frac{k^2}{a^2 + k^2} \gtrless 0.$$

Introducing the new variable $x = \cos\theta$ in this equation, we obtain

$$\dot{x}^2 = F(1 - x^2) - \alpha x(1 - x^2) - (\varepsilon - \beta x)^2 = f(x). \tag{5.13}$$

The curve $f(x)$ is a cubic parabola and cannot, therefore, intersect the abscissa in more than three points. It follows from (5.13) that $f(-\infty) < 0$, $f(+1) < 0$, and $f(+\infty) > 0$, and, furthermore, that the interval $(-1, +1)$ contains values $x = \bar{x}$ for which $f(\bar{x}) > 0$.

In the general case, the curve $f(x)$ therefore has the form shown in Figure 2.14. The value of x, which only has a physical meaning for values in the interval $(-1, +1)$, lies in the interval (x_1, x_2). Thus, as the heavy sphere moves on the fixed sphere, the angle θ varies, fluctuating between the values θ_1 and θ_2. Here, as in the case of a top, three types of motion can occur, for which the trajectory of the center of mass (or the point of contact of the two spheres) has the form shown in Figures 2.15a, 2.15b, and 2.15c, respectively. Apart from these cases, there will be a motion similar to that of a "sleeping" top. Let us find the condition for the stability of the motion of the sphere in this case.

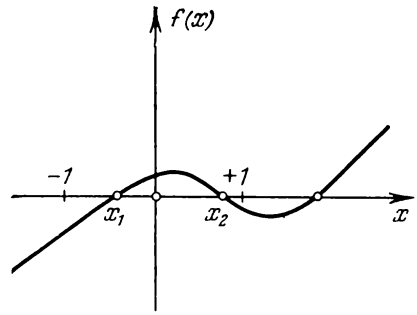

FIGURE 2.14

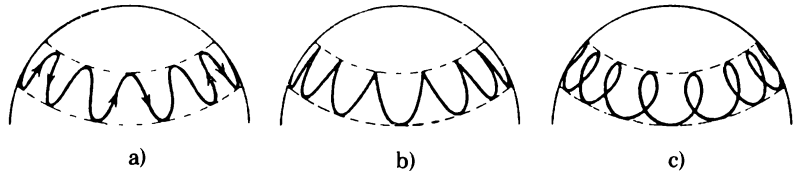

FIGURE 2.15

Using the initial conditions $\theta = \dot{\theta} = 0$ to determine the constants of integration F and ϵ from (5.11) and (5.12), we find $F = \alpha$ and $\epsilon = \beta$. Substituting these values into (5.13), we obtain

$$f(x) = (1 - x^2)\,[\alpha(1 + x) - \beta^2]. \tag{5.14}$$

This situation is characterized by the double root $x_{1,2} = 1$. Hence the graph of the curve $f(x)$ in this case may have the two forms shown in Figures 2.16a and 2.16b. In the first case, the curve $f(x)$ touches the

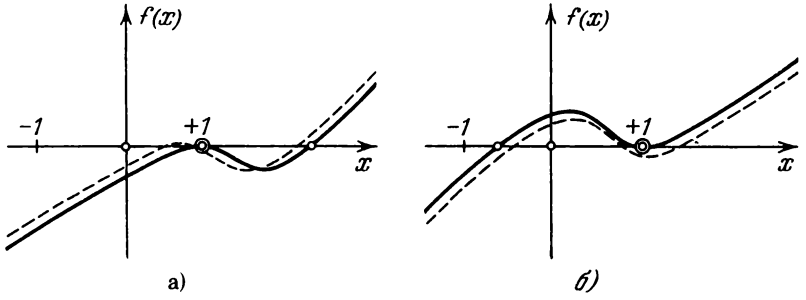

FIGURE 2.16

abscissa at the point $x = 0$ from below, and in the second case, from above. The continuous curve represents an unperturbed motion. Considering the dashed curve, which corresponds to a perturbed motion, one can easily see that in the case of Figure 2.16a the interval of positive values of the function $f(x)$ lying between -1 and $+1$ may be arbitrarily small if the perturbation is sufficiently small. On the other hand, in the case of Figure 2.16b, the interval is finite (and may be large) for an arbitrarily small perturbation. Hence, the case of Figure 2.16a corresponds to a stable motion and the case of Figure 2.16b to an unstable motion. The form of the curve $f(x)$ in the case of Figure 2.16a is such that at the point $x = 1$ we have $f''(1) < 0$, from which we obtain the desired stability condition. Using the expression (5.14), we find $f''(1) = 2(2\alpha - \beta^2) < 0$ or $\beta < \sqrt{2\alpha}$. Substituting the values of α and

β, we obtain

$$|\omega_3| > \frac{2}{k^2}\sqrt{gb(a^2+k^2)} = \omega_{cr}. \tag{5.15}$$

Thus, a necessary condition for the motion to be stable is that the angular velocity of rotation of the sphere exceeds a certain "critical" velocity ω_{cr}.

EXAMPLE 3. *Rolling of a ball on a moving sphere.* Suppose that a sphere on which a ball rolls rotates with a constant angular velocity Ω about a fixed diameter directed along the Oz axis. Then, instead of (5.3), we have

$$u - a\omega_2 = 0, \quad v + a\omega_1 = c\Omega\sin\theta,$$

where c is the radius of the sphere; θ is the angle between Oz and the common normal OO_1 of the ball and the sphere, the $O_1\xi$ axis lying in the plane zOO_1. Equations (5.1) and (5.2) remain valid in this case.

If the sphere can rotate freely about its fixed center, its motion is described by an equation identical with equation (5.1), in which the quantities ω_1, ω_2, ω_3, a, and mk^2 must be replaced by Ω_1, Ω_2, Ω_3, c, and Mk^2, respectively. In this case, the equations (5.3) are replaced by

$$u - a\omega_2 = c\Omega_2, \quad v + a\omega_1 = -c\Omega_1.$$

EXAMPLE 4. *Rolling of a sphere on a cylindrical surface.* In this case, the lines of curvature are the generators and the curves of the transverse section of the cylindrical surface. We take the $O_1\xi$ axis along a generator. Then $\rho_1 = \infty$ and the radius of curvature of a curve of a transverse section is $\rho_2 - a$. Hence, $\theta_1 = -v/\rho_2$, $\theta_2 = 0$ and, consequently, $\chi_2 = 0$ and $\theta_3 = 0$. Thus, equations (5.4) and (5.7) take the form

$$\dot{u} = \frac{a^2}{a^2+k^2}X - \frac{k^2}{a^2+k^2}\frac{v}{\rho_2}a\omega_3, \quad \dot{v} = \frac{a^2}{a^2+k^2}Y, \quad \frac{d(a\omega_3)}{dt} = \frac{uv}{\rho_2}. \tag{5.16}$$

These equations determine the motions of the sphere. The second of the equations determines the motion of the sphere in the direction perpendicular to the generators of the cylinder. If Y does not depend on the position on a given generator, this equation can be solved independently of the remaining equations. Let $v = v(t)$ be the solution of this equation. We introduce φ, the angle between the normal plane through the generator and the center of the sphere and some fixed plane through the generator of the cylindrical surface. Then, $v = \rho_2 d\varphi/dt$. If $d\varphi/dt \neq 0$, the first and the third equations of (5.16) can be transformed to the form

$$\frac{du}{d\varphi} + \frac{k^2}{a^2+k^2}\, a\omega_3 = \frac{a^2}{a^2+k^2}\,\frac{\rho_2}{v}\,X, \quad u = \frac{d(a\omega_3)}{dt}. \tag{5.17}$$

If X does not depend on the position on a given generator, the equations can be solved without difficulty. If ρ_2 and v are known functions of φ, these equations are linear in u and $a\omega_3$. If $X \equiv 0$ and $k^2 = 2a^2/5$, then

$$a\omega_3 = A\sin\left(\sqrt{\frac{2}{7}}\,\varphi + B\right), \quad u = A\sqrt{\frac{2}{7}}\cos\left(\sqrt{\frac{2}{7}}\,\varphi + B\right),$$

where A and B are arbitrary constants.

Now suppose X depends on the position on the generator. We introduce ξ, the coordinate of the center of the sphere along the generator; then

$$u = \frac{d\xi}{dt} = \frac{v}{\rho_2}\,\frac{d\xi}{d\varphi}.$$

Substituting this expression for u into (5.17), we obtain two equations for the determination of ξ and $a\omega_3$ as functions of φ.

EXAMPLE 5. *Rolling of a sphere on a rough cone.* The lines of curvature are the generators and the curves orthogonal to them. We take the $O_1\xi$ axis parallel to a generator; then, $\rho_1 = \infty$ and $\rho_2 - a$ is the radius of curvature of the normal section perpendicular to the generator; $\theta_1 = -v/\rho_2$ and $\theta_2 = 0$. We specify the position of the sphere by the distance r of the center of the sphere from the apex of the cone on which the center is always situated and by the angle φ, chosen such that $d\varphi > 0$ along the positive direction of the $O_1\eta$ axis. If the cone is spread out into a plane, r and φ evidently become polar coordinates. We have

$$\theta_3 = \frac{d\varphi}{dt}, \quad u = \frac{dr}{dt}, \quad v = r\,\frac{d\varphi}{dt},$$

so that equations (5.4) and (5.7) take the form

$$\ddot{r} - r\ddot{\varphi} = \frac{a^2}{a^2+k^2}\,X - \frac{k^2}{a^2+k^2}\,\frac{r}{\rho_2}\,a\omega_3\dot{\varphi},$$
$$\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\varphi}{dt}\right) = \frac{a^2}{a^2+k^2}\,Y,$$
$$\frac{d(a\omega_3)}{dt} = \frac{r}{\rho_2}\,\frac{d\varphi}{dt}\,\frac{dr}{dt}.$$

If $Y = 0$, we obtain an integral of the motion $r^2\dot{\varphi} = h = \text{const}$. If, in addition, $X = X(r)$, we obtain an energy integral

$$\dot{r}^2 + r^2\dot{\varphi}^2 + \frac{k^2}{a^2+k^2}\,a^2\omega_3^2 = \frac{2a^2}{a^2+k^2}\int X(r)\,dr + h'.$$

If, finally, the cone is circular, then $\rho_2 = r\tan\alpha$, where α is the angle between the axis of the cone and a generator, and we obtain a further integral of the motion:

$$a\omega_3 = -\frac{h\cot\alpha}{r} + h''.$$

The equations of motion of the center of the sphere are similar to the equations of motion of a particle in a field of central forces: $r(t)$ and $\varphi(t)$ can be regarded as the coordinates of a particle moving in a plane under the influence of the central force

$$\frac{a^2}{a^2+k^2}\left(X - k^2\omega_3\frac{d\omega_3}{dr}\right),$$

where ω_3 takes the values just found.

EXAMPLE 6. *Rolling of a sphere on a surface of revolution.* We shall consider the case when the axis of the surface of revolution is vertical, the apex is upward, and the only external force is the force of gravity. The lines of curvature are meridians and parallels. We take the Oz axis along the axis of the figure and denote by θ the angle between $O_1\zeta$ and Oz and by ψ the angle between the plane containing Oz and $O_1\zeta$ and any fixed vertical plane.

We have

$$\theta_1 = -\frac{d\psi}{dt}\sin\theta, \quad \theta_2 = \frac{d\theta}{dt}, \quad \theta_3 = \frac{d\psi}{dt}\cos\theta. \tag{5.18}$$

In our case, $X = g\sin\theta$, $Y = 0$, and equations (5.4) take the form

$$\left.\begin{aligned} \frac{du}{dt} - \theta_3 v &= \frac{a^2}{a^2+k^2}g\sin\theta + \frac{k^2}{a^2+k^2}\theta_1 a\omega_3, \\ \frac{dv}{dt} + \theta_3 u &= \frac{k^2}{a^2+k^2}\theta_2 a\omega_3. \end{aligned}\right\} \tag{5.19}$$

Since $d\varphi = -gdz = g\sin\theta d\theta$, equation (5.8) can be written in the form

$$u^2 + v^2 + \frac{k^2}{a^2+k^2}a^2\omega_3^2 = \varepsilon + 2g\frac{a^2}{a^2+k^2}\int\rho\sin\theta\, d\theta, \tag{5.20}$$

where ϵ is an arbitrary constant and ρ is the radius of curvature of the meridian. In accordance with (5.7), we have

$$\frac{d\omega_3}{dt} = -\frac{uv}{a}\left(\frac{1}{\rho} - \frac{\sin\theta}{r}\right),$$

where r is the distance of the center of the sphere from the axis of the surface of revolution. Using the geometrical equations (5.5)

$$\theta_1 = -\frac{v\sin\theta}{r}, \quad \theta_2 = \frac{u}{\rho},$$

we can reduce equations (5.18) to the form

$$u = \rho \frac{d\theta}{dt}, \quad v = r \frac{d\psi}{dt}, \quad \theta_3 = \frac{v}{r} \cos\theta, \tag{5.21}$$

and equations (5.19) to the form

$$\frac{dv}{d\theta} + \rho \frac{\cos\theta}{r} v = \frac{k^2}{a^2 + k^2} a\omega_3.$$

Hence

$$\frac{d}{d\theta}\left(\frac{dv}{d\theta} + \frac{\rho\cos\theta}{r} v\right) + \frac{k^2}{a^2 + k^2}\left(1 - \frac{\rho\sin\theta}{r}\right) v = 0. \tag{5.22}$$

However, ρ and r can be expressed by means of the equation of the curve of the meridian section as functions of θ; therefore, (5.22) is a linear equation in v. Finding the solution of this equation, $v = v(\theta)$, we can then find ω_3 from the equation

$$\frac{d\omega_3}{d\theta} = -\frac{v}{a}\left(1 - \frac{\rho\sin\theta}{r}\right),$$

after which we can determine u from (5.20) and θ and ψ from (5.21).

EXAMPLE 7. *Rolling of a sphere on a paraboloid.* In this case, equation (5.22) takes the form

$$\frac{d^2v}{d\theta^2} + \frac{1}{\sin\theta\cos\theta}\frac{dv}{d\theta} + \left(\frac{1 - m^2}{\cos^2\theta} - \frac{1}{\sin^2\theta} + m^2\right) v = 0, \tag{5.23}$$

where $m^2 = k^2/(a^2 + k^2)$. We seek the solution of this equation in the form

$$v = \sin^p\theta \cos^q\theta.$$

Substitution into equation (5.23) shows that $p = -1$ and $q = 1 \pm m$; hence

$$\frac{v}{a} = \operatorname{ctg}\theta\,(L\cos^m\theta + M\cos^{-m}\theta), \quad m\omega_3 = -L\cos^m\theta + M\cos^{-m}\theta,$$

$$\frac{u^2}{a^2} = -\frac{1}{\sin^2\theta}(L\cos^m\theta + M\cos^{-m}\theta)^2 + \frac{2gbn^2}{k^2}\frac{1}{\cos^2\theta} + \frac{E}{a^2} + 4LM, \tag{5.24}$$

where $2b$ is the parameter of the generator of the parabola and L, M, and E are constants of integration, which can be determined from the given values of u, v, and ω_3.

If the initial value of u is negative, the sphere commences by moving upward until the value of u equals zero. The right-hand side of the third equation of (5.24) is positive when θ takes its initial value and

tends to $-\infty$ as $\theta \to 0$. It follows that during its motion the sphere cannot rise above a certain horizontal circle on the paraboloid situated above the initial position of the sphere. Since the value of v is finite (with the exception of the case $M = -L$) and $u = 0$ when this level is reached, the trajectory of the center of the sphere touches the limiting circle.

Suppose that the sphere is originally placed at the apex of the paraboloid and rotates about the vertical diameter with an angular velocity $\omega_3 = n$. For these initial conditions, equations (5.24) take the form

$$\frac{v}{a} = \frac{mn}{2}\frac{\cos^{1-m}\theta - \cos^{1+m}\theta}{\sin\theta}, \qquad \omega_3 = \frac{n}{2}(\cos^{-m}\theta + \cos^{m}\theta),$$

$$\frac{4u^2}{a^2m^2} = -n^2\left(\frac{\cos^{-m}\theta - \cos^{m}\theta}{\sin\theta}\right)^2 + \frac{8gb}{k^2}\tan\theta.$$

Hence, the motion is stable provided $n^2 \geqq 8gb/k^2m^2$.

If the sphere commences its motion from any point on the paraboloid with initial conditions u, v, $\omega_3 = n$, it will fall steadily if $u > 0$ or rise if $u < 0$, attaining its highest position as $t \to \infty$ if $n^2 \leqq 8gb/k^2m^2$. However, if $n^2 > 8gb/k^2m^2$, the sphere will first rise and then, having passed through its highest position, steadily descend.

CHAPTER III

ANALYTICAL DYNAMICS OF NONHOLONOMIC SYSTEMS

§1. Principle of virtual displacements and the d'Alembert-Lagrange equations

The content of mechanics as a science and, in particular, the modern approach to the derivation of the equations of motion of a mechanical system are the outcome of a long historical process.

The first investigations of Galileo and Newton concerned the motion of heavy bodies on the Earth and of the planets around the Sun (objects whose dimensions are very small compared with the distances between them), i.e., with what we now call material points. Thus, equations of motion were first set up for material points.

In the study of the laws of motion of more complex mechanical systems, recourse was taken to the device of regarding the systems as made up of material points. This approach led to a point representation of a mechanical system, i.e., a mechanical system came to be regarded as a collection of material points subject to definite relationships.

This approach was even adopted in hydrodynamics in the form of the so-called Lagrangian specification of the motion of a liquid, but was then largely displaced by the concept of a continuous medium and the corresponding Eulerian method of specification of the motion of a liquid.

As every student learns, the motion of a material point is completely determined by the totality of the forces applied to it. In writing down the equations of motion of a system of material points, we must introduce the forces that act on each of the points forming the system. These forces will include forces that arise because of the existence of connections between the individual points of the system. Thus, the equations of motion of the system can be written in the form

$$m_i \boldsymbol{w}_i = \boldsymbol{F}_i + \boldsymbol{R}_i, \tag{1.1}$$

where m_i is the mass of the ith point; $\boldsymbol{w}_i$ is its acceleration; $\boldsymbol{F}_i + \boldsymbol{R}_i$ is the resultant force acting on the point, $\boldsymbol{R}_i$ being the resultant of the forces acting on the ith point because of the connections.

Since we are interested in only the motion of the system, the forces of the connections $\boldsymbol{R}_i$ are merely auxiliary quantities which we are forced to introduce and then eliminate. The reactions of the connections are not given directly, although the forces $\boldsymbol{R}_i$ must be such that the motion of a point of the system following from (1.1) does not violate the connections.

The problem of the elimination of unknown forces of reactions arises already in statics when one is finding the equilibrium conditions of a system of material points. The most general principle that permits one to obtain the equilibrium conditions of a system of material points is the principle of virtual displacements (or virtual work). As we have already discussed in §3 of Chapter I, a virtual displacement of a system is the displacement executed by the system as the result of a virtual variation of its generalized coordinates. By a virtual variation, we mean an infinitely small variation of the coordinates which is compatible with the constraints imposed on the system and which is carried out at a given instant of time. The principle of virtual displacements is usually formulated for a special, but fairly large class of constraints, which are called ideal constraints. By definition, a constraint is ideal if the reaction forces of this constraint in any virtual displacement of the system perform no work, i.e.,

$$\delta A = \sum_{i=1}^{N} \boldsymbol{R}_i \delta \boldsymbol{r}_i = 0, \tag{1.2}$$

where N is the number of material points in the system. We shall now consider some examples of ideal constraints.

A. All the internal reaction forces between the material points forming an absolutely rigid body are forces of ideal constraints. For in this case the reaction forces of the constraints serve to keep the distance between any two points constituting the rigid body fixed, i.e., we always have

$$(\boldsymbol{r}_i - \boldsymbol{r}_j)^2 = \text{const}, \tag{1.3}$$

where $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are the radius vectors from a fixed center to the two points of the rigid body. Denoting by $\boldsymbol{R}_{ij}$ the force with which the jth point acts on the ith point, we obtain from Newton's third law

$$\boldsymbol{R}_{ij} = -\boldsymbol{R}_{ji}, \tag{1.4}$$

where, since these forces are directed along the straight line joining the points,

$$\boldsymbol{R}_{ij} = \lambda(\boldsymbol{r}_i - \boldsymbol{r}_j). \tag{1.5}$$

In order to prove that equation (1.2) is satisfied, we write down an expression for the virtual work of the reaction forces acting on these points

$$\delta A = \boldsymbol{R}_{ij}\delta\boldsymbol{r}_i + \boldsymbol{R}_{ji}\delta\boldsymbol{r}_j.$$

Using the expressions (1.4) and (1.5), we find

$$\delta A = \boldsymbol{R}_{ij}(\delta\boldsymbol{r}_i - \delta\boldsymbol{r}_j) = \lambda(\boldsymbol{r}_i - \boldsymbol{r}_j)(\delta\boldsymbol{r}_i - \delta\boldsymbol{r}_j) = 0$$

by virtue of the fact that the condition (1.3) is satisfied.

B. We have an ideal constraint when a body slides on a smooth surface. For then the vectors $\boldsymbol{R}_i$ and $\delta\boldsymbol{r}_i$ are orthogonal, and, hence, $\delta A = \boldsymbol{R}_i\delta\boldsymbol{r}_i = 0$.

The rolling of a body without sliding also gives rise to an ideal constraint. At the point of application of the reaction $\boldsymbol{R}_i$, the vector $\delta\boldsymbol{r}_i$ is a null vector if the surface on which the body rolls is fixed. If the body rolls without sliding on a moving body, the vector $\delta\boldsymbol{r}_i$ at the point of contact is not equal to the null vector. However, in this case too, the condition (1.2) is satisfied. For suppose the ith point of the first body touches the kth point of the second body. Then, since there is no sliding, we have $\delta\boldsymbol{r}_i = \delta\boldsymbol{r}_j$ and the reaction forces satisfy equation (1.4). Using this in the expression for the virtual work, we obtain

$$\delta A = \boldsymbol{R}_{ij}\delta\boldsymbol{r}_i + \boldsymbol{R}_{ji}\delta\boldsymbol{r}_j = \delta\boldsymbol{r}_i(\boldsymbol{R}_{ij} - \boldsymbol{R}_{ji}) = 0.$$

We now turn to the formulation of the principle of virtual displacements; a necessary and sufficient condition for equilibrium of a system of material points with ideal constraints is the vanishing of the virtual work of the applied forces, i.e.,

$$\sum_{i=1}^{N} \boldsymbol{F}_i\delta\boldsymbol{r}_i = 0. \tag{1.6}$$

Equation (1.6) does not contain the reaction forces. Thus, in statics, the problem of eliminating the reactions of ideal constraints is solved as follows. Applying to each point the reaction due to the imposed constraints, we replace the system by a set of "free" points. The equilibrium condition for each of the points is:

$$\boldsymbol{F}_i + \boldsymbol{R}_i = 0 \qquad (i = 1, 2, \cdots, N). \tag{1.7}$$

Now, forming the scalar product of each of the equations of (1.7) with

$\delta \boldsymbol{r}_i$ and adding all the expressions obtained, we have

$$\sum_{i=1}^{N} \boldsymbol{F}_i \delta \boldsymbol{r}_i = -\sum_{i=1}^{N} \boldsymbol{R}_i \delta \boldsymbol{r}_i, \tag{1.8}$$

which is identical with (1.6), since the right-hand side vanishes because the constraints are ideal [the condition (1.2)].

Reverting to dynamics, it is natural to solve the problem of eliminating the constraint reactions in the first place for systems with ideal constraints. Considering, therefore, systems with ideal constraints, we note that equations (1.1) have the same form as equations (1.7). This analogy was already noted by d'Alembert, who formulated his well-known principle, which enables one formally to reduce the problem of setting up the equations of dynamics to the setting up of the equations of statics. Thus, multiplying the equations of (1.1) by $\delta \boldsymbol{r}_i$ and adding the resulting expressions, we obtain an equation which is known as the d'Alembert-Lagrange equation:

$$\sum_{i=1}^{N} (m_i \boldsymbol{w}_i - \boldsymbol{F}_i)\, \delta \boldsymbol{r}_i = 0. \tag{1.9}$$

This equation plays as important a role in dynamics as the principle of virtual displacements (1.6) in statics.

Thus, we conclude that equation (1.9) holds for any virtual displacement of an arbitrary mechanical system with ideal constraints. Written out in coordinates, the d'Alembert-Lagrange equation (1.9) takes the form

$$\sum_{j=1}^{3N} (m_j \ddot{x}_j - X_j)\, \delta x_j = 0. \tag{1.10}$$

Let $q_1, q_2, \cdots, q_n$ be the generalized coordinates of the system under consideration, i.e., the radius vectors $\boldsymbol{r}_i$ of the material points of the system at any moment of time t are determined by the values of the quantities q_k:

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \cdots, q_n, t) \quad (i = 1, 2, \cdots, N) \quad (k = 1, 2, \cdots, n). \tag{1.11}$$

In the d'Alembert-Lagrange equation (1.9), we express the virtual variations $\delta \boldsymbol{r}_i$ of the radius vectors in terms of the virtual variations δq_k of the generalized coordinates. From the relations (1.11), we obtain

$$\delta \boldsymbol{r}_i = \sum_{k=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_k} \delta q_k,$$

which, after substitution into equation (1.9), gives

$$\sum_{i=1}^{N}\sum_{k=1}^{n}\left(m_i\boldsymbol{w}_i\frac{\partial \boldsymbol{r}_i}{\partial q_k}-\boldsymbol{F}_i\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)\delta q_k=0.$$

We shall denote the expression $\sum_{i=1}^{N}\boldsymbol{F}_i\partial \boldsymbol{r}_i/\partial q_k$, which is called the kth generalized force, by Q_k. Noting that

$$\frac{d}{dt}\frac{\partial \boldsymbol{r}_i}{\partial q_k}=\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k},\qquad \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k}=\frac{\partial \boldsymbol{r}_i}{\partial q_k},$$

we can transform the first sum in the resulting equation as follows:

$$m_i\boldsymbol{w}_i\frac{\partial \boldsymbol{r}_i}{\partial q_k}=m_i\ddot{\boldsymbol{r}}_i\frac{\partial \boldsymbol{r}_i}{\partial q_k}=\frac{d}{dt}\left(m_i\dot{\boldsymbol{r}}_i\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)-m_i\dot{\boldsymbol{r}}_i\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)$$

$$=\frac{d}{dt}\left(m_i\dot{\boldsymbol{r}}_i\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k}\right)-m_i\dot{\boldsymbol{r}}_i\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k}=\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\frac{m_i\dot{r}_i^2}{2}\right)-\frac{\partial}{\partial q_k}\left(\frac{m_i\dot{r}_i^2}{2}\right).$$

Denoting by T the kinetic energy of the system $\frac{1}{2}m_i\dot{r}_i^2$, we write the d'Alembert-Lagrange equation in the form

$$\sum_{k=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k}-Q_k\right)\delta q_k=0 \tag{1.12}$$

or, if the forces have a potential, when

$$Q_k=\partial U/\partial q_k,$$

in the form

$$\sum_{k=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}\right)\delta q_k=0, \tag{1.13}$$

where $L=T+U$ is the so-called Lagrange function. We note that only in the case when the variations δq_k are independent, i.e., in the case of a holonomic system, do the relations (1.12) or (1.13) yield the well-known Lagrange equations of the second kind:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k}=Q_k\qquad (k=1,2,\ldots,n) \tag{1.14}$$

or, respectively,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=0\qquad (k=1,2,\ldots,n). \tag{1.15}$$

In the case of a nonholonomic system, it is impossible to obtain equations of motion in the form of Lagrange equations of the second kind directly from the relation (1.12) since the variations $\delta q_1,\delta q_2,\cdots,\delta q_n$

are dependent for any choice of the generalized coordinates $q_1, q_2, \cdots, q_n$. If there are $n - m$ nonholonomic constraints of the form

$$\sum_{k=1}^{n} a_{ik}\dot{q}_k + a_{i0} = 0 \qquad (i = 1, 2, \ldots, n - m) \tag{1.16}$$

the variations of the generalized coordinates satisfy a system of $n - m$ linear homogeneous equations

$$\sum_{k=1}^{n} a_{ik}\delta q_k = 0 \qquad (i = 1, 2, \ldots, n - m). \tag{1.17}$$

The transition from the relations (1.12) to differential equations of motion can be effected either by means of the so-called method of undetermined multipliers (or Lagrangian multipliers) or by directly eliminating the dependent variations, the latter being expressed in terms of any m independent variations. In the first case, we obtain a system of n differential equations containing, apart from the n desired functions $q_1, q_2, \cdots, q_n$, a further $n - m$ unknown multipliers. Together with (1.16), these equations form a complete system of $2n - m$ equations for $2n - m$ unknowns. In the second case, we obtain m equations, which, together with (1.16), form a complete system. Integration of these differential equations enables us to determine the motion of the mechanical system from any preassigned given initial state. In addition, in the first case one can also find the reaction forces of the constraints imposed on the system.

§2. Equations of motion of nonholonomic systems with Lagrangian multipliers. The reactions of ideal nonholonomic constraints

We multiply each of the equations (1.17) by a quantity λ_i, which does not, in general, vanish, and add the resulting expressions. Adding the double sum

$$\sum_{i=1}^{n-m} \sum_{k=1}^{n} \lambda_i a_{ik}\delta q_k,$$

which necessarily vanishes, to the d'Alembert-Lagrange equation (1.12), we obtain an equation that is equivalent to (1.12)

$$\sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k - \sum_{i=1}^{n-m} \lambda_i a_{ik} \right) \delta q_k = 0, \tag{2.1}$$

in which the Lagrangian multipliers λ_i are new auxiliary unknowns. Since the equations (1.17) are linearly independent, i.e., the rank of

the matrix $\|a_{ik}\|$ is $n - m$, one of its minors must be nonzero. Suppose, to be specific,

$$\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1,\,n-m} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n-m,1}, & \dots, & & a_{n-m,\,n-m} \end{vmatrix} \neq 0. \tag{2.2}$$

The variations $\delta q_{n-m+1}, \dots, \delta q_n$ of the generalized coordinates may be assumed to be arbitrary, since the system of equations (1.17) can be satisfied for all values of these variations by the choice of appropriate and quite definite values of the remaining variations $\delta q_1, \delta q_2, \dots, \delta q_{n-m}$. We now choose the values of the Lagrangian multipliers λ_i in such a way that the expressions in the brackets in equation (2.1) vanish for the dependent variations $\delta q_1, \delta q_2, \dots, \delta q_{n-m}$. The values of λ_i can be chosen in this way, since the system of $n - m$ linear equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k - \sum_{i=1}^{n-m} \lambda_i a_{ik} = 0 \qquad (k = 1, 2, \dots, n-m)$$

for λ_i is solvable by virtue of the assumption (2.2). However, if this choice of the values of λ_i is made, equation (2.1) takes the form:

$$\sum_{k=n-m+1}^{n} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k - \sum_{i=1}^{n-m} \lambda_i a_{ik} \right) \delta q_k = 0,$$

where the variations $\delta q_{n-m+1}, \dots, \delta q_n$ are independent and, therefore, each of the expressions in the brackets must also vanish.

Thus, we have arrived at equations of motion with Lagrangian multipliers

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k + \sum_{i=1}^{n-m} \lambda_i a_{ik} \qquad (k = 1, 2, \dots, n), \tag{2.3}$$

which, together with the constraint equations (1.16), form a complete system of $n + (n - m)$ equations for the $2n - m$ unknown functions q_k and λ_i. The expressions $\sum_{i=1}^{n-m} \lambda_i a_{ik}$ have a simple mechanical meaning: they are the generalized reaction forces of the nonholonomic constraints. For if R_k are the generalized reaction forces, the motion of the nonholonomic system will be the same as the motion of the holonomic system with the n coordinates $q_1, q_2, \dots, q_n$, the kinetic energy T, and the generalized forces $Q_k + R_k$, i.e., the equations of motion of the nonholonomic system can be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k + R_k \qquad (k = 1, 2, \dots, n).$$

Comparing these equations with (2.3), we see that

$$R_k = \sum_{i=1}^{n-m} \lambda_i a_{ik}. \tag{2.4}$$

EXAMPLE 1. Let us consider the free motion of a solid body on a horizontal plane in the case when the projection of the center of mass coincides with the point of contact of a sharp wheel and the plane (a special case of Čaplygin's sleigh). In this case, the Lagrange function L has the form

$$L = T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2\dot{\varphi}^2),$$

where x and y are the coordinates of the center of mass; φ is the angle between the plane of the wheel and the fixed axis Ox; and k is the radius of gyration, the body being assumed to have unit mass. Since the wheel has a sharp rim, the velocity of the center of mass is always directed along the plane of the wheel. This is equivalent to the nonholonomic constraint

$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0. \tag{2.5}$$

Writing down the equations of motion (2.3), we obtain

$$\ddot{x} = \lambda\sin\varphi, \quad \ddot{y} = -\lambda\cos\varphi, \quad k^2\ddot{\varphi} = 0. \tag{2.6}$$

The equations (2.5) and (2.6) form a system for the determination of the functions x, y, φ, and λ. Integrating this system, we note that the last equation yields

$$\dot{\varphi} = \omega = \text{const}, \quad \varphi = \omega t,$$

if $\varphi(0) = 0$ at the initial instant of time. Eliminating λ from the first two equations of (2.6), we obtain

$$\ddot{x}\cos\varphi + \ddot{y}\sin\varphi = 0,$$

and, hence, taking into account (2.5), we obtain a first integral of the motion

$$\dot{x}\cos\varphi + \dot{y}\sin\varphi = A, \tag{2.7}$$

where A is an arbitrary constant determined by the initial conditions. From the system of equations (2.5) and (2.7), we obtain

$$\dot{x} = A\cos\varphi, \quad \dot{y} = A\sin\varphi; \tag{2.8}$$

hence

$$x = \frac{A}{\omega}\sin\omega t + x_0, \quad y = -\frac{A}{\omega}\cos\omega t + y_0.$$

The same result could be obtained by using the integral expressing the conservation of mechanical energy. In the case under consideration, this integral expresses the constancy of the kinetic energy

$$\dot{x}^2 + \dot{y}^2 + k^2\dot{\varphi}^2 = 2h.$$

Using (2.5) and $\dot{\varphi} = \omega$, we obtain the expressions (2.8), where $A = \sqrt{2h - k^2\omega^2}$. It follows from (2.6) that $\lambda = -\omega A$ and, consequently, the generalized forces are

$$R_x = -\omega A\sin\omega t; \quad R_y = \omega A\cos\omega t.$$

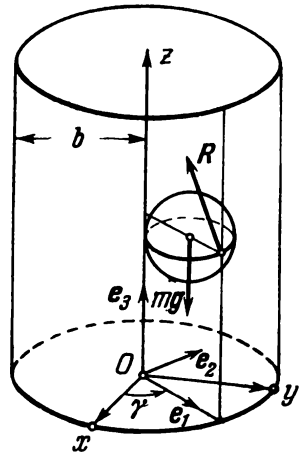

FIGURE 3.1

EXAMPLE 2. We consider the free motion of a heavy sphere of radius a which rolls (and rotates) on the rough inside wall of a fixed vertical cylinder of radius b.

We introduce the generalized coordinates γ, z, φ, θ, and ψ. In accordance with Figures 3.1 and 1.24, the projections of the instantaneous angular velocity of the sphere can be written in the form

$$\left.\begin{aligned} \omega_x &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = \omega_1\cos\gamma - \omega_2\sin\gamma, \\ \omega_y &= -\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi = \omega_1\sin\gamma + \omega_2\cos\gamma, \\ \omega_z &= \dot{\psi} + \dot{\varphi}\cos\theta = \omega_3, \end{aligned}\right\} \tag{2.9}$$

where

$$\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \omega_3\boldsymbol{e}_3.$$

Hence,

$$\left.\begin{aligned} \omega_1 &= \dot{\theta}\cos(\psi - \gamma) + \dot{\varphi}\sin\theta\sin(\psi - \gamma), \\ \omega_2 &= \dot{\theta}\sin(\psi - \gamma) - \dot{\varphi}\sin\theta\cos(\psi - \gamma), \\ \omega_3 &= \dot{\psi} + \dot{\varphi}\cos\theta. \end{aligned}\right\} \tag{2.10}$$

The nonholonomic constraints which arise because the sphere rolls without sliding are

$$\frac{d}{dt}(z\boldsymbol{e}_3 + \rho\boldsymbol{e}_1) + \boldsymbol{\omega} \times a\boldsymbol{e}_1 = 0 \quad (\rho = b - a)$$

or, in terms of the projections onto the axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$,

$$\rho\dot{\gamma} + a\omega_3 = 0, \quad \dot{z} - a\omega_2 = 0. \tag{2.11}$$

The Lagrange function has the form (the mass of the sphere is $m = 1$)

$$L = \frac{1}{2}(\dot{z}^2 + \rho^2\dot{\gamma}^2) + \frac{1}{2}k^2(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) - gz.$$

Substituting (2.10) into equations (2.11), we obtain

$$\begin{array}{l} \rho\dot{\gamma} + a\dot{\psi} + a\dot{\varphi}\cos\theta = 0, \\ \dot{z} - a\dot{\theta}\sin(\psi - \gamma) - a\dot{\varphi}\sin\theta\cos(\psi - \gamma) = 0. \end{array} \left| \begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} \right. \tag{2.12}$$

Defining

$$\mathcal{L}_q = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i},$$

we obtain

$$\begin{array}{l} \mathcal{L}_\gamma = \rho\lambda_1, \quad \mathcal{L}_z = \lambda_2, \quad \mathcal{L}_\varphi = a\lambda_1\cos\theta - a\lambda_2\sin\theta\cos(\psi - \gamma), \\ \mathcal{L}_\psi = a\lambda_1, \quad \mathcal{L}_\theta = -a\lambda_2\sin(\psi - \gamma). \end{array} \tag{2.13}$$

Eliminating λ_1 and λ_2, we obtain the equations of motion

$$\begin{array}{l} a\mathcal{L}_\gamma = \rho\mathcal{L}_\psi, \quad \mathcal{L}_\theta + a\mathcal{L}_z\sin(\psi - \gamma) = 0, \\ \rho\mathcal{L}_\varphi = a\mathcal{L}_\gamma\cos\theta - a\rho\mathcal{L}_z\sin\theta\cos(\psi - \gamma), \end{array} \tag{2.14}$$

which, together with (2.12), form a complete system, in which

$$\mathcal{L}_\gamma = \rho^2\ddot{\gamma}, \quad \mathcal{L}_\psi = k^2\frac{d}{dt}(\dot{\psi} + \dot{\varphi}\cos\theta),$$

$$\mathcal{L}_\theta = k^2(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta), \quad \mathcal{L}_z = \ddot{z} + g,$$

$$\mathcal{L}_\varphi = k^2\frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta).$$

Substituting these expressions into (2.14) and combining them with the constraint equations (2.12), we obtain the complete system of equations of motion

$$a\rho\ddot{\gamma} = k^2\frac{d}{dt}(\dot{\psi} + \dot{\varphi}\cos\theta), \tag{2.15}$$

$$k^2(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta) + a(\ddot{z} + g)\sin(\psi - \gamma) = 0, \tag{2.16}$$

$$k^2\frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta) = a\rho\ddot{\gamma}\cos\theta - a(\ddot{z} + g)\sin\theta\cos(\psi - \gamma), \tag{2.17}$$

$$\rho\dot{\gamma} + a(\dot{\psi} + \dot{\varphi}\cos\theta) = 0, \tag{2.18}$$

$$\dot{z} = a\,[\dot{\theta}\sin(\psi-\gamma) + \dot{\varphi}\sin\theta\cos(\psi-\gamma)]. \tag{2.19}$$

From (2.15) and (2.18), we find

$$\dot{\gamma} = \Omega = \text{const}, \quad \dot{\psi} + \dot{\varphi}\cos\theta = \omega_3 = -\frac{\rho}{a}\Omega. \tag{2.20}$$

Equation (2.19) is an expression of the relationship

$$\dot{z} = a\omega_2, \tag{2.21}$$

and equations (2.16) and (2.17) reduce to the two equations

$$\dot{\omega}_1 = \Omega\omega_2, \tag{2.22}$$

$$k^2(\dot{\omega}_2 + \Omega\omega_1) = -a(\ddot{z} + g). \tag{2.23}$$

In order to obtain equation (2.22), we multiply (2.16) by $\sin\theta\cos(\psi-\gamma)$ and equation (2.17) by $-\sin(\psi-\gamma)$ and add the resulting equations. Using (2.20), we obtain

$$(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta)\cos(\psi-\gamma) + \left(\ddot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\cos\theta - \frac{\rho\Omega}{a}\dot{\theta}\right)\sin(\psi-\gamma) = 0,$$

in which we separate out the term

$$\dot{\omega}_1 = \ddot{\theta}\cos(\psi-\gamma) + (\ddot{\varphi}\sin\theta + \dot{\varphi}\dot{\theta}\cos\theta)\cos(\psi-\gamma) - \dot{\theta}(\dot{\psi}-\dot{\gamma})\sin(\psi-\gamma) + \varphi(\dot{\psi}-\dot{\gamma})\sin\theta\cos(\psi-\gamma),$$

and also the expression $\omega_2\dot{\gamma} = \Omega\omega_2$. The remaining terms in the sum vanish by virtue of (2.20). Similarly, multiplying (2.16) by $\sin\theta\sin(\psi-\gamma)$ and equation (2.17) by $\cos(\psi-\gamma)$ and adding the resulting expressions, we obtain an equation which can be transformed into (2.23). With allowance for (2.21), we can reduce the system (2.22) and (2.23) to the equation

$$\ddot{\omega}_2 + \frac{k^2\Omega^2}{a^2+k^2}\omega_2 = 0.$$

Hence

$$\omega_2 = A\sin\left(\frac{k\Omega}{\sqrt{a^2+k^2}}t - \mathscr{E}\right), \tag{2.24}$$

where A and $\mathscr{E}$ are arbitrary constants and also

$$\omega_1 = -\frac{A}{k}\sqrt{a^2+k^2}\cos\left(\frac{k\Omega}{\sqrt{a^2+k^2}}t - \mathscr{E}\right) - \frac{ga}{k^2\Omega}. \tag{2.25}$$

Finally, substituting (2.24) into (2.21) and then integrating the latter, we find

$$z = -\frac{aA\sqrt{a^2+k^2}}{k\Omega}\cos\left(\frac{k\Omega}{\sqrt{a^2+k^2}}t - \mathscr{E}\right) + z_0.$$

Thus, a homogeneous heavy sphere rolling on the rough inside wall of a vertical circular cylinder moves, in the general case, in such a way that the coordinate z of its center moves on a sinusoid, i.e., the sphere does not fall. This result has been obtained under the assumption that the constraints are not violated; it is therefore necessary to find the conditions of nonviolation of the constraints and, in particular, to establish whether or not the constraints are violated during the motion of the sphere on the cylinder. To this end, we write down an expression for the Lagrange function, assuming that ρ is a variable quantity,

$$L = \frac{1}{2}(\dot{z}^2 + \rho^2\dot{\gamma}^2 + \dot{\rho}^2) + \frac{k^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) - gz,$$

and set up an equation with Lagrangian multipliers for the coordinate ρ, taking into account the existence of the constraint

$$\rho = b - a \text{ or } \dot{\rho} = 0. \tag{2.26}$$

Denoting by λ_3 the Lagrangian multiplier corresponding to the constraint (2.26), we obtain

$$\mathscr{L}_\rho \equiv \ddot{\rho} - \rho\dot{\gamma}^2 = \lambda_3,$$

and, hence, by virtue of (2.26) the force of the normal pressure R_1 is

$$R_1 = \lambda_3 = -(b-a)\quad \Omega^2 = \text{const}.$$

Consequently, the constraint is not violated provided $\Omega \neq 0$, a condition which is generally satisfied. The relationships (2.13) show that the components of the reaction forces due to dry friction are, respectively,

$$R_2 = \rho\lambda_1 = \mathscr{L}_\gamma = \rho^2\ddot{\gamma} = 0,$$

$$R_3 = \lambda_2 = \mathscr{L}_z = \ddot{z} + g = \frac{akA\Omega}{\sqrt{a^2+k^2}}\cos\left(\frac{k\Omega}{\sqrt{a^2+k^2}}t - \mathscr{E}\right) + g.$$

If μ is the coefficient of dry friction, the condition that the sphere always rolls without sliding is the inequality

$$\frac{akA\Omega}{\sqrt{a^2+k^2}} + g < \mu\rho\Omega^2,$$

which, for given parameters of the system (i.e., for given values of a, k, μ, ρ and g) imposes restrictions on the initial conditions which determine the arbitrary constants A and Ω.

EXAMPLE 3. Let us consider a system[1] consisting of three rough homogeneous cylinders, of which two, of radius r, are identical and roll on a horizontal plane and the third, of radius R, rolls on the first two cylinders (Figure 3.2).

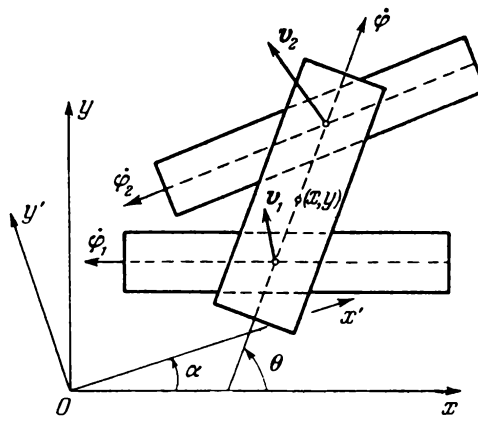

FIGURE 3.2

We introduce fixed coordinate systems $Oxyz$ and $Ox'y'z'$ whose Ox and Ox' axes are directed parallel to the generators of the lower cylinders, which are inclined to each other at an angle α. The angle α remains constant by virtue of the condition of rolling without sliding. We introduce the following generalized coordinates: x and y, the coordinates of the center of mass of the upper cylinder (the coordinate $z = 2r + R$ remains constant); θ, the angle between a generator of the upper cylinder and the Ox axis; φ, the angle of rotation of the upper cylinder about its axis; and φ_1 and φ_2, the corresponding angles of rotation of the lower cylinders. We shall denote by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ the radius vectors from the center of mass of the upper cylinder to the points of contact of the cylinders. Using the relations

$$x = x' \cos\alpha - y' \sin\alpha, \quad x' = x\cos\alpha + y\sin\alpha,$$
$$y = x' \sin\alpha + y' \cos\alpha, \quad y' = -x\sin\alpha + y\cos\alpha,$$

we obtain the following expressions for the projections of the vectors onto the $Oxyz$ axes

$$\boldsymbol{r}_1\{(r\varphi_1 - y)\cot\theta,\ r\varphi_1 - y,\ -R\},$$
$$\boldsymbol{r}_2\left\{(r\varphi_2 + x\sin\alpha - y\cos\alpha)\frac{\cos\theta}{\sin(\theta-\alpha)},\right.$$
$$\left.(r\varphi_2 + x\sin\alpha - y\cos\alpha)\frac{\sin\theta}{\sin(\theta-\alpha)},\ -R\right\},$$

$$\left.\begin{aligned} &\boldsymbol{v}_1\{0, 2r\dot\varphi_1, 0\} \\ &\boldsymbol{v}_2\{-2a\dot\varphi_2\sin\alpha, 2a\dot\varphi_2\cos\alpha, 0\} \end{aligned}\right\} \text{ the velocities of the points of contact of the cylinders;}$$

[1] The present example, proposed by A. Ju. Išlinskiĭ, is a nonholonomic system which is not a Čaplygin system (see the following section).

$v\{\dot{x}, \dot{y}, 0\}$, the velocity of the center of mass of the upper cylinder;

$\omega\{\dot{\varphi}\cos\theta, \dot{\varphi}\sin\theta, \dot{\theta}\}$, the instantaneous angular velocity of the upper cylinder.

The condition of rolling without sliding has the form

$$v + \omega \times r_1 = v_1, \quad v + \omega \times r_2 = v_2$$

or, projected onto the $Pxyz$ axes,

$$\begin{aligned}
&\dot{x} - R\dot{\varphi}\sin\theta - \dot{\theta}(r\varphi_1 - y) = 0,\\
&\dot{y} + R\dot{\varphi}\cos\theta + \dot{\theta}(r\varphi_1 - y)\cot\theta - 2a\dot{\varphi}_1 = 0,\\
&\dot{x}\sin(\theta-\alpha) - R\dot{\varphi}\sin\theta\sin(\theta-\alpha) + 2a\dot{\varphi}_2\sin\alpha\sin(\theta-\alpha)\\
&\qquad - \dot{\theta}(r\varphi_2 + x\sin\alpha - y\cos\alpha)\sin\theta = 0,\\
&\dot{y}\sin(\theta-\alpha) + R\dot{\varphi}\cos\theta\sin(\theta-\alpha) - 2a\dot{\varphi}_2\cos\alpha\sin(\theta-\alpha)\\
&\qquad + \dot{\theta}(r\varphi_2 + x\sin\alpha - y\cos\alpha)\cos\theta = 0.
\end{aligned} \tag{2.27}$$

Thus, the system has six generalized coordinates x, y, θ, φ, φ_1, and φ_2, and two degrees of freedom.

We now set up the equations of motion for this system. The kinetic energy of the system can be expressed in the form

$$T = \frac{1}{2}[m(\dot{x}^2 + \dot{y}^2) + A\dot{\theta}^2 + C\dot{\varphi}^2 + J_1\dot{\varphi}_1^2 + J_2\dot{\varphi}_2^2], \tag{2.28}$$

where m, A, and C are the mass and principal central moments of inertia of the upper cylinder and J_1 and J_2 are the moments of inertia of the lower cylinders about a generator.

The equations of motion with undetermined multipliers have the form

$$\begin{aligned}
&m\ddot{x} = \lambda_1 + \lambda_3\sin(\theta-\alpha), \quad m\ddot{y} = \lambda_2 + \lambda_4\sin(\theta-\alpha),\\
&A\ddot{\theta} = -\lambda_1(r\varphi_1 - y) + \lambda_2(r\varphi_1 - y)\cot\theta - \lambda_3(r\varphi_2 + x\sin\alpha\\
&\quad - y\cos\alpha)\sin\theta + \lambda_4(r\varphi_2 + x\sin\alpha - y\cos\alpha)\cos\theta = 0,\\
&J_1\ddot{\varphi}_1 = -2a\lambda_2, \quad J_2\ddot{\varphi}_2 = (\lambda_3\sin\alpha - \lambda_4\cos\alpha)2a\sin(\theta-\alpha),\\
&C\ddot{\varphi} = -R\lambda_1\sin\theta + R\lambda_2\cos\theta - R\lambda_3\sin\theta\sin(\theta-\alpha)\\
&\quad + R\lambda_4\cos\theta\sin(\theta-\alpha).
\end{aligned}$$

§3. Nonholonomic Čaplygin systems. Čaplygin's equations. Vorenec's equations

It was pointed out by Čaplygin that in many conservative nonholonomic systems the generalized coordinates $q_1, \cdots, q_m, q_{m+1}, \cdots, q_n$ can be chosen in such a way that the variations of the first m coordinates

may be assumed to be independent and the remaining $n - m$ coordinates $q_{m+1}, \cdots, q_n$ occur neither in the coefficients $b_{s,m+k}$ of the equations of the nonintegrable kinematic constraints, written in the form

$$\dot{q}_{m+k} = \sum_{s=1}^{m} b_{s,m+k}\dot{q}_s \qquad (k = 1, 2, \ldots, n-m), \tag{3.1}$$

nor in the expressions for the Lagrange function L, written down without allowance for the constraints (3.1). Henceforth we shall refer to these as Čaplygin systems.[2] Such systems are remarkable in that their dynamical equations of motion can be separated from the nonintegrable constraint equations.

We shall now give some examples of nonholonomic Čaplygin systems.

EXAMPLE 1. *Disc on a rough plane.* Suppose that a homogeneous disc of radius a rolls without slipping on a horizontal plane. The position of the disc is determined by five generalized coordinates; the Cartesian coordinates x and y of the point of contact of the disc on the plane; the angle φ between the Ox axis and the straight line in which the plane of the disc intersects the horizontal plane; the angle θ between the plane of the disc and the vertical; and the angle of rotation ψ of the disc, measured in the plane of the disc from the radius through the point of contact to a fixed radius on the disc (see Figure 2.4). The condition of rolling without sliding can be expressed by the two differential equations

$$dx = -a\cos\varphi\, d\psi, \qquad dy = -a\sin\varphi\, d\psi. \tag{3.2}$$

We now find an expression for the kinetic energy of the disc. The coordinates of the center of the disc, which coincides with the center of mass, are

$$x_C = x - a\sin\theta\sin\varphi, \quad y_C = y + a\sin\theta\cos\varphi, \quad z_C = a\cos\theta.$$

Hence, one readily obtains the expressions $v_C^2 = \dot{x}_C^2 + \dot{y}_C^2 + \dot{z}_C^2$. Writing down the kinetic energy of the rotational motion, we have

$$T_r = \frac{1}{2}A(\dot{\theta}^2 + \dot{\varphi}^2\cos^2\theta) + \frac{1}{2}C(\dot{\psi} + \dot{\varphi}\sin\theta)^2,$$

where A and C are the principal moments of inertia of the disc. The total kinetic energy of the disc, written down without allowance for

[2] We shall give the same name to nonconservative systems or to systems with inhomogeneous kinematic constraints if the expressions for the generalized forces and the coefficients of the equations of the nonholonomic constraints do not contain the coordinates $q_{m+1}, q_{m+2}, \cdots, q_n$.

the nonholonomic constraints (3.2), is

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + a^2\dot{\theta}^2 + a^2\dot{\varphi}^2 \sin^2\theta) - ma [\dot{\theta} \cos\theta (\dot{x} \sin\varphi - \dot{y} \cos\varphi) + \dot{\varphi} \sin\theta (\dot{x} \cos\varphi + \dot{y} \sin\varphi)] + \frac{1}{2} A (\dot{\theta}^2 + \dot{\varphi}^2 \cos^2\theta) + \frac{1}{2} C (\dot{\psi} + \dot{\varphi} \sin\theta)^2. \tag{3.3}$$

The potential energy of the disc is

$$V = mga \cos\theta. \tag{3.4}$$

In this example, the expressions (3.2) and (3.4) contain only the coordinates q and θ; hence, as the independent variations of the coordinates q_1, q_2, and q_3 (the system has three degrees of freedom) one must take $\delta\varphi$, $\delta\theta$, and one of the variations of the variables ψ, x, or y.

EXAMPLE 2. *Sleigh on an inclined plane.* We now consider a special case of nonholonomic motion on a plane when the center of mass lies on the straight line along the knife edge at a distance a from the point (x, y) of contact of the knife edge and the plane. In accordance with the notation of Figure 3.3, the kinetic energy of the system can be written in the form

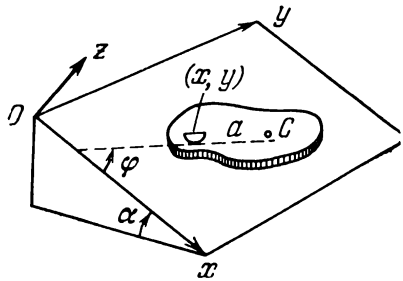

FIGURE 3.3

$$T = \frac{1}{2} m [\dot{x}^2 + \dot{y}^2 + J\dot{\varphi}^2 + 2a\dot{\varphi} (\dot{y} \cos\varphi - \dot{x} \sin\varphi)], \tag{3.5}$$

where m is the mass of the sleigh and J is the moment of inertia about the axis passing through the point (x, y). The potential energy is

$$V = -mgx \sin\alpha. \tag{3.6}$$

The equation of the nonholonomic constraint is

$$dy - \tan\varphi\, dx = 0. \tag{3.7}$$

Of the three generalized coordinates x, y, and φ, the expressions for the Lagrange function and the nonholonomic constraints (3.7) contain only x and φ. Hence, if the variations of these coordinates are assumed to be independent, we have a nonholonomic Čaplygin system.

EXAMPLE 3. *Two-wheeled carriage on a rough plane.* Figure 3.4 shows

that the position of the system is determined by five generalized coordinates: the Cartesian coordinates x and y of the point of intersection of the symmetry axis of the carriage with the axle on which the wheels are mounted; the angle θ between the symmetry axis of the carriage and the Ox axis; and the angles of rotation φ_2 of the left-hand and φ_1 of the right-hand wheel. We shall assume that the body of the carriage does not undergo vertical displacements. The condition of rolling without sliding leads to three kinematic constraints (a is the radius of the wheels and b is half the length of the axle). For the right-hand wheel, we have

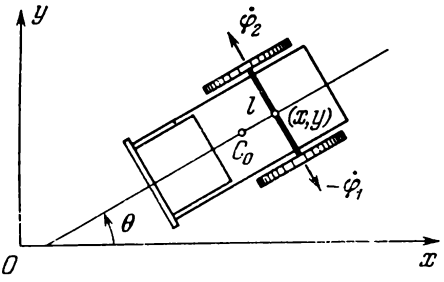

FIGURE 3.4

$$\dot x\cos\theta + \dot y\sin\theta + b\dot\theta + a\dot\varphi_1 = 0. \tag{3.8}$$

For the left-hand wheel,

$$\ddot x\cos\theta + \ddot y\sin\theta - b\ddot\theta + a\ddot\varphi_2 = 0. \tag{3.9}$$

Finally, the absence of lateral sliding of both wheels yields

$$\ddot x\sin\theta - \ddot y\cos\theta = 0. \tag{3.10}$$

Thus, the system has two degrees of freedom.

Let the center of mass C_0 be situated on the symmetry axis at a distance l from the point (x,y). Then the kinetic energy of the body of the carriage is

$$T_0 = \frac{m_0}{2}[\dot x^2 + \dot y^2 + 2l\dot\theta(\dot y\cos\theta - \dot x\sin\theta) + k_0^2\dot\theta^2],$$

where m_0 is the mass of the body and k_0 is the radius of gyration of the body about the vertical passing through (x,y).

We now write down an expression for the kinetic energy of the wheels. Denoting by A the moment of inertia of a wheel about a diameter and by C the axial moment of inertia, we obtain

$$T_1 = \frac{m_1}{2}[\dot x^2 + \dot y^2 + 2b\dot\theta(\dot x\cos\theta + \dot y\sin\theta) + b^2\dot\theta^2] + \frac{1}{2}A\dot\theta^2 + \frac{1}{2}C\dot\varphi_1^2$$

for the right-hand wheel and

$$T_2 = \frac{m_1}{2}[\dot{x}^2 + \dot{y}^2 - 2b\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta) + b^2\dot{\theta}^2] + \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}C\dot{\varphi}_2^2$$

for the left-hand wheel, where m_i is the mass of each wheel (the wheels are assumed to be identical). The total kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + m_0 l\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}C(\dot{\varphi}_1^2 + \dot{\varphi}_2^2), \tag{3.11}$$

where $m = m_0 + 2m_1$ is the mass of the complete system and $J = m_0 k_0^2 + 2m_1 b^2 + 2A$ is the moment of inertia of the system when it rotates as a whole about the vertical axis passing through the point (x, y). The constraint equations (3.8) – (3.10) and the kinetic energy (3.11) contain only the coordinate θ. Therefore, taking $\delta\theta$ as one of the two independent variations of the coordinates, we obtain a Čaplygin system.

EXAMPLE 4. *Homogeneous heavy sphere rolling on the inner surface of a vertical rough cone.* In accordance with Figure 3.5, we introduce the following generalized coordinates: r and γ, the coordinates of the point of contact of the sphere and the cone, where r is the distance of this point from the apex measured along the corresponding generator of the cone and γ is the angle between the plane passing through this point and the axis of the cone and a fixed plane passing through the axis. We take the three Eulerian angles φ, ψ, and θ as the remaining generalized coordinates. The expression for the Lagrange function can be written down without difficulty:

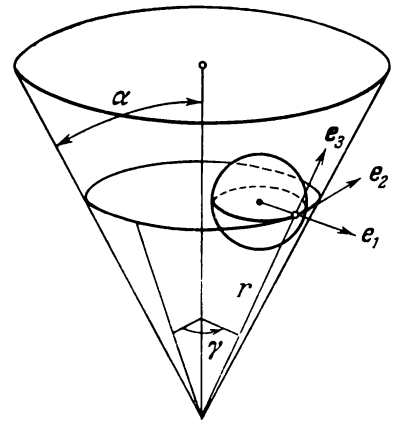

FIGURE 3.5

$$L = \frac{m}{2}[\dot{r}^2 + (r\sin\alpha - a\cos\alpha)^2\dot{\gamma}^2] + \frac{mk^2}{2}(\dot{\varphi}^2 + \dot{\psi}^2 + \dot{\theta}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) - mgr\cos\alpha, \tag{3.12}$$

where m is the mass of the sphere; α is the angle between the axis and a generator of the cone; and k is the radius of gyration of the sphere.

In order to obtain the kinematic equations that express the condition of rolling without sliding, we introduce moving coordinate axes, as shown in Figure 3.5. Let the projections of the instantaneous angular

velocity ω of the sphere onto these axes be ω_1, ω_2, ω_3. The condition of rolling without sliding can then be expressed by the equations

$$\begin{aligned} &r\dot{\gamma}\sin\alpha + a\omega_3 = 0, \\ &\dot{r} - a\omega_2 = 0, \end{aligned} \tag{3.13}$$

where a is the radius of the sphere. In these equations, the quantities ω_2 and ω_3 must be expressed in terms of the Eulerian angles and their derivatives. It follows from Figure 3.6 that

$$\begin{aligned} \omega_x &= (\omega_1\cos\alpha + \omega_3\sin\alpha)\cos\gamma - \omega_2\sin\gamma, \\ \omega_y &= (\omega_1\cos\alpha + \omega_3\sin\alpha)\sin\gamma + \omega_2\cos\gamma, \\ \omega_z &= -\omega_1\sin\alpha + \omega_3\cos\alpha. \end{aligned}$$

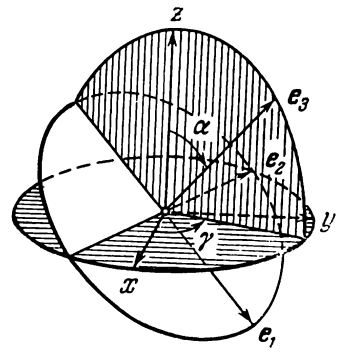

FIGURE 3.6

Comparing these expressions with the well-known Eulerian kinematic formulas, we obtain

$$\begin{aligned} \omega_1 &= \dot{\theta}\cos\alpha\cos(\psi-\gamma) \\ &\quad + \dot{\varphi}\,[\sin\theta\sin(\psi-\gamma)\cos\alpha - \sin\alpha\cos\theta] - \dot{\psi}\sin\alpha; \\ \omega_2 &= \dot{\theta}\sin(\psi-\gamma) - \dot{\varphi}\sin\theta\cos(\psi-\gamma), \\ \omega_3 &= \dot{\theta}\cos(\psi-\gamma)\sin\alpha + \dot{\varphi}\,[\sin\theta\sin(\psi-\gamma)\sin\alpha \\ &\quad + \cos\alpha\cos\theta] + \dot{\psi}\cos\alpha. \end{aligned}$$

Substituting these into (3.13), we obtain

$$\begin{aligned} &r\dot{\gamma}\sin\alpha + a\dot{\theta}\cos(\psi-\gamma)\sin\alpha \\ &\quad + a\dot{\varphi}\,[\sin\theta\sin(\psi-\gamma)\sin\alpha + \cos\theta\cos\alpha] + a\dot{\psi}\cos\alpha = 0, \\ &\quad \dot{r} - a\dot{\theta}\sin(\psi-\gamma) + a\dot{\varphi}\sin\theta\cos(\psi-\gamma) = 0. \end{aligned} \tag{3.14}$$

The equations of the kinematic constraints (3.14) contain the four coordinates r, γ, ψ, and θ, although the system has only three degrees of freedom. However, if the coordinates are suitably chosen, the system can be seen to be a Čaplygin system. For instead of ψ, we introduce the new variable χ by the relation

$$\psi = \chi + \gamma.$$

The expressions (3.12) and (3.14) then take the form

$$\begin{aligned} L &= \frac{m}{2}\,[\dot{r}^2 + (r\sin\alpha - a\cos\alpha)^2\dot{\gamma}^2] \\ &\quad + \frac{mk^2}{2}\,[\dot{\varphi}^2 + 2\dot{\varphi}(\dot{\chi}+\dot{\gamma})\cos\theta + (\dot{\chi}+\dot{\gamma})^2 + \dot{\theta}^2] - mgr\cos\alpha, \end{aligned} \tag{3.12′}$$

$$
\begin{aligned}
&r\dot{\gamma}\sin\alpha + a\dot{\theta}\cos\chi\sin\alpha + a\dot{\varphi}\,(\sin\theta\sin\chi\sin\alpha + \\
&\qquad + \cos\theta\cos\alpha) + a\,(\dot{\chi}+\dot{\gamma})\cos\alpha = 0, \\
&\dot{r} - a\dot{\theta}\sin\chi + a\dot{\varphi}\sin\theta\cos\chi = 0.
\end{aligned}
\tag{3.14'}
$$

Of the five generalized coordinates r, γ, φ, θ, and χ, the expressions for the kinematic constraints (3.14′) and the Lagrange function (3.12′) contain only the three coordinates r, θ, and χ, whose variations may be assumed to be independent.

We now turn to the derivation of Čaplygin's dynamical equations.

It is evident from the form of the constraint equations (3.1) that the virtual variations $\delta q_1, \delta q_2, \cdots, \delta q_m$ may be assumed to be independent. The remaining variations $\delta q_{m+1}, \delta q_{m+2}, \cdots, \delta q_n$ can then be expressed in terms of them by the relations

$$
\delta q_{m+k} = \sum_{s=1}^{m} b_{s,\,m+k}\delta q_s \qquad (k = 1, 2, \ldots, n-m). \tag{3.15}
$$

Using the relations (3.15) to express the dependent variations of the coordinates in terms of the independent variations in the d'Alembert-Lagrange equation, we arrive at the equation

$$
\sum_{s=1}^{m}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} - Q_s + \sum_{k=1}^{n-m} b_{s,\,m+k}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+k}}\right)\delta q_s = 0,
$$

from which, by virtue of the independence of the variations δq_s, we obtain m equations

$$
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} - Q_s + \sum_{k=1}^{n-m} b_{s,\,m+k}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+k}} = 0 \qquad (s = 1, 2, \ldots, m). \tag{3.16}
$$

We shall affix an asterisk to an expression in which the dependent generalized velocities $\dot{q}_{m+1}, \dot{q}_{m+2}, \cdots, \dot{q}_n$ (or rather, the generalized velocities which are regarded as dependent) have been eliminated by means of the constraint equations (3.1).

The following relations follow immediately:

$$
\frac{\partial T^*}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s} + \sum_{k=1}^{n-m}\frac{\partial T}{\partial \dot{q}_{m+k}}\frac{\partial \dot{q}_{m+k}}{\partial \dot{q}_s}, \qquad \frac{\partial T^*}{\partial q_s} = \frac{\partial T}{\partial q_s} + \sum_{k=1}^{n-m}\frac{\partial T}{\partial \dot{q}_{m+k}}\frac{\partial \dot{q}_{m+k}}{\partial q_s},
$$

where, in accordance with (3.1),

$$\frac{\partial \dot{q}_{m+k}}{\partial \dot{q}_s} = b_{s,\, m+k}, \qquad \frac{\partial \dot{q}_{m+k}}{\partial q_s} = \sum_{r=1}^{m} \frac{\partial b_{r,\, m+k}}{\partial q_s} \dot{q}_r.$$

By virtue of these relations, we have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} + \sum_{k=1}^{n-m} b_{s,\, m+k} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+k}} = \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_s} - \sum_{k=1}^{n-m} \frac{\partial T}{\partial \dot{q}_{m+k}} \frac{db_{s,\, m+k}}{dt},$$

where

$$\frac{db_{s,\, m+k}}{dt} = \sum_{r=1}^{m} \frac{\partial b_{s,\, m+k}}{\partial q_r} \dot{q}_r.$$

Using these expressions, we can rewrite equation (3.16) in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_s} - \frac{\partial T^*}{\partial q_s} + \sum_{k=1}^{n-m} \frac{\partial T}{\partial \dot{q}_{m+k}} \left[\sum_{r=1}^{m} \left(\frac{\partial b_{r,\, m+k}}{\partial q_s} - \frac{\partial b_{s,\, m+k}}{\partial q_r} \right) \dot{q}_r \right] = Q_s$$

$$(s = 1, 2, \ldots, m). \tag{3.17}$$

These are the equations that Čaplygin laid before the Society of the Friends of Natural Science on October 25, 1895 and which were published in the Proceedings of the Society in 1897 [**82**].[3)]

If the dependent velocities $\dot{q}_{m+1}, \dot{q}_{m+2}, \cdots, \dot{q}_n$ in the expressions $\partial T/\partial \dot{q}_{m+k}$ in equations (3.17) are eliminated by means of the constraint equations (3.1), a system of m equations for the m unknown functions $q_1, q_2, \cdots, q_m$ is obtained. Thus, Čaplygin's dynamical equations, whose number is equal to the number of degrees of freedom of the system, enable one to determine $q_1(t), q_2(t), \cdots, q_m(t)$ independently of the

[3)] Čaplygin derived his equations for true coordinates. However, when solving the problem of nonholonomic motion on a plane, for which he used his equations of motion, he in fact introduced a quasi-coordinate, the arc length, as an independent parameter. Čaplygin did not notice this fact. The reason why Čaplygin was able to use his equations in this manner is connected with the fact that his equations retain their form if some of the first m coordinates (whose variations are assumed to be independent) occur neither in the constraint equations nor in the Lagrange function L and these coordinates are replaced by quasi-coordinates. Usually, the quasi-coordinates are introduced by means of relations (generally linear) between the derivatives of the quasi-coordinates and the generalized velocities, the quasi-coordinates themselves being prevented by their very nature from occurring in the relations. If l ($l < m$) is the number of coordinates occurring in the function L and the constraint equations, not more than $m - l$ quasi-coordinates can be introduced if one wishes to use Čaplygin's equations.

kinematic constraints (3.1). If it is necessary, one can then determine the remaining $q_{m+1}(t), \cdots, q_n(t)$ from equations (3.1). It is readily seen that if the constraints (3.1) are integrable, i.e., if

$$\frac{\partial b_{r,\,m+k}}{\partial q_s} - \frac{\partial b_{s,\,m+k}}{\partial q_r} = 0 \qquad (s, r = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n-m), \tag{3.18}$$

then Čaplygin's equations take the form of ordinary Lagrange equations of the second kind.

If there are also conservative forces in the system, then, introducing the Lagrange function $L = T - V$, we arrive at Čaplygin's equations in the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} + \sum_{k=1}^{n-m}\left(\frac{\partial L}{\partial \dot{q}_{m+k}}\right)^*\left[\sum_{r=1}^{m}\left(\frac{\partial b_{r,\,m+k}}{\partial q_s} - \frac{\partial b_{s,\,m+k}}{\partial q_r}\right)\dot{q}_r\right] = Q_s \qquad (s = 1, 2, \ldots, m). \tag{3.19}$$

We shall now give some examples of the derivation of Čaplygin's equations.

EXAMPLE 5. *Plate with a knife edge on an inclined plane.* For the sake of simplicity, we shall consider the case when the center of mass of the plate coincides with the point (x, y) of contact of the knife edge and the plane. Then, in accordance with Figure 3.7, the Lagrange function is

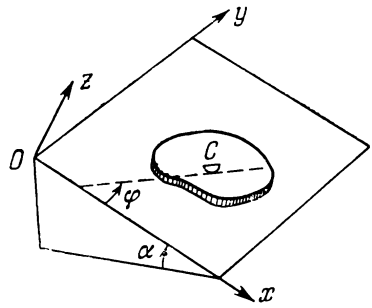

FIGURE 3.7

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}k^2\dot{\varphi}^2 + gx\sin\alpha, \tag{3.20}$$

where the mass of the plate is assumed equal to unity.

The kinematic constraint can be expressed by the equation

$$\dot{y} = \dot{x}\tan\varphi. \tag{3.21}$$

Let $q_1 = x$, $q_2 = \varphi$, and $q_3 = y$. As independent variations, we choose δx and $\delta\varphi$. Equation (3.21) is then the kinematic constraint (3.1), in which $b_{13} = \tan\varphi$ and $b_{23} \equiv 0$. Hence, all the derivatives

$$\partial b_{r,m+k}/\partial q_s \quad \text{and} \quad \partial b_{s,m+k}/\partial q_r$$

vanish with the exception of

$$\partial b_{13}/\partial q_2 = 1/\cos^2\varphi.$$

Čaplygin's equations (3.19), in which $Q = 0$, take the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \left(\frac{\partial L}{\partial \dot{y}}\right)^* \left(-\frac{1}{\cos^2 \varphi}\dot{\varphi}\right) = 0,$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{\varphi}} - \frac{\partial L^*}{\partial \varphi} + \left(\frac{\partial L}{\partial \dot{y}}\right)^* \left(\frac{1}{\cos^2 \varphi}\dot{x}\right) = 0.$$

In this problem

$$L^* = \frac{\dot{x}^2}{2\cos^2 \varphi} + \frac{1}{2}k^2\dot{\varphi}^2 + gx\sin\alpha, \qquad \left(\frac{\partial L}{\partial \dot{y}}\right)^* = \dot{x}\tan\varphi.$$

Hence, we obtain the following equations of motion:

$$\ddot{x} + \dot{x}\dot{\varphi}\tan\varphi = g\sin\alpha\cos^2\varphi, \quad \ddot{\varphi} = 0. \tag{3.22}$$

From the second equation of (3.22), we immediately obtain

$$\dot{\varphi} = \omega = \text{const}, \quad \varphi = \omega t + \varphi_0,$$

where the constants ω and φ_0 are determined by the initial conditions. Rather than integrate the first equation of (3.22) immediately, we shall exploit the energy integral, which exists in this case:

$$\frac{\dot{x}^2}{2\cos^2 \varphi} + \frac{1}{2}k^2\omega^2 - gx\sin\alpha = h = \text{const}. \tag{3.23}$$

In order to be specific, we shall take some definite initial conditions. Suppose that at the instant of time $t = 0$ the plate has only an angular velocity, i.e., $\varphi_0 = x_0 = \dot{x}_0 = y_0 = \dot{y}_0 = 0$, $\dot{\varphi}_0 = \omega$. We then obtain from (3.23)

$$x = a^2\sin^2\omega t,$$

where

$$a^2 = \frac{g}{2\omega^2}\sin\alpha,$$

and, finally, from (3.1) we obtain

$$y = a^2(\omega t - \tfrac{1}{2}\sin 2\omega t).$$

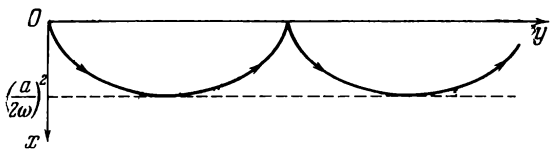

FIGURE 3.8

It follows from the solution obtained that the plate rotates with a constant angular velocity during its motion on the inclined plane, the point of contact of the knife edge and the plane describing the cycloid depicted in Figure 3.8. A characteristic and remarkable feature

of the motion is the fact that, on the average, the plate moves along a horizontal line, i.e., the force of gravity does not, on the average, appear to affect the plate in the normal manner.

Let us consider the case $\alpha = 0$. In this case of motion on a horizontal plane, the potential energy of the system is constant. Since the coordinate x now disappears from the relevant equations, it can be replaced by a single quasi-coordinate, for example, the arc length s, by means of the relation

$$\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2}, \tag{3.24}$$

which, together with (3.21), forms a system equivalent to the two linear equations:

$$\dot{x} = \dot{s}\cos\varphi, \quad \dot{y} = \dot{s}\sin\varphi. \tag{3.25}$$

Now, introducing the notation $q_1 = s$, $q_2 = \varphi$, $q_3 = x$, and $q_4 = y$, we obtain equations (3.15), in which $b_{13} = \cos\varphi$, $b_{14} = \sin\varphi$, and $b_{ij} \equiv 0$ for the remaining b_{ij}. Hence, of all the $\partial b_{r,m+k}/\partial q_s$ and $\partial b_{s,m+k}/\partial q_r$, only $\partial b_{13}/\partial q_2 = -\sin\varphi$ and $\partial b_{14}/\partial q_2 = \cos\varphi$ do not vanish. Čaplygin's equations take the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{s}} = 0, \qquad \frac{d}{dt}\frac{\partial L^*}{\partial \dot{\varphi}} = 0,$$

where

$$L^* = \frac{1}{2}(\dot{s}^2 + k^2\dot{\varphi}^2).$$

Hence,

$$\dot{\varphi} = \omega = \text{const}, \quad \dot{s} = v_0 = \text{const}$$

and, consequently, the point of contact of the knife edge moves around a circle with a constant velocity v_0.

We shall now derive Čaplygin's equations for the case when quasi-coordinates are chosen as the free parameters.

We consider a nonholonomic system whose position is determined by n generalized coordinates $q_1, q_2, \cdots, q_n$. Suppose that the system has m degrees of freedom ($m < n$), i.e., the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ are connected by $n - m$ nonintegrable equations, which we shall assume are linear and homogeneous. Suppose that the coefficients in these equations, as also the Lagrange function L, depend on only the first m coordinates $q_1, q_2, \cdots, q_m$. As independent parameters, we introduce the quasi-coordinates $\pi_1, \pi_2, \cdots, \pi_m$ by means of the linear equations[4]

[4] As usual, summation is to be understood over indices that occur twice in a product.

$$\dot{\pi}_\alpha = a_{\alpha\sigma}\dot{q}_\sigma \quad (\alpha, \sigma = 1, 2, \ldots, m), \tag{3.26}$$

where the coefficients $a_{\alpha\sigma}$ are functions of $q_1, q_2, \cdots, q_m$. Using (3.26) and the $n - m$ equations of the nonholonomic constraints, we express all the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ in terms of the m independent quasi-velocities $\dot{\pi}_1, \dot{\pi}_2, \cdots, \dot{\pi}_m$:

$$\dot{q}_i = b_{i\beta}\dot{\pi}_\beta \ (i = 1, 2, \ldots, n; \ \beta = 1, 2, \ldots, m). \tag{3.27}$$

Hence, the variations of the generalized velocities can be expressed in terms of the variations of the free parameters by means of the relations

$$\delta q_i = b_{i\alpha}\delta\pi_\alpha \ (i = 1, 2, \ldots, n; \ \alpha = 1, 2, \ldots, m). \tag{3.28}$$

We shall take the d'Alembert-Lagrange equation as our starting point, expressing it in the generalized coordinates

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right)\delta q_i = 0 \quad (i = 1, 2, \ldots, n).$$

Substituting the expressions (3.28) instead of δq_i, we obtain a sum, which, by virtue of the independence of the variations $\delta\pi_\alpha$, can be split into m equations of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)b_{i\alpha} - \frac{\partial L}{\partial q_i}b_{i\alpha} = 0 \ (i = 1, 2, \ldots, n; \ \alpha = 1, 2, \ldots, m). \tag{3.29}$$

In these equations, the Lagrange function L depends on the m coordinates $q_1, q_2, \cdots, q_m$ and the n velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, i.e., $L = L(q_\beta, \dot{q}_i)$. Substituting the expressions (3.27) instead of the $\dot{q}_i$, we obtain a function L^* given by

$$L^*(q_\beta, \dot{\pi}_\alpha) = L(q_\beta, a_{i\sigma}\dot{\pi}_\sigma). \tag{3.30}$$

On the basis of (3.30), one can readily find the following relations

$$\frac{\partial L^*}{\partial q_\beta} = \frac{\partial L}{\partial q_\beta} + \frac{\partial L}{\partial \dot{q}_i}\dot{\pi}_\sigma\frac{\partial b_{i\sigma}}{\partial q_\beta}, \quad \frac{\partial L}{\partial q_i}b_{i\alpha} = \frac{\partial L^*}{\partial q_\beta}b_{\beta\alpha} - \frac{\partial L}{\partial \dot{q}_i}\frac{\partial b_{i\sigma}}{\partial q_\beta}\cdot\dot{\pi}_\sigma b_{\beta\alpha},$$

$$\frac{\partial L^*}{\partial \dot{\pi}_\sigma} = \frac{\partial L}{\partial \dot{q}_i}b_{i\sigma}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)b_{i\alpha} = \frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_\alpha} - \frac{\partial L}{\partial \dot{q}_i}\frac{\partial b_{i\alpha}}{\partial q_\beta}\dot{\pi}_\sigma b_{\beta\sigma}$$

$$(i = 1, 2, \ldots, n; \ \alpha, \beta, \sigma = 1, 2, \ldots, m).$$

Substituting these expressions into (3.29) and defining

$$\frac{\partial b_{i\alpha}}{\partial q_\beta}b_{\beta\sigma} = \frac{\partial b_{i\alpha}}{\partial \pi_\sigma}, \quad \frac{\partial b_{i\sigma}}{\partial q_\beta}b_{\beta\alpha} = \frac{\partial b_{i\sigma}}{\partial \pi_\alpha}, \quad \frac{\partial L^*}{\partial q_\beta}b_{\beta\alpha} = \frac{\partial L^*}{\partial \pi_\alpha}, \tag{3.31}$$

we arrive at the following equations:

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_\alpha} - \frac{\partial L^*}{\partial \pi_\alpha} + \left(\frac{\partial L}{\partial \dot{q}_i}\right)^* \left(\frac{\partial b_{i\sigma}}{\partial \pi_\alpha} - \frac{\partial b_{i\alpha}}{\partial \pi_\sigma}\right)\dot{\pi}_\sigma = 0 \tag{3.32}$$

$$(i = 1, 2, \ldots, n;\ \alpha, \beta, \sigma = 1, 2, \ldots, m).$$

The equations (3.32) are Čaplygin's equations in quasi-coordinates. It is readily seen that the expressions (3.32) are identical with the Čaplygin's equations (3.19) if $\pi_1, \pi_2, \cdots, \pi_m$ are true coordinates, i.e., $\pi_\alpha = q_\alpha$ $(\alpha = 1, 2, \cdots, m)$. For in this case $b_{\beta\alpha} = \delta_{\beta\alpha}$, $b_{\beta\sigma} = \delta_{\beta\sigma}$ ($\delta_{\beta\alpha}$ and $\delta_{\beta\sigma}$ are the Kronecker deltas) and equations (3.32) take the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_\alpha} - \frac{\partial L^*}{\partial q_\alpha} + \left(\frac{\partial L}{\partial \dot{q}_i}\right)^* \left(\frac{\partial b_{i\sigma}}{\partial q_\alpha} - \frac{\partial b_{i\alpha}}{\partial q_\sigma}\right)\dot{q}_\sigma = 0 \tag{3.33}$$

$$(i = 1, 2, \ldots, n;\ \alpha, \sigma = 1, 2, \ldots, m),$$

found by Čaplygin.

Equations (3.32) differ in form from the Boltzmann-Hamel equations in quasi-coordinates (see §5) by the manner in which the nonholonomy terms are treated. In the Boltzmann-Hamel equations, the coefficients γ_{kij} contain the coefficients of the transformation matrix connecting the true velocities and the quasi-velocities and also the inverse of this matrix. In writing down the analogous terms in equations (3.32), we have made use of only the coefficients of a single matrix, which is rectangular (n rows and m columns) and does not, in general, have an inverse.

We now turn to an exposition of the results of Voronec, who, together with Čaplygin, Appell, and others, was one of the founders of the mechanics of nonholonomic systems. In his paper [**472**], which appeared in 1901, Voronec derived equations of motion without making the restrictive assumptions which lead to a Čaplygin system. Voronec's equations can therefore be applied to a larger class of nonholonomic systems than Čaplygin's equations. Following Voronec's paper, we consider the motion of a constrained system of material points subjected to forces that have a potential. We shall denote by $q_1, q_2, \cdots, q_{m+k}$ the generalized coordinates of the system and assume that the equations of the nonholonomic constraints have the form

$$\dot{q}_{m+\nu} - \alpha_{\nu i}\dot{q}_i = 0 \qquad (i = 1, 2, \ldots, m;\ \nu = 1, 2, \ldots, k), \tag{3.34}$$

where the coefficients $a_{\nu i}$ are, in general, functions of all the generalized coordinates $q_1, q_2, \cdots, q_n$. Since the equations (3.34) are nonintegrable, the quantities

(3.35) $$\beta_{ij}^{\nu} = \frac{\partial \alpha_{\nu i}}{\partial q_j} + \sum_{\mu=1}^{k} \frac{\partial \alpha_{\nu i}}{\partial q_{m+\mu}} \alpha_{\mu j} - \frac{\partial \alpha_{\nu j}}{\partial q_i} - \sum_{\mu=1}^{k} \frac{\partial \alpha_{\nu j}}{\partial q_{m+\mu}} \alpha_{\mu i}$$

cannot all vanish simultaneously.

If we denote by T the kinetic energy of the system and by U the potential function, the equations of motion, containing the undetermined multipliers λ_ν, are

(3.36)
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial (T + U)}{\partial q_i} - \lambda_\nu \alpha_{\nu i},$$
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+\nu}} = \frac{\partial (T + U)}{\partial q_{m+\nu}} + \lambda_\nu$$
$$(i = 1, 2, \ldots, m; \ \nu = 1, 2, \ldots, k).$$

In the expression for T, we replace $\dot{q}_{m+\nu}$ by their expressions (3.34), denoting the resulting function by Θ, i.e.,

$$T(t, q_1, \ldots, q_{m+k}, \dot{q}_1, \ldots, \dot{q}_{m+k}) = \Theta(t, q_1, \ldots, q_{m+k}, \dot{q}_1, \ldots, \dot{q}_m).$$

It is easily seen that

$$\frac{\partial \Theta}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} + \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \alpha_{\nu i}$$

and, hence,

$$\frac{d}{dt}\left(\frac{\partial \Theta}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \sum_{\nu=1}^{k} \left(\alpha_{\nu i} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+\nu}} + \frac{d\alpha_{\nu i}}{dt}\frac{\partial T}{\partial \dot{q}_{m+\nu}}\right).$$

Replacing the derivatives

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+\nu}}$$

by means of the formulas (3.36), we obtain

$$\frac{d}{dt}\left(\frac{\partial \Theta}{\partial \dot{q}_i}\right) = \frac{\partial (T + U)}{\partial q_i} + \sum_{\nu=1}^{k} \left(\frac{\partial (T + U)}{\partial q_{m+\nu}} \alpha_{\nu i} + \frac{d\alpha_{\nu i}}{dt}\frac{\partial T}{\partial \dot{q}_{m+\nu}}\right).$$

Eliminating $\partial T/\partial q_i$ and $\partial T/\partial q_{m+\nu}$ by means of relations of the form

$$\frac{\partial T}{\partial q_s} + \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \sum_{j=1}^{m} \frac{\partial \alpha_{\nu j}}{\partial q_s} \dot{q}_j = \frac{\partial \Theta}{\partial q_s} \qquad (s = 1, 2, \ldots, m + k),$$

we obtain

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_l} = \frac{\partial (\Theta + U)}{\partial q_l} + \sum_{\nu=1}^{k} \frac{\partial (\Theta + U)}{\partial q_{m+\nu}} \alpha_{\nu l} + \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \left[\frac{d\alpha_{\nu l}}{dt} - \sum_{j=1}^{m} \left(\frac{\partial \alpha_{\nu j}}{\partial q_l} + \sum_{\mu=1}^{k} \frac{\partial \alpha_{\nu j}}{\partial q_{m+\mu}} \alpha_{\mu l} \right) \dot{q}_j \right].$$

By virtue of (3.34) and (3.35), the expression in the square brackets is equal to $\sum_{j=1}^{m} \beta_{ij}^{\nu} \dot{q}_j$. We therefore obtain finally

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_l} = \frac{\partial (\Theta + U)}{\partial q_l} + \sum_{\nu=1}^{k} \frac{\partial (\Theta + U)}{\partial q_{m+\nu}} \alpha_{\nu l} + \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \beta_{ij}^{\nu} \dot{q}_j. \tag{3.37}$$

Equations (3.37) are Voronec's equations. In the special case when the coordinates $q_{m+1}, q_{m+2}, \cdots, q_{m+k} = q_n$ corresponding to the eliminated velocities do not appear explicitly in the expressions for the kinetic energy and the potential function or in the equations of the nonholonomic constraints, Voronec's equations (3.37) are identical with Čaplygin's equations (3.17). If the forces acting on the system do not have potentials, equations (3.37) can be written in the form

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_i} - \frac{\partial \Theta}{\partial q_i} - \sum_{\nu=1}^{k} \frac{\partial \Theta}{\partial q_{m+\nu}} \alpha_{\nu i} - \sum_{\nu=1}^{k} \sum_{j=1}^{m} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \beta_{ij}^{\nu} \dot{q}_j = Q_i + \sum_{\nu=1}^{k} Q_{m+\nu} \alpha_{\nu i} \tag{3.38}$$
$$(i = 1, 2, \ldots, m).$$

In the same paper [472], Voronec derived equations (3.37) by another method based on the Hamilton-Ostrogradskiĭ variational principle, which he generalized and extended to nonholonomic systems. In subsequent papers, Voronec also obtained equations of motion of nonholonomic systems in quasi-coordinates.

§4. Equations of Volterra and Maggi

In 1898, Volterra [469] published equations of motion of holonomic and nonholonomic systems in variables which he called characteristics of the motion. As the characteristics of the motion of a system of N material points, Volterra took arbitrary parameters $p_1, p_2, \cdots, p_\nu$ ($\nu \leqq 3N$), which are related to the components $\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_{3N}$ of the velocities of the points of the system by means of linear relations

$$\dot{x}_l = \xi_{ls} p_s \quad (i = 1, 2, \ldots, 3N;\ s = 1, 2, \ldots, \nu), \tag{4.1}$$

where the coefficients ξ_{is} are functions of the coordinates $x_1, x_2, \cdots, x_{3N}$ of the points of the system. In the case $\nu < 3N$, the relations (4.1) can

always be written down if x_i and $\dot{x}_i$ $(i = 1, 2, \cdots, 3N)$ are connected by $3N - \nu$ equations that are linear in the components of the velocities $\dot{x}_i$; this situation obtains, for example, in nonholonomic systems. In the case $\nu = 3N$, the relations (4.1) may be regarded as the transition from the quantities $\dot{x}_i$ to the parameters p_s, which are quasi-coordinates if the relations (4.1) are nonintegrable. Thus, Volterra actually derived equations in quasi-coordinates (although the actual concept of quasi-coordinates was introduced much later).

The relations (4.1) show that the variations of the coordinates δx_i can be expressed in terms of the characteristics of the virtual displacements δw_s of the system by the relations

$$\delta x_i = \xi_{is}\delta w_s \quad (i = 1, 2, \ldots, 3N;\ s = 1, 2, \ldots, \nu). \tag{4.2}$$

It follows directly from (4.1) and (4.2) that

$$\delta x_i = \frac{\partial \dot{x}_i}{\partial p_s}\delta w_s. \tag{4.3}$$

The kinetic energy T of the system can also be expressed in terms of the characteristics of the motion

$$T = \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} E_{is} p_r p_s, \tag{4.4}$$

where the quantities

$$E_{rs} = E_{sr} = m_i \xi_{is}\xi_{ir} \tag{4.5}$$

are known functions of the coordinates $x_1, x_2, \cdots, x_{3N}$.

In deriving Volterra's equations, we shall take as our starting point the d'Alembert-Lagrange equation

$$(m_i \ddot{x}_i - X_i)\,\delta x_i = 0 \quad (i = 1, 2, \ldots, 3N),$$

which we transform to the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\delta x_i\right) - \delta T + \frac{\partial T}{\partial \dot{x}_i}\left(\delta \dot{x}_i - \frac{d}{dt}\delta x_i\right) = X_i \delta x_i. \tag{4.6}$$

Using the relations (4.2), we can calculate the expressions $\delta\dot{x}_i - d\delta x_i/dt$ which occur on the left-hand side of equation (4.6). From (4.1) and (4.2) we obtain

$$\delta \dot{x}_i = \frac{\partial \xi_{is}}{\partial x_j}\xi_{jr} p_s \delta w_r + \xi_{is}\delta p_s,$$

$$\frac{d}{dt}\delta x_i = \frac{\partial \xi_{is}}{\partial x_j}\xi_{jr} p_r \delta w_s + \xi_{is}\frac{d}{dt}\delta w_s.$$

Hence

$$\delta\dot{x}_i - \frac{d}{dt}\delta x_i = \left(\frac{\partial \xi_{ir}}{\partial x_j}\xi_{js} - \frac{\partial \xi_{is}}{\partial x_j}\xi_{jr}\right) p_r \delta w_s + \xi_{is}\left(\delta p_s - \frac{d}{dt}\delta w_s\right). \tag{4.7}$$

By virtue of the independence of the characteristics of the motion p_s, the following relations hold:

$$\delta p_s - \frac{d}{dt}\delta w_s = 0. \tag{4.8}$$

Therefore, from (4.7) we obtain

$$\delta\dot{x}_i - \frac{d}{dt}\delta x_i = \left(\frac{\partial \xi_{ir}}{\partial x_j}\xi_{js} - \frac{\partial \xi_{is}}{\partial x_j}\xi_{jr}\right) p_r \delta w_s. \tag{4.9}$$

In accordance with the expression (4.4) for the kinetic energy T, we find

$$\frac{\partial T}{\partial \dot{x}_i}\delta x_i = \frac{\partial T}{\partial \dot{x}_i}\xi_{is}\delta w_s = \frac{\partial T}{\partial p_s}\delta w_s,$$

$$(i = 1, 2, \ldots, 3N;\ s, k = 1, 2, \ldots, \nu),$$

$$\frac{\partial T}{\partial \dot{x}_i} = m_i\dot{x}_i = m_i\xi_{ik}p_k.$$

We denote by

$$P_s = X_i\xi_{is}$$

the generalized force corresponding to the given quasi-coordinate. We now introduce the operation δT:

$$\delta T = \frac{\partial T}{\partial x_j}\delta x_j + \frac{\partial T}{\partial p_s}\delta p_s = \frac{\partial T}{\partial x_j}\xi_{js}\delta w_s + \frac{\partial T}{\partial p_s}\delta p_s = T_s\delta w_s + \frac{\partial T}{\partial p_s}\delta p_s,$$

where

$$T_s \equiv \frac{\partial T}{\partial w_s} = \frac{\partial T}{\partial x_j}\xi_{js}.$$

Thus, the d'Alembert-Lagrange equation (4.6) takes the form

$$\left[\frac{d}{dt}\frac{\partial T}{\partial p_s} + \left(m_i\xi_{ik}\frac{\partial \xi_{ir}}{\partial x_j}\xi_{js} - m_i\xi_{ik}\frac{\partial \xi_{is}}{\partial x_j}\xi_{jr}\right)p_k p_r - T_s - P_s\right]\delta w_s = 0.$$

In writing down this sum, we have used the relations (4.8). By virtue of the independence of the variations δw_s, this sum can be split into new equations

$$\frac{d}{dt}\frac{\partial T}{\partial p_s} + (b_{rs}^{(k)} - b_{sr}^{(k)})\,p_k p_r - T_s = P_s \qquad (s = 1, 2, \ldots, \nu), \tag{4.10}$$

in which we have introduced the Volterra notation

$$(4.11)\quad b_{rs}^{(k)} = m_i \xi_{ik} \frac{\partial \xi_{ir}}{\partial x_j} \xi_{js} \quad (i, j = 1, 2, \ldots, 3N;\ k, r, s = 1, 2, \ldots, \nu).$$

Equations (4.10) are Volterra's equations. We now transform these equations, noting that

$$T_s \equiv \frac{\partial T}{\partial x_j} \xi_{js} = \frac{1}{2} m_i \left(\frac{\partial \xi_{ik}}{\partial x_j} \xi_{ir} + \xi_{ik} \frac{\partial \xi_{ir}}{\partial x_j} \right) \xi_{js} p_k p_r = \frac{1}{2} (b_{ks}^{(r)} + b_{rs}^{(k)}) p_k p_r.$$

Substituting this expression into (4.10), we obtain

$$\frac{d}{dt} \frac{\partial T}{\partial p_s} + \left(b_{rs}^{(k)} - b_{sr}^{(k)} - \frac{1}{2} b_{ks}^{(r)} - \frac{1}{2} b_{rs}^{(k)} \right) p_k p_r = P_s.$$

Since k and r are dummy indices, the sum on the left-hand side of this equation can be transformed as follows:

$$\left(b_{rs}^{(k)} - b_{sr}^{(k)} - \frac{1}{2} b_{ks}^{(r)} - \frac{1}{2} b_{rs}^{(k)} \right) p_k p_r = \left(b_{ks}^{(r)} - b_{sk}^{(r)} - \frac{1}{2} b_{ks}^{(r)} - \frac{1}{2} b_{ks}^{(r)} \right) p_k p_r = - b_{sk}^{(r)} p_k p_r.$$

Therefore, the equations (4.10) are equivalent to the equations

$$(4.12)\quad \frac{d}{dt} \frac{\partial T}{\partial p_s} - b_{sk}^{(r)} p_k p_r = P_s \quad (s, r, k = 1, 2, \ldots, \nu).$$

The equations obtained by Volterra in his paper [**469**] on p. 458 have the form

$$(4.13)\quad \frac{d}{dt} \frac{\partial T}{\partial p_s} = a_{sk}^{(u)} \frac{\partial T}{\partial p_u} p_k + T_s + P_s \quad (u, s, k = 1, 2, \ldots, \nu),$$

where

$$a_{sk}^{(u)} = e_{gu} (b_{sk}^{(g)} - b_{ks}^{(g)}), \quad e_{gu} E_{ur} = \begin{cases} 1, & \text{if } g = r, \\ 0, & \text{if } g \neq r. \end{cases}$$

Using the expression $\partial T / \partial p_u = E_{ur} p_r$, we transform the first term on the right-hand side of equation (4.13)

$$a_{sk}^{(u)} \frac{\partial T}{\partial p_u} p_k = e_{gu} E_{ur} (b_{sk}^{(g)} - b_{ks}^{(g)}) p_k p = (b_{sk}^{(r)} - b_{ks}^{(r)}) p_k p_r = (b_{sr}^{(k)} - b_{rs}^{(k)}) p_k p_r.$$

Substituting these expressions into (4.13), we readily see that equations (4.13) are identical with equations (4.10). Thus, the equations (4.13), obtained by Volterra by means of invalid transpositional relations, are nevertheless correct.

As is well known [326], Volterra, when deriving his equations (4.13), erroneously used the transpositional relations

$$\delta\dot{x}_i - \frac{d}{dt}\delta x_i = 0 \tag{4.13}$$

in conjunction with the expression (4.1), which led him to the equations

$$\xi_{is}\left(\delta p_s - \frac{d}{dt}\delta\omega_s\right) = \xi_{jr}\frac{\partial\xi_{is}}{\partial x_j}(p_r\delta\omega_s - p_s\delta\omega_r), \tag{4.15}$$

which form an inconsistent system. The fact that the relations (4.1) and (4.14) cannot be used simultaneously is illustrated most simply by the example of a nonholonomic system consisting of two points M_1 and M_2 which can move on a plane xOy, but only in such a way that the velocity of the point M_1 is always directed along the straight line M_1M_2 joining the points M_1 and M_2. For if we denote by (x_1, y_1) and (x_2, y_2) the Cartesian coordinates of the points M_1 and M_2, the equation of this constraint can be written in the form

$$\frac{\dot{x}_1}{x_2 - x_1} = \frac{\dot{y}_1}{y_2 - y_1}. \tag{4.16}$$

It follows from (4.16) that any two displacements (dx_1, dy_1, dx_2, dy_2) and $(\delta x_1, \delta y_1, \delta x_2, \delta y_2)$ of the system compatible with the nonholonomic constraint (4.16) satisfy

$$dx_1 = \frac{x_2 - x_1}{y_2 - y_1}dy_1, \qquad \delta x_1 = \frac{x_2 - x_1}{y_2 - y_1}\delta y_1. \tag{4.17}$$

We now show that the conditions (4.17) and Volterra's transpositional relations (4.14)

$$d\,\delta x_1 - \delta\,dx_1 = d\,\delta y_1 - \delta\,dy_1 = d\,\delta x_2 - \delta\,dx_2 = d\,\delta y_2 - \delta\,dy_2 = 0 \tag{4.18}$$

form a system which is not compatible with the degree of arbitrariness allowed by (4.17) in the choice of dy_1, dx_2, and dy_2 and δy_1, δx_2, and δy_2. From (4.17) we find δdx_1 and $d\delta x_1$, and then the difference

$$\begin{aligned} d\,\delta x_1 - \delta\,dx_1 = -\frac{1}{(y_2 - y_1)^2}\{&(y_2 - y_1)\,\delta x_2\,dy_1 \\ &- (x_2 - x_1)(\delta y_2 - \delta y_1)\,dy_1 - (y_2 - y_1)\,\delta x_1\,dy_1 - (y_2 - y_1)\,dx_2\,\delta y_1 \\ &+ (x_2 - x_1)(dy_2 - dy_1)\,\delta y_1 + (y_2 - y_1)\,dx_1\delta y_1\} + \frac{x_2 - x_1}{y_2 - y_1}(d\,\delta y_1 - \delta\,dy_1). \end{aligned}$$

Expressing dx_1 and δx_1 in this equation in terms of dy_1 and δy_1 in accordance with (4.17), and using (4.18), we arrive at the equation

$$(y_2 - y_1)(\delta x_2\,dy_1 - dx_2\delta y_1) + (x_2 - x_1)(dy_2\delta y_1 - \delta y_2 dy_1) = 0,$$

which cannot be satisfied for all displacements of the system dx_1, dx_2, dy_1, dy_2 and δx_1, δx_2, δy_1, δy_2 permitted by the constraint (4.16).

In the special case when the distance between the points M_1 and M_2 is constant (a further constraint is imposed), the relations (4.15) take the form

$$\begin{aligned}
&(x_2 - x_1)(\delta\, ds - d\,\delta s) = -(y_2 - y_1)(d\varphi\,\delta s - \delta\varphi\, ds),\\
&(y_2 - y_1)(\delta\, ds - d\,\delta s) = (x_2 - x_1)(d\varphi\,\delta s - \delta\varphi\, ds),\\
&(y_2 - y_1)(\delta\, d\varphi - d\,\delta\varphi) = 0,\\
&(x_2 - x_1)(\delta\, d\varphi - d\,\delta\varphi) = 0,
\end{aligned} \tag{4.19}$$

where φ is the angle between the segment M_1M_2 and the Ox axis and s is the arc length traversed by the point M_1. It is readily seen that the system (4.19) is also contradictory.

In the same year, Volterra published a correction to his paper, in which he showed that the relations that determine the difference $\delta p_s - d\delta w_s/dt$ can be derived without recourse to the contradictory system of equations (4.15). However, in his correction, Volterra used the transpositional relations (4.14) in conjunction with the expressions (4.1) and (4.2). It follows that the intermediate equations in this case, too, are contradictory, as one can readily verify by carrying out Volterra's calculations for a concrete example. Nevertheless, as we have shown above, Volterra's equations are correct and can be derived rigorously.

In 1901, Maggi published a note [284] in which he showed that Volterra's equations as well as Appell's equations (see §8 of this chapter) may be derived from equations that he, Maggi, had already proposed in 1896 in his book on mechanics. We shall now derive Maggi's equations.

Let $q_1, q_2, \cdots, q_n$ be the generalized coordinates of a system whose motion is restricted by constraints expressed by equations of the form

$$a_{ij}\,\dot{q}_j + b_i = 0 \quad (i = 1,\ 2,\ldots,\ n - m;\ j = 1,\ 2,\ldots,\ n;\ m < n).$$

Hence,

$$a_{ij}\,\delta q_j = 0,$$

and, consequently, the variations of the generalized coordinates δq_j can be expressed in terms of certain independent parameters $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$:

$$\delta q_j = E_{jr}\epsilon_r \qquad (j = 1, 2, \cdots, n;\ r = 1, 2, \cdots, m). \tag{4.20}$$

Taking as our starting point the d'Alembert-Lagrange equation

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j\right)\delta q_j = 0 \quad (j = 1, 2, \ldots, n),$$

we replace the quantities δq_j by the expression (4.20) for them and then, writing $E_r = Q_j E_{jr}$, we obtain

$$\left[\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right)E_{jr} - E_r\right]\varepsilon_r = 0.$$

By virtue of the independence of the parameters ϵ_r, we obtain the m equations

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right)E_{jr} = E_r \qquad (j = 1, 2, \ldots, n;\ r = 1, 2, \ldots, m), \tag{4.21}$$

which are Maggi's equations.

It is readily seen that for a system of n material points whose coordinates and velocity components satisfy the relations (4.1), Maggi's equations are identical with Volterra's equations. For in this case equations (4.21) take the form

$$m_i \ddot{x}_i \xi_{is} = P_s \qquad (i = 1, 2, \ldots, 3N;\ s = 1, 2, \ldots, \nu). \tag{4.22}$$

We transform the left-hand side of equations (4.22):

$$m_i \ddot{x}_i \xi_{is} = \frac{d}{dt}(m_i \dot{x}_i \xi_{is}) - m_i \dot{x}_i \frac{d\xi_{is}}{dt} = \frac{d}{dt}\frac{\partial T}{\partial p_s} - m_i \xi_{ik}\frac{\partial \xi_{is}}{\partial x_j}\xi_{ir} p_r p_k$$
$$= \frac{d}{dt}\frac{\partial T}{\partial p_s} - b_{sr}^{(k)} p_r p_k = \frac{d}{dt}\frac{\partial T}{\partial p_s} - b_{sk}^{(r)} p_k p_r.$$

Substituting this expression into (4.22), we obtain equations identical with Volterra's equations (4.12).

§5. Equations of motion in quasi-coordinates

The main advantage derived by the introduction of the concept of quasi-coordinates into mechanics and the generalization of Lagrange's equations to quasi-coordinates is the unification of the form of the ordinary Lagrange equations, the equations of motion of nonholonomic systems, and also equations such as Euler's dynamical equations of motion of a rigid body with a fixed point.[5] In order to emphasize the importance of this generalization from the practical as well as formal point of view, we may mention that a felicitous choice of the unknown parameters (the generalized coordinates and quasi-coordinates determining the motion) plays a very important role in the investigation

[5] The so-called method of moving axes (coordinates) is a special case of the choice of quasi-coordinates. The equations derived in this method may be regarded as Lagrange equations in quasi-coordinates.

of the motion of specific mechanical systems. It is, for example, well known that Euler's problem of the inertial motion of a rigid body with a fixed point was formulated and studied by means of quasi-coordinates. Chaplygin used quasi-coordinates to study the problem of nonholonomic motion on a plane and the difficult problem of the rolling of an inhomogeneous sphere on a plane. Quasi-coordinates, as certain kinematic characteristics of the motion determining the velocities of the points of a system, have been employed for a long time in mechanics. However, it was only at the beginning of the twentieth century that generalized coordinates and the kinematic parameters were brought together in the single general concept of quasi-coordinates. In his paper [186], Hamel summarized this field of mechanics, deriving equations of motion in quasi-coordinates in a form similar to Lagrange's equation, but applicable to both holonomic and nonholonomic systems.[6] Although quasi-coordinates as such do not occur explicitly in the Lagrange function or in the constraint equations, the operation of differentiation with respect to a quasi-coordinate is fully defined; namely, let $q_1, q_2, \cdots, q_n$ be generalized (true) coordinates of a system and $\pi_1, \pi_2, \cdots, \pi_n$ be the quasi-coordinates, whose derivatives are connected to the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ by the following linear equations

$$\dot{\pi}_s = a_{sk}\dot{q}_k, \qquad \dot{q}_k = b_{kl}\dot{\pi}_l \qquad (s, k, l = 1, 2, \ldots, n), \tag{5.1}$$

where a_{sk} and b_{kl} are functions of only the generalized coordinates. Obviously,

$$a_{sk}b_{kl} = \delta_{sl}, \tag{5.2}$$

where δ_{sl} is the Kronecker delta, i.e., equal to unity when $s = l$ and zero if $s \neq l$. It then follows from the second relation of (5.1) that

$$\frac{\partial \dot{q}_k}{\partial \dot{\pi}_l} = \frac{\partial q_k}{\partial \pi_l} = b_{kl}, \tag{5.3}$$

and for any function $f(q_1, q_2, \cdots, q_n, t)$ the expression $\partial f/\partial \pi_l$ means, in accordance with (5.3), the operation

$$\frac{\partial f}{\partial \pi_l} = \frac{\partial f}{\partial q_k}\frac{\partial q_k}{\partial \pi_l} = \frac{\partial f}{\partial q_k}b_{kl}. \tag{5.4}$$

We now turn to the generalization of Lagrange's equations for holonomic

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[6] Hamel gave the name Lagrange-Euler equations (Lagrange-Eulersche Gleichungen) to these equations.

systems to quasi-coordinates. It follows from the equations (5.1) that

$$\delta q_s = b_{sk}\delta\pi_k,$$

and, therefore, expressing the variations of the generalized coordinates in terms of the variations of the quasi-coordinates in the d'Alembert-Lagrange equations (1.12), we arrive at the relation

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} - Q_s\right) b_{sk}\delta\pi_k = 0,$$

from which, by virtue of the independence of the variations $\delta\pi_k$ of the quasi-coordinates, we obtain n equations of the form

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} - Q_s\right) b_{sk} = 0. \tag{5.5}$$

We shall denote by T^* the function obtained from T by replacing the generalized velocities $\dot{q}_s$ by $\dot{\pi}_k$ by means of (5.1). These functions are obviously related:

$$T(q_1, q_2, \ldots, q_n, b_{1k}\dot{\pi}_k, \ldots, b_{nk}\dot{\pi}_k) = T^*(q_1, \ldots, q_n, \dot{\pi}_1, \ldots, \dot{\pi}_n),$$
$$T^*(q_1, q_2, \ldots, q_n, a_{1s}\dot{q}_s, \ldots, a_{ns}\dot{q}_s) = T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n).$$

Using these equations, we perform a number of transformations, taking into account (5.1) – (5.4):

$$b_{sk}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} = \frac{d}{dt}\left(b_{sk}\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial \dot{q}_s}\frac{\partial b_{sk}}{\partial q_l}\dot{q}_l = \frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} + \frac{\partial T^*}{\partial \dot{\pi}_i}\frac{\partial a_{is}}{\partial q_l}b_{sk}b_{lj}\dot{\pi}_j,$$
$$b_{sk}\frac{\partial T}{\partial q_s} = b_{sk}\frac{\partial T^*}{\partial q_s} + \frac{\partial T^*}{\partial \dot{\pi}_i}\dot{q}_l\frac{\partial a_{il}}{\partial q_s}b_{sk} = \frac{\partial T^*}{\partial \pi_k} + \frac{\partial T^*}{\partial \dot{\pi}_i}\frac{\partial a_{il}}{\partial q_s}b_{sk}b_{lj}\dot{\pi}_j.$$

Substituting these expressions into (5.5), we obtain

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} - \frac{\partial T^*}{\partial \pi_k} + b_{sk}b_{lj}\left(\frac{\partial a_{is}}{\partial q_l} - \frac{\partial a_{il}}{\partial q_s}\right)\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j = Q_s b_{sk}.$$

The quantity $\Pi_k = Q_s b_{sk}$ is the generalized force corresponding to the virtual displacement $\delta\pi_k$. We introduce the notation

$$\gamma_{kij} = b_{sk}b_{lj}\left(\frac{\partial a_{is}}{\partial q_l} - \frac{\partial a_{il}}{\partial q_s}\right). \tag{5.6}$$

This expression depends only on the relations between the true coordinates and the derivatives of the quasi-coordinates and is quite independent of the structure and motion of the system. Finally, we obtain

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} - \frac{\partial T^*}{\partial \pi_k} + \gamma_{kij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j = \Pi_k \quad (k = 1, 2, \ldots, n). \tag{5.7}$$

These n equations are the equations of motion in quasi-coordinates. The quantities

$$p_k = \frac{\partial T^*}{\partial \dot{\pi}_k}$$

are the generalized momenta corresponding to the quasi-coordinates.[7]

If $\pi_1, \pi_2, \cdots, \pi_n$ are true coordinates, then the relations (5.1) are integrable, by virtue of which all the coefficients γ_{kij} vanish and equations (5.7) take the form of the ordinary Lagrange equations of the second kind.

We shall now show that Euler's dynamical equations of motion of a rigid body with a fixed point can be derived from equations (5.7). As true coordinates, we take the Eulerian angles θ, φ, and ψ, which determine the position of the system of axes $O\xi\eta\zeta$ attached to the body and moving with it with respect to the system of axes $Oxyz$, which are fixed in space. An arbitrary displacement $(\delta\theta, \delta\varphi, \delta\psi)$ of the body is equivalent to three infinitely small rotations $\delta\pi_1, \delta\pi_2, \delta\pi_3$ about the axes $O\xi$, $O\eta$, $O\zeta$. Defining $\dot{\pi}_1 = p$, $\dot{\pi}_2 = q$, $\dot{\pi}_3 = r$ and taking the axes $O\xi$, $O\eta$, and $O\zeta$ along the principal axes of the ellipsoid of inertia, we obtain

$$T^* = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2), \tag{5.8}$$

where A, B, and C, the corresponding principal moments of inertia, are constants. The quantities p, q, and r are related to the angles θ, φ, and ψ by the well-known Eulerian kinematic formulas

$$\begin{aligned} p &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \\ q &= \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \\ r &= \dot{\psi}\cos\theta + \dot{\varphi}. \end{aligned} \tag{5.9}$$

Solving these equations for $\dot{\theta}$, $\dot{\psi}$ and $\dot{\varphi}$, we obtain

$$\begin{aligned} \dot{\theta} &= p\cos\varphi - q\sin\varphi, \\ \dot{\psi} &= p\frac{\sin\varphi}{\sin\theta} + q\frac{\cos\varphi}{\sin\theta}, \\ \dot{\varphi} &= -p\sin\varphi\cot\theta - q\cos\varphi\cot\theta + r. \end{aligned} \tag{5.10}$$

If we now take $q_1 = \theta$, $q_2 = \psi$, and $q_3 = \varphi$, these equations take the form of the relations (5.1), in which

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[7] We mention that if a virtual displacement $\delta\pi_1 = 0, \cdots, \delta\pi_k \neq 0, \delta\pi_{k+1} = 0, \cdots, \delta\pi_n = 0$ corresponds to a rotation about an instantaneous axis, the generalized force Π_k and the generalized momentum p_k are, as usual, the moment of the force and the angular momentum about this axis, respectively.

$$a_{11} = \cos\varphi, \qquad a_{12} = \sin\theta\sin\varphi, \qquad a_{13} = 0,$$
$$a_{21} = -\sin\varphi, \qquad a_{22} = \sin\theta\cos\varphi, \qquad a_{23} = 0,$$
$$a_{31} = 0, \qquad a_{32} = \cos\theta, \qquad a_{33} = 1$$

and, respectively,

$$b_{11} = \cos\varphi, \qquad b_{12} = -\sin\varphi, \qquad b_{13} = 0,$$
$$b_{21} = \frac{\sin\varphi}{\sin\theta}, \qquad b_{22} = \frac{\cos\varphi}{\sin\theta}, \qquad b_{23} = 0,$$
$$b_{31} = -\sin\varphi\cot\theta, \qquad b_{32} = -\cos\varphi\cot\theta, \qquad b_{33} = 1.$$

Taking these values, we calculate γ_{kij} by formula (5.6):

$$\gamma_{1ij} = b_{1l}\left[b_{11}\left(\frac{\partial a_{i1}}{\partial q_l} - \frac{\partial a_{jl}}{\partial\theta}\right) + b_{21}\frac{\partial a_{i2}}{\partial q_l} - b_{31}\frac{\partial a_{jl}}{\partial\varphi}\right]$$
$$= b_{1j}\frac{\sin\varphi}{\sin\theta}\left(\frac{\partial a_{i2}}{\partial\theta} + \cos\theta\cdot\frac{\partial a_{i1}}{\partial\varphi}\right) - b_{2j}\left[b_{11}\frac{\partial a_{i2}}{\partial\theta} + b_{31}\frac{\partial a_{i2}}{\partial\varphi}\right]$$
$$+ b_{3j}\left[b_{11}\frac{\partial a_{i1}}{\partial\varphi} + b_{21}\frac{\partial a_{i2}}{\partial\varphi}\right].$$

Hence, $\gamma_{123} = -1$, $\gamma_{132} = 1$ and $\gamma_{1ij} \equiv 0$ for the remaining values of i and j. Making similar calculations for $k = 2$ and $k = 3$, we obtain the following values: $\gamma_{213} = -\gamma_{231} = 1$; $\gamma_{321} = -\gamma_{312} = 1$, and $\gamma_{2ij} \equiv 0$ and $\gamma_{3ij} \equiv 0$ for the remaining values of i and j. In addition, since the function T^* defined by the expression (5.8) does not depend on the coordinates, we have $\partial T^*/\partial\pi_k \equiv 0$. Now, writing down equations (5.7), we find

$$A\frac{dp}{dt} + (C - B)qr = \Pi_1,$$

$$B\frac{dq}{dt} + (A - C)rp = \Pi_2,$$

$$C\frac{dr}{dt} + (B - A)pq = \Pi_3,$$

which are Euler's dynamical equations of motion of a rigid body with a fixed point.

In the case of a nonholonomic system with constraints of the form

$$\sum_{j=1}^{n} a_{m+i,j}\dot{q}_j = 0 \qquad (i = 1, 2, \ldots, n - m), \tag{5.11}$$

it is, in a number of cases, convenient to introduce quasi-coordinates such that the last $n - m$ quasi-coordinates vanish by virtue of the constraint equations (5.11). To this end, one can assume, in particular,

that the last quasi-coordinates are related to the true coordinates by the equations

$$\dot{\pi}_{m+i} = a_{m+i,j}\dot{q}_j \quad (i = 1, 2, \ldots, n-m;\ j = 1, 2, \ldots, n), \tag{5.12}$$

and the first m quasi-coordinates by arbitrary linear equations

$$\dot{\pi}_k = a_{kj}\dot{q}_j \quad (k = 1, 2, \ldots, m), \tag{5.13}$$

but such that the matrix connecting the quasi-coordinates and the true coordinates has a nonvanishing determinant (in particular, the first m quasi-coordinates may also be true coordinates). In accordance with (5.12), the variations $\delta\pi_i$ and δq_i are related as follows:

$$\delta\pi_{m+i} = a_{m+i,j}\,\delta q_j \quad (i = 1, 2, \ldots, n-m;\ j = 1, 2, \ldots, n) \tag{5.12'}$$

and, by virtue of the equations of the nonholonomic constraints (5.11), these variations vanish:

$$\delta\pi_{m+i} = 0 \quad (i = 1, 2, \ldots, n-m). \tag{5.14}$$

Therefore, in the d'Alembert-Lagrange equation

$$\left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} - \frac{\partial T^*}{\partial \pi_k} + \gamma_{kij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j - \Pi_k\right)\delta\pi_k = 0 \quad (k = 1, 2, \ldots, n),$$

transformed to quasi-coordinates before allowance is made for the equations of the nonholonomic constraints, the last $n - m$ terms vanish in the case of a nonholonomic system by virtue of the conditions (5.14), and the remaining sum splits up into m equations, since the first m variations $\delta\pi_k$ are independent:

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} - \frac{\partial T^*}{\partial \pi_k} + \gamma_{kij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j = \Pi_k \ (i = 1, 2, \ldots, n;\ \ k, j = 1, 2, \ldots, m). \tag{5.15}$$

Here, γ_{kij} are given by

$$\gamma_{kij} = b_{sk}b_{lj}\left(\frac{\partial a_{is}}{\partial q_l} - \frac{\partial a_{il}}{\partial q_s}\right)$$
$$(i = 1, 2, \ldots, n;\ \ l, s = 1, 2, \ldots, n; k = 1, 2, \ldots, m). \tag{5.16}$$

The matrix $\|b_{sr}\|$ is the inverse of the matrix $\|a_{rl}\|$ and the subscripts k and j, in contrast to (5.6), take the values from 1 to m. These m equations (5.15), which do not contain the unknown reaction forces of the constraints, together with the $n - m$ constraint equations (5.11) and the m relations (5.13), are the $n + m$ equations of motion for the non-

holonomic system in quasi-coordinates. For given initial conditions, these equations determine the $n + m$ unknown functions of the time $\pi_1, \pi_2, \cdots, \pi_m, q_1, q_2, \cdots, q_n$. It should be noted that the kinetic energy T^* in equations (5.15) is, in general, a function of all n quasi-velocities π_k and the relations (5.11) need only be used after equations (5.15) have been written down (in accordance with the motivation of the derivation of these equations). The coefficients γ_{kij} occurring in the equations of motion of nonholonomic systems are functions of the true coordinates $q_1, q_2, \cdots, q_n$. By virtue of the definition (5.6), these coefficients satisfy the relations

$$\gamma_{kij} = -\gamma_{jik}, \tag{5.17}$$

i.e., for any value of the central subscript i, they form a skew-symmetric matrix with respect to the outer subscripts k and j. Another property of the coefficients γ_{kij} is intimately related to the so-called transpositional relations, which play a very important role in the analytical mechanics of nonholonomic systems. The transpositional relations are used not only to calculate the coefficients γ_{kij}, but, in a number of cases, the derivation of the equations of motion of nonholonomic systems are based on them. What then are transpositional relations? Suppose that the differentials of the quasi-coordinates and the true coordinates are related by (nonintegrable) equations of the general form

$$d\pi_s = a_{sk}dq_k + a_s dt \quad (s, k = 1, 2, \cdots, n), \tag{5.18}$$

where a_{sk} and a_s are functions of the true coordinates and the time. For the sake of uniformity of the notation, we shall denote the time t by q_{n+1} in what follows. Obviously, $\dot{q}_{n+1} = 1$ and $\delta q_{n+1} = 0$. Then, replacing $n + 1$ by n, we obtain, instead of (5.18), a system of homogeneous equations of the form

$$d\pi_s = a_{sk}dq_k \quad (s, k = 1, 2, \cdots, n), \tag{5.19}$$

and for the virtual variations

$$\delta\pi_s = a_{sk}\delta q_k \quad (s, k = 1, 2, \cdots, n). \tag{5.20}$$

Solving this system for dq_k, we obtain

$$dq_s = b_{sk}d\pi_k \quad (s, k = 1, 2, \cdots, n) \tag{5.21}$$

and, respectively,

$$\delta q_s = b_{sk}\delta\pi_k \quad (s, k = 1, 2, \cdots, n). \tag{5.22}$$

We now apply the operation d to the equation (5.20):[8]

$$d\,\delta\pi_s = a_{sk} d\,\delta q_k + \frac{\partial a_{sk}}{\partial q_l} dq_l\,\delta q_k.$$

In this expression, we replace dq_l and δq_k by the expressions (5.21) and (5.22), respectively, obtaining

$$d\,\delta\pi_s = a_{sk} d\,\delta q_k + \frac{\partial a_{sk}}{\partial q_l} b_{li} b_{kj}\, d\pi_i \delta\pi_j. \tag{5.23}$$

We now apply the operation δ to equation (5.19) in the same way, obtaining

$$\delta\, d\pi_s = a_{sk}\delta\, dq_k + \frac{\partial a_{sk}}{\partial q_l} b_{li} b_{kj}\, \delta\pi_i\, d\pi_j$$

or, exchanging the positions of the dummy indices l and k and i and j in the second sum

$$\delta\, d\pi_s = a_{sk}\delta\, dq_k + \frac{\partial a_{sl}}{\partial q_k} b_{kj} b_{li}\, d\pi_i \delta\pi_j.$$

Subtracting this equation from (5.23), we obtain

$$d\,\delta\pi_s - \delta\, d\pi_s = a_{sk}(d\,\delta q_k - \delta\, dq_k) + b_{li} b_{kj}\left(\frac{\partial a_{sk}}{\partial q_l} - \frac{\partial a_{sl}}{\partial q_k}\right) d\pi_i \delta\pi_j$$

or, in accordance with the notation (5.6),

$$d\,\delta\pi_s - \delta\, d\pi_s = a_{sk}(d\,\delta q_k - \delta\, dq_k) + \gamma_{jsi}\, d\pi_i \delta\pi_j. \tag{5.24}$$

These relations are called transpositional relations.

Assuming here

$$d\delta q_k - \delta dq_k = 0, \tag{5.25}$$

which, as we shall show below, is always possible, we arrive at the following expression for the transpositional relations:

$$d\delta\pi_s - \delta d\pi_s = \gamma_{jsi} d\pi_i \delta\pi_j, \tag{5.26}$$

which are generally used to calculate the values of the coefficients γ_{jsi}. In order to obtain the second property of the coefficients γ_{kij}, we replace the symbol of variation δ in (5.26) by the differential d. In this case,

[8] Without, as yet, going into the meaning of the operations $d\delta$ and δd, we shall use these symbols formally, desiring only to establish their connection with the ordinary differentials and variations of the quantities occurring in the given relations. The question of the meaning of these operations and the transpositional relations associated with them will be considered separately in the following section.

the left-hand side of (5.26) vanishes, and, consequently,

$$\gamma_{jsi}d\pi_i d\pi_j = 0. \tag{5.27}$$

These properties of the coefficients γ_{kij} permit one to establish the gyroscopic nature of the nonholonomy terms in the equations of a nonholonomic system. For the additional terms $\gamma_{kij}(\partial T^*/\partial\dot{\pi}_i)\dot{\pi}_j$, contained in equations (5.7), which may be called nonholonomy terms, can be regarded as generalized forces. Like gyroscopic forces, they possess the following two properties: 1) By virtue of the property (5.17) of the coefficients γ_{kij}, the terms $\gamma_{kij}(\partial T^*/\partial\dot{\pi}_i)\dot{\pi}_j$ form a skew-symmetric matrix with respect to the subscripts k and j; 2) the work of these forces for any real displacement vanishes.

For let us write down an expression for the infinitesimal work of these forces for an arbitrary real displacement

$$dA = \gamma_{kij}\frac{\partial T^*}{\partial\dot{\pi}_i}\dot{\pi}_j d\pi_k = \frac{\partial T^*}{\partial\dot{\pi}_i}(\gamma_{kij}\dot{\pi}_k\dot{\pi}_j)dt. \tag{5.28}$$

Now, by virtue of (5.27) the expression in the brackets vanishes, and, hence, $dA = 0$.

We shall now give some examples of the use of the transpositional relations in the calculation of the coefficients γ_{kij}. We note that by writing down the so-called bilinear form $d\delta\pi_i = \delta d\pi_i$ we can immediately find several of the γ_{kij}. This is the first advantage of this method of finding the γ_{kij} compared with the direct calculations by means of (5.6). The second simplification in the calculations, as we shall show in the examples, arises because, in order to obtain γ_{kij}, it is in practice sufficient to write down only a single form, for example, $d\delta\pi_i$, in an expression in terms of $d\pi_k\delta\pi_j$.

EXAMPLE 1. *Nonholonomic motion on a plane.* If there is a knife edge, we have the constraint $dy = \tan\varphi\, dx$, where x, y and φ are true coordinates. We introduce the quasi-coordinates φ_1, φ_2, φ_3 by means of the relations

$$\begin{aligned} d\pi_1 &= \cos\varphi\, dx + \sin\varphi\, dy = v\, dt, & \delta\pi_1 &= \cos\varphi\,\delta x + \sin\varphi\,\delta y, \\ d\pi_2 &= d\varphi, & \delta\pi_2 &= \delta\varphi, \\ d\pi_3 &= \cos\varphi\, dy - \sin\varphi\, dx, & \delta\pi_3 &= \cos\varphi\,\delta y - \sin\varphi\,\delta x. \end{aligned}$$

Hence,

$$\begin{aligned} dx &= \cos\varphi\ d\pi_1 - \sin\varphi\, d\pi_3, & \delta x &= \cos\varphi\,\delta\pi_1 - \sin\varphi\,\delta\pi_3, \\ dy &= \sin\varphi\ d\pi_1 + \cos\varphi\ d\pi_3, & \delta y &= \sin\varphi\,\delta\pi_1 + \cos\varphi\ \delta\pi_3, \\ d\varphi &= d\pi_2, & \delta\varphi &= \delta\pi_2. \end{aligned}$$

Writing down the expression for the bilinear form, we obtain

$$\begin{aligned} d\,\delta\pi_1 &= \cos\varphi\, d\,\delta x + \sin\varphi\, d\,\delta y + d\varphi\,(-\sin\varphi\,\delta x + \cos\varphi\,\delta y) \\ &= \cos\varphi\, d\,\delta x + \sin\varphi\, d\,\delta y + d\pi_2\,\delta\pi_3, \\ \delta\, d\pi_1 &= \cos\varphi\,\delta\, dx + \sin\varphi\,\delta\, dy + \delta\pi_2\, d\pi_3. \end{aligned}$$

Hence,

$$d\,\delta\pi_1 - \delta\, d\pi_1 = (d\,\delta x - \delta\, dx)\cos\varphi + (d\,\delta y - \delta\, dy)\sin\varphi + d\pi_2\,\delta\pi_3 - d\pi_3\delta\pi_2$$

and, consequently, $\gamma_{213} = -\gamma_{312} = -1$ and $\gamma_{j1i} = 0$ for the remaining values of j and i. For the remaining quasi-coordinates, the calculations are similar:

$$\begin{aligned} d\,\delta\pi_2 - \delta\, d\pi_2 &= d\,\delta\varphi - \delta\, d\varphi, \\ d\,\delta\pi_3 - \delta\, d\pi_3 &= (d\,\delta y - \delta\, dy)\cos\varphi - (d\,\delta x - \delta\, dx)\sin\varphi \\ &\quad - d\pi_2\,\delta\pi_1 + \delta\pi_2\, d\pi_1. \end{aligned}$$

Hence, $\gamma_{231} = -\gamma_{132} = 1$ and $\gamma_{j3i} = 0$ for the remaining values of j and i and, in addition, $\gamma_{j2i} = 0$ for all j and i, since $\pi_3 = \varphi$ is a true coordinate.

EXAMPLE 2. *Disc on a rough plane.* We have five coordinates x, y, θ, φ, and ψ and two nonholonomic constraints

$$dx = a\cos\theta\, d\varphi, \quad dy = a\sin\theta\, d\varphi.$$

We introduce the quasi-coordinates $\pi_1, \pi_2, \cdots, \pi_5$ by the following relations

$$\begin{gathered} d\pi_1 = \cos\theta\, dx + \sin\theta\, dy, \quad d\pi_2 = d\theta, \quad d\pi_3 = d\psi, \\ d\pi_4 = dx - a\cos\theta\, d\varphi, \\ d\pi_5 = dy - a\sin\theta\, d\varphi. \end{gathered}$$

Solving these equations for the differentials of the true coordinates, we obtain

$$\begin{aligned} dx &= \cos\theta\, d\pi_1 + \sin^2\theta\, d\pi_4 - \sin\theta\cos\theta\, d\pi_5, \\ dy &= \sin\theta\, d\pi_1 - \sin\theta\cos\theta\, d\pi_4 + \cos^2\theta\, d\pi_5, \\ d\theta &= d\pi_2, \quad d\psi = d\pi_3, \quad a\, d\varphi = d\pi_1 - \cos\theta\, d\pi_4 - \sin\theta\, d\pi_5. \end{aligned}$$

We now write down expressions for $d\delta\pi_s$ $(s = 1, 2, \cdots, 5)$, omitting, in accordance with (5.27), the terms containing $d\delta q_k$ $(k = 1, 2, 3, 4, 5)$:

$$\begin{aligned} d\,\delta\pi_1 &= d\theta\,(-\sin\theta\,\delta x + \cos\theta\,\delta y) = d\pi_2\,(-\sin\theta\,\delta\pi_4 + \cos\theta\,\delta\pi_5), \\ d\,\delta\pi_4 &= d\pi_2\,(\delta\pi_1 + \cos\theta\,\delta\pi_4 - \sin\theta\,\delta\pi_5)\sin\theta, \\ d\,\delta\pi_5 &= -d\pi_2\,(\delta\pi_1 + \cos\theta\,\delta\pi_4 - \sin\theta\,\delta\pi_5)\cos\theta. \end{aligned}$$

There would be no point in writing down a bilinear form for π_2 and π_3, since they are true coordinates. From these expressions, using (5.26), we obtain the following values of the coefficients γ_{kij}:

$$\gamma_{214} = -\gamma_{412} = \sin\theta, \qquad \gamma_{215} = -\gamma_{512} = -\cos\theta,$$
$$\gamma_{241} = -\gamma_{142} = -\sin\theta,$$
$$\gamma_{244} = -\gamma_{442} = -\sin\theta\cos\theta, \qquad \gamma_{245} = -\gamma_{542} = \sin^2\theta,$$
$$\gamma_{251} = -\gamma_{152} = \cos\theta,$$
$$\gamma_{254} = -\gamma_{452} = \cos^2\theta, \qquad \gamma_{255} = -\gamma_{552} = -\sin\theta\cos\theta.$$

The remaining γ_{kij} vanish.

We shall now show how the transpositional relations can be used in the derivation of equations of motion in quasi-coordinates. Applying the identity transformation

$$m_\nu\ddot{x}_\nu\delta x_\nu = \frac{d}{dt}(m_\nu\dot{x}_\nu\delta x_\nu) - m_\nu\dot{x}_\nu\delta\dot{x}_\nu + m_\nu\dot{x}_\nu\left(\delta\dot{x}_\nu - \frac{d}{dt}\delta x_\nu\right), \tag{5.29}$$

we transform the d'Alembert-Lagrange equation (1.10) to the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_\nu}\delta x_\nu\right) - \delta T + \frac{\partial T}{\partial\dot{x}_\nu}\left(\delta\dot{x}_\nu - \frac{d}{dt}\delta x_\nu\right) = X_\nu\delta x_\nu$$

or, going over to generalized coordinates $q_1, q_2, \cdots, q_n$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\delta q_k\right) - \delta T + \frac{\partial T}{\partial\dot{q}_k}\left(\delta\dot{q}_k - \frac{d}{dt}\delta q_k\right) = Q_k\delta q_k. \tag{5.30}$$

By virtue of (5.25), one can assume $\delta dq_k = d\delta q_k = 0$ in this equation. One then has

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\delta q_k\right) - \delta T = Q_k\delta q_k. \tag{5.31}$$

Introducing quasi-coordinates by means of the relations (5.19) – (5.22) and the function $T^* = T^*(q, \dot{\pi})$, which is obtained from $T = T(q, \dot{q})$ by replacing all the $\dot{q}_k$ by the $\dot{\pi}_s$, we transform equation (5.31) to the following form:

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial\dot{\pi}_s}\delta\pi_s\right) - \frac{\partial T^*}{\partial\dot{\pi}_s}\delta\dot{\pi}_s - \frac{\partial T^*}{\partial q_k}\delta q_k = \Pi_s\delta\pi_s$$

or, after performing the above operations,

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial\dot{\pi}_s}\right)\delta\pi_s + \frac{\partial T^*}{\partial\dot{\pi}_s}\left(\frac{d}{dt}\delta\pi_s - \delta\dot{\pi}_s\right) - \frac{\partial T^*}{\partial q_k}b_{ks}\delta\pi_s = \Pi_s\delta\pi_s.$$

We now use the transpositional relations (5.26):

$$\frac{d}{dt}\delta\pi_s - \delta\dot{\pi}_s = \gamma_{jsi}\dot{\pi}_i\delta\pi_j.$$

Changing the letters used for the dummy subscripts in the second sum, we obtain

$$\left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_s} + \gamma_{sij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j - \frac{\partial T^*}{\partial \pi_s}\right)\delta\pi_s = \Pi_s\delta\pi_s$$

and, by virtue of the independence of the variations $\delta\pi_s$, we arrive at the equations of motion in quasi-coordinates

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_s} + \gamma_{sij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j - \frac{\partial T^*}{\partial \pi_s} = \Pi_s \quad (s = 1, 2, \ldots, n).$$

We now consider an example of the derivation of the equations of motion in quasi-coordinates when there are nonholonomic constraints expressed by linear inhomogeneous equations in the generalized velocities. This example will illustrate the gyroscopic nature of the terms connected with the nonholonomy of the system.

EXAMPLE 3. *Homogeneous sphere on a horizontal plane rotating with a constant angular velocity* Ω. Let the Oz axis of the fixed system of coordinates $Oxyz$ coincide with the axis of rotation. Then, as the sphere moves on the plane, it has the kinetic energy

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k^2(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta), \tag{5.32}$$

where k is the radius of gyration; the mass of the sphere is $m = 1$; x and y are the coordinates of the center of the sphere; and θ, ψ, and φ are the Eulerian angles. Denoting by ω the instantaneous angular velocity of the sphere and by a its radius, we write down the condition of rolling without sliding of the sphere on the rotating plane:

$$\dot{x} - a\omega_y + \Omega y = 0, \quad \dot{y} + a\omega_x - \Omega x = 0 \tag{5.33}$$

or, using Euler's kinematic equations (2.9),

$$\begin{aligned} &\dot{x} - a\dot{\theta}\sin\psi + a\dot{\varphi}\sin\theta\cos\psi + \Omega y = 0, \\ &\dot{y} + a\dot{\theta}\cos\psi + a\dot{\varphi}\sin\theta\sin\psi - \Omega x = 0. \end{aligned} \tag{5.34}$$

Since we have three degrees of freedom and the coordinates x, y, θ, and ψ all occur in the constraint equations (5.34) and in the expression for the kinetic energy (5.32), we have an example of a system which is not a Čaplygin system.

We now derive the equations of motion in quasi-coordinates. For the sake of uniformity of the notation, we introduce a coordinate $q_6 = t$, together with the coordinates $q_1 = \theta$, $q_2 = \varphi$, $q_3 = \psi$, $q_4 = x$, and $q_5 = y$ and the quasi-coordinates π_s, whose differentials are related to the differentials of the true coordinates by the following equations:

$$
\begin{aligned}
d\pi_1 &\equiv \omega_x\, dt = \cos\psi\, d\theta + \sin\theta \sin\psi\, d\varphi,\\
d\pi_2 &\equiv \omega_y\, dt = \sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi,\\
d\pi_3 &\equiv \omega_z\, dt = d\psi + \cos\theta\, d\varphi,\\
d\pi_4 &= -dx + a\sin\psi\, d\theta - a\sin\theta\cos\psi\, d\varphi - \Omega y\, dt,\\
d\pi_5 &= dy + a\cos\psi\, d\theta + a\sin\theta\sin\psi\, d\varphi - \Omega x\, dt,\\
d\pi_6 &= dt.
\end{aligned}
\tag{5.35}
$$

Accordingly, the variations are related by the equations

$$
\begin{aligned}
\delta\pi_1 &= \cos\psi\, \delta\theta + \sin\theta\sin\psi\, \delta\varphi,\\
\delta\pi_2 &= \sin\psi\, \delta\theta - \sin\theta\cos\psi\, \delta\varphi,\\
\delta\pi_3 &= \delta\psi + \cos\theta\, \delta\varphi,\\
\delta\pi_4 &= -\delta x + a\sin\psi\, \delta\theta - a\sin\theta\cos\psi\, \delta\varphi,\\
\delta\pi_5 &= \delta y + a\cos\psi\, \delta\theta + a\sin\theta\sin\psi\, \delta\varphi,\\
\delta\pi_6 &= 0.
\end{aligned}
\tag{5.36}
$$

Solving these equations for the differentials of the true coordinates, we find

$$
\begin{aligned}
d\theta &= \cos\psi\, d\pi_1 + \sin\psi\, d\pi_2,\\
d\varphi &= \frac{\sin\psi}{\sin\theta}\, d\pi_1 - \frac{\cos\psi}{\sin\theta}\, d\pi_2,\\
d\psi &= -\sin\psi\cot\theta\, d\pi_1 + \cos\psi\cot\theta\, d\pi_2 + d\pi_3,\\
dx &= a\, d\pi_2 - d\pi_4 - \Omega y\, d\pi_6,\\
dy &= -a\, d\pi_1 + d\pi_5 + \Omega x\, d\pi_6, \qquad dt = d\pi_6
\end{aligned}
\tag{5.37}
$$

and, respectively, for the variations

$$
\begin{aligned}
\delta\theta &= \cos\psi\, \delta\pi_1 + \sin\psi\, \delta\pi_2,\\
\delta\rho &= \frac{\sin\psi}{\sin\theta}\, \delta\pi_1 - \frac{\cos\psi}{\sin\theta}\, \delta\pi_2,\\
\delta\psi &= -\sin\psi\cot\theta\, \delta\pi_1 + \cos\psi\cot\theta\, \delta\pi_2 + \delta\pi_3,\\
\delta x &= a\, \delta\pi_2 - \delta\pi_4, \quad \delta y = -a\, \delta\pi_1 + \delta\pi_5, \quad \delta = 0.
\end{aligned}
\tag{5.38}
$$

In order to determine the values of the coefficients γ_{kij} we find expressions for the bilinear forms of the quasi-coordinates

$$
\begin{aligned}
d\,\delta\pi_1 - \delta\, d\pi_1 &= d\pi_2\delta\pi_3 - \delta\pi_2 d\pi_3,\\
d\,\delta\pi_2 - \delta\, d\pi_2 &= \delta\pi_1 d\pi_3 - d\pi_1\delta\pi_3,\\
d\,\delta\pi_3 - \delta\, d\pi_3 &= d\pi_1\delta\pi_2 - \delta\pi_1 d\pi_2,\\
d\,\delta\pi_4 - \delta\, d\pi_4 &= a\,(\delta\pi_1 d\pi_3 - d\pi_1\delta\pi_3) - a\Omega\,\delta\pi_1 d\pi_6 + \Omega\,\delta\pi_5 d\pi_6,\\
d\,\delta\pi_5 - \delta\, d\pi_5 &= a\,(d\pi_2\delta\pi_3 - \delta\pi_2 d\pi_3) + a\Omega\,\delta\pi_2 d\pi_6 - \Omega\,\delta\pi_4 d\pi_6,\\
d\,\delta\pi_6 - \delta\, d\pi_6 &= 0.
\end{aligned}
\tag{5.39}
$$

For, applying the operation d to the first equation of (5.36), we obtain

$$d\,\delta\pi_1 = \cos\psi\, d\,\delta\theta + \sin\theta\sin\psi\, d\,\delta\varphi - d\psi\,(\sin\psi\,\delta\theta - \sin\theta\cos\psi\,\delta\varphi) + \cos\theta\sin\psi\, d\theta\,\delta\varphi.$$

Replacing $d\theta$ and $d\varphi$ in this equation by their expressions (5.38), and $d\theta$ and $d\psi$ by their expressions (5.37), we obtain

$$\begin{aligned} d\,\delta\pi_1 &= \delta\pi_2\,(\sin\psi\cot\theta\, d\pi_1 - d\pi_3) \\ &\sin\psi\cot\theta\,(\cos\psi\, d\pi_1 + \sin\psi\, d\pi_2)(\sin\psi\,\delta\pi_1 - \cos\psi\,\delta\pi_2) \\ &= -\delta\pi_2 d\pi_3 + (d\pi_1\,\delta\pi_2 + \delta\pi_1\, d\pi_2)\sin^3\psi\cot\theta, \end{aligned} \tag{5.40}$$

where the terms containing $d\delta\theta$, $d\delta\varphi$, $d\pi_1\delta\pi_1$, and $d\pi_2\delta\pi_2$ have been omitted because the corresponding terms obtained by writing down the difference $d\delta\varphi_1 - \delta d\pi_1$ vanish by virtue of the condition (5.25). In order to obtain the expression for $\delta d\pi_1$, it is sufficient to reverse the order of the letters d and δ in (5.40):

$$\delta\, d\pi_1 = -d\pi_2\delta\pi_3 + (\delta\pi_1 d\pi_2 + d\pi_1\delta\pi_2)\sin^3\psi\cot\theta.$$

Subtracting this expression from (5.40), we find

$$d\,\delta\pi_1 - \delta\, d\pi_1 = d\pi_2\delta\pi_3 - \delta\pi_2 d\pi_3.$$

Similarly, we can calculate the remaining bilinear forms (5.39). We must, however, bear in mind that $\delta\pi_6 = 0$, but $d\pi_6 \neq 0$, since $\dot\pi_6 = 1$.

In accordance with the expressions (5.39), the nonholonomy coefficients have the following values:

$$\begin{aligned} &\gamma_{312} = -\gamma_{213} = 1, \quad \gamma_{123} = -\gamma_{321} = 1, \quad \gamma_{231} = -\gamma_{132} = 1, \\ &\gamma_{143} = -\gamma_{341} = a, \quad \gamma_{146} = -a\Omega, \quad \gamma_{546} = \Omega, \\ &\gamma_{352} = -\gamma_{253} = a, \quad \gamma_{256} = a\Omega, \quad \gamma_{456} = -\Omega. \end{aligned}$$

The remaining coefficients vanish, $\gamma_{kij} = 0$. Using these values, we write down the equations in quasi-coordinates (5.7)

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T^*}{\partial\dot\pi_1} + \frac{\partial T^*}{\partial\dot\pi_2}\dot\pi_3 - \frac{\partial T^*}{\partial\dot\pi_3}\dot\pi_2 + a\frac{\partial T^*}{\partial\dot\pi_4}(\dot\pi_3 - \Omega\dot\pi_6) + a\frac{\partial T^*}{\partial y} = 0, \\ &\frac{d}{dt}\frac{\partial T^*}{\partial\dot\pi_2} + \frac{\partial T^*}{\partial\dot\pi_3}\dot\pi_1 - \frac{\partial T^*}{\partial\dot\pi_1}\dot\pi_3 - a\frac{\partial T^*}{\partial\dot\pi_5}(\dot\pi_3 - \Omega\dot\pi_6) - a\frac{\partial T^*}{\partial x} = 0, \\ &\frac{d}{dt}\frac{\partial T^*}{\partial\dot\pi_3} + \frac{\partial T^*}{\partial\dot\pi_1}\dot\pi_2 - \frac{\partial T^*}{\partial\dot\pi_2}\dot\pi_1 - a\frac{\partial T^*}{\partial\dot\pi_4}\dot\pi_1 + a\frac{\partial T^*}{\partial\dot\pi_5}\dot\pi_2 = 0. \end{aligned} \tag{5.41}$$

Here

$$T^* = \frac{1}{2}\left[(a\dot{\pi}_2 - \dot{\pi}_4 - \Omega y\dot{\pi}_6)^2 + (a\dot{\pi}_1 - \dot{\pi}_5 - \Omega x\dot{\pi}_6)^2\right] + \frac{k^2}{2}(\dot{\pi}_1^2+\dot{\pi}_2^2+\dot{\pi}_3^2),$$

$$\frac{\partial T^*}{\partial \dot{\pi}_1} = (a^2 + k^2)\,\dot{\pi}_1 - a\,(\dot{\pi}_5 + \Omega x\dot{\pi}_6), \quad \frac{\partial T^*}{\partial \dot{\pi}_3} = k^2\dot{\pi}_3,$$

$$\frac{\partial T^*}{\partial \dot{\pi}_2} = (a^2 + k^2)\,\dot{\pi}_2 - a\,(\dot{\pi}_4 + \Omega y\dot{\pi}_6),$$

$$\frac{\partial T^*}{\partial \dot{\pi}_4} = -a\dot{\pi}_2 + \dot{\pi}_4 + \Omega y\dot{\pi}_6, \quad \frac{\partial T^*}{\partial x} = -\Omega\dot{\pi}_6\,(a\dot{\pi}_1 - \dot{\pi}_5 - \Omega x\dot{\pi}_6),$$

$$\frac{\partial T^*}{\partial \dot{\pi}_5} = -a\dot{\pi}_1 + \dot{\pi}_5 + \Omega x\dot{\pi}_6, \quad \frac{\partial T^*}{\partial y} = -\Omega\dot{\pi}_6\,(a\dot{\pi}_2 - \dot{\pi}_4 - \Omega y\dot{\pi}_6).$$

Substituting this into (5.41) and then eliminating the variables $\dot{x}$ and $\dot{y}$ by means of the equations of the nonholonomic constraints (5.33) and setting $\dot{\pi}_4 = \dot{\pi}_5 = 0$ and $\dot{\pi}_6 = 1$, we obtain the equations

$$\begin{aligned}(a^2 + k^2)\,\dot{\omega}_x - a\Omega\,(a\omega_y - \Omega y) &= 0,\\ (a^2 + k^2)\,\dot{\omega}_y + a\Omega\,(a\omega_x - \Omega x) &= 0,\\ k^2\dot{\omega}_z &= 0,\end{aligned} \tag{5.42}$$

which, together with the constraint equations (5.33), form a complete system for the determination of x, y, ω_x, ω_y, and ω_z as functions of the time. From the last equation of (5.42), we find $\omega_z = \text{const}$, and, from the first two equations of (5.42) in conjunction with (5.33), we obtain a further two first integrals

$$\begin{aligned}(a^2 + k^2)\,\omega_x - a\Omega x &= C_1\,(a^2 + k^2),\\ (a^2 + k^2)\,\omega_y - a\Omega y &= C_2\,(a^2 + k^2),\end{aligned}$$

where C_1 and C_2 are arbitrary constants, or, writing $b = a\Omega/(a^2 + k^2)$,

$$\omega_x - bx = C_1, \qquad \omega_y - by = C_2. \tag{5.43}$$

Eliminating ω_x and ω_y from (5.33) by means of (5.43), we arrive at the following equations for x and y:

$$\ddot{x} + \frac{\Omega k^2}{a^2 + k^2}\,\dot{y} = 0, \quad \ddot{y} - \frac{\Omega k^2}{a^2 + k^2}\,\dot{x} = 0. \tag{5.44}$$

The gyroscopic nature of the nonholonomy terms can be seen directly from these equations. [We should mention that we could have obtained equations (5.44) directly by writing down the equations in quasi-coordinates with $\pi_1 = x$ and $\pi_2 = y$.] From (5.44) for the determination of x and y as functions of the time, we obtain the equations

$$\ddot{x} + \left(\frac{\Omega k^2}{a^2 + k^2}\right)^2 x = C_3, \quad \ddot{y} + \left(\frac{\Omega k^2}{a^2 + k^2}\right)^2 y = C_4, \tag{5.45}$$

where C_3 and C_4 are arbitrary constants.

It follows from equations (5.45) that, in the general case, a sphere on a rough rotating horizontal plane describes an ellipse whose parameters depend on the initial conditions and which does not move in the fixed system of coordinates.

§6. Transpositional relations in the analytical mechanics of nonholonomic systems

We shall now dwell in more detail on the question of transpositional relations. We mention first of all that, besides (5.24) and (5.26), there exist other forms of transpositional relations. For example, suppose that the position of a system is determined by n generalized coordinates $q_1, q_2, \cdots, q_n$, whose differentials are connected by $n - m$ $(m < n)$ nonintegrable equations of the form

$$dq_{m+i} = a_{m+i,s}\, dq_s \quad (i = 1, 2, \ldots, m - n;\ s = 1, 2, \ldots, m),$$

where the coefficients $a_{m+i,s}$ are, in general, functions of all the coordinates $q_1, q_2, \cdots, q_n$. In some investigations, transpositional relations for such a system are taken in the form

$$d\,\delta q_{m+i} - \delta\, dq_{m+i} = \beta_{ks}^{m+i}\, dq_k \delta q_s \quad (i = 1, 2, \ldots, n - m;\ k, s = 1, 2, \ldots, m), \tag{6.1}$$

where

$$\beta_{ks}^{m+i} = \frac{\partial a_{m+i,\,s}}{\partial q_k} - \frac{\partial a_{m+i,\,k}}{\partial q_s} + \frac{\partial a_{m+i,\,s}}{\partial q_{m+l}}\, a_{m+l,\,k} - \frac{\partial a_{m+i,\,k}}{\partial q_{m+l}}\, a_{m+l,\,s}, \tag{6.2}$$

and it is assumed that the operations δ and d commute only for independent coordinates, i.e., (5.27) is satisfied only for $k = 1, 2, \cdots, m$. It is clear that the adoption of these transpositional relations, which differ from the preceding, must lead to a different form of the equations of motion of nonholonomic systems. For assuming

$$\delta\dot{q}_k - \frac{d}{dt}\delta q_k = 0 \qquad (k = 1, 2, \ldots, m)$$

in the transformed d'Alembert-Lagrange equation (5.30) and using (6.1) for the remaining $n - m$ bilinear forms, we arrive at the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\,\delta q_k\right) - \delta T - \frac{\partial T}{\partial \dot{q}_{m+i}}\,\beta_{rs}^{m+i}\dot{q}_r \delta q_s = Q_k \delta q_k$$

$$(k = 1, 2, \ldots, n;\ r, s = 1, 2, \ldots, m;\ i = 1, 2, \ldots, n - m).$$

Further, introducing the function

$$T^0(q_1, q_2, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_m) = T[q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_m, \dot{q}_{m+1}(q_i, \dot{q}_s), \ldots, \dot{q}_n(q_i, \dot{q}_s)]$$

and replacing the dependent variations δq_{m+1} by means of the equations of the nonholonomic constraints, we obtain, by virtue of the independence of the variations δq_s, the following equations of motion:

$$\frac{d}{dt}\frac{\partial T^0}{\partial \dot{q}_s} - \frac{\partial T^0}{\partial q_s} - \frac{\partial T^0}{\partial q_{m+i}} a_{m+i,s} - \beta_{rs}^{m+i}\frac{\partial T}{\partial \dot{q}_{m+i}}\dot{q}_r = Q_s + Q_{m+i}a_{m+i,s}, \tag{6.3}$$

which are identical with equations (3.38) of Voronec, which he obtained by eliminating directly the undetermined Lagrangian multipliers from equations (3.36). It is readily seen that in the case of a Čaplygin system, i.e., when the coordinates $q_{m+1}, \cdots, q_n$ occur neither in the expression for the kinetic energy T nor in the coefficients $a_{m+1,s}$ of the equations of the nonholonomic constraints, the equations obtained are identical with Čaplygin's equations (3.17).

What are then the correct transpositional relations? Until recently, there has not been, even in the text books, a common point of view concerning the commutativity of the operations of differentiation with respect to the time d/dt and virtual variation δ (or, putting it briefly, of the operations d and δ) when there are nonintegrable kinematic constraints. Two standpoints have been maintained. According to one (supported, for example, by Volterra and Hamel), the operations d and δ commute for all true coordinates $q_1, q_2, \cdots, q_n$, irrespective of whether the system if holonomic or nonholonomic. According to the other point of view (Suslov, Levi-Civita and Amaldi), the operations d and δ only commute for holonomic systems. In the case of nonholonomic systems, the relation

$$d\delta q_i - \delta dq_i = 0 \tag{6.4}$$

is assumed only for those generalized coordinates whose variations (in accordance with the equations of the nonholonomic constraints) are independent. For the remaining coordinates, the transpositional relations are to be derived on the basis of the equations of the nonholonomic constraints, and they differ from (6.4). The second point of view acquired general acceptance and its adherents assumed that the first point of view was erroneous. However, Hamel [**193**] showed that the first point of view cannot be dismissed as erroneous, but rather that it too can be shown to be tenable. Namely, the contradiction can be resolved if the

variations of the quasi-coordinates of the corresponding constraints are not assumed to vanish. At the same time, Hamel did not justify his argument or explain why such an assumption should be made. New light was cast on this problem recently in the paper [**324**], in which it was shown that both points of view can be maintained and that they do not contradict each other, as had seemed to be the case. Analysis of the problem shows that the contradiction arises because no definition had been made of the original concepts, i.e., of the operations d and δ, which occur in the transpositional relations.

For, performing the identity transformation (5.29) in the d'Alembert-Lagrange equation and reducing the latter to the form (5.30), we are lead naturally to ask what is meant by the expressions

$$\delta\dot{q}_i - \frac{d}{dt}\delta q_i.$$

However, one cannot answer this question if it is formulated in this manner. Moreover, the transformation (5.29) is meaningless, since the operations performed in it are not defined. For what is $\delta\dot{q}_i$? The quantity $\ddot{q}_i$ is defined only on the curve of the motion and the operation δ is, in general, a variation in a direction that differs from the actual direction of motion.

Thus, the problem must be attacked at the very beginning. Precise definitions are required of the meaning of the operations d, δ, $d\delta$, and δd. At the same time, it is important to note that it is sufficient to define these operations on an arbitrary curve $x_i = x_i(t)$ of an actually or kinematically possible motion. Furthermore, the definition is linked to this particular curve, so that, at a given point $x_1, x_2, \cdots, x_{3N}$, the operations may be different for different curves $x_i = x_i(t)$ passing through the given point.

We shall now formulate some definitions needed for the further study of the problem of transpositional relations.

1. A differential operation ω is defined at a point $q_1, q_2, \cdots, q_n$ of the configuration space of a system if a vector $\boldsymbol{a}(\omega q_1, \omega q_2, \cdots, \omega q_n)$ is defined at this point. If a vector $\boldsymbol{a}$ has been prescribed, the operation of ω on any function $f(q_1, q_2, \cdots, q_n)$ can be defined by the following limiting process:

$$\omega f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [f(q_1 + \varepsilon\omega q_1, \ldots) - f(q_1, \ldots)]$$

$$= \frac{\partial f}{\partial q_\alpha}\omega q_\alpha \quad (\alpha = 1, 2, \ldots, n). \tag{6.5}$$

2. If at all points of a region of the configuration space of a system two differential operations ω_1 and ω_2 have been defined, i.e., in this region the quantities $\omega_1 q_1, \cdots, \omega_1 q_n$ and $\omega_2 q_1, \cdots, \omega_2 q_n$ have been defined as functions of $q_1, q_2, \cdots, q_n$, the more complex operations $\omega_1\omega_2$ and $\omega_2\omega_1$ are, by the same token, also defined.

For $\omega_1 f$ is a well-defined function of the generalized coordinates $q_1, q_2, \cdots, q_n$ in the region under consideration and, hence, the operation ω_2 can be applied to the function $\omega_1 f$:

$$\omega_2(\omega_1 f) = \omega_2\left(\frac{\partial f}{\partial q_\alpha}\omega_1 q_\alpha\right) = \frac{\partial^2 f}{\partial q_\alpha \partial q_\beta}\omega_1 q_\alpha \omega_2 q_\beta + \frac{\partial f}{\partial q_\alpha}\frac{\partial(\omega_1 q_\alpha)}{\partial q_\beta}\omega_2 q_\beta .$$

In mechanics, the operation d consists of differentiating with respect to the time. It is therefore defined only for points $q_i = q_i(t)$ of the curve of the motion of the system, the definition being linked to this curve. The vector field corresponding to this operation is $\dot{q}_1 dt, \cdots, \dot{q}_n dt$. A virtual variation δ in mechanics is taken to mean any operation from the innumerable set of operations for which the corresponding vectors are all possible virtual displacements of the system, i.e., all possible linear combinations of the m linearly independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m$ obtained in a definite way from the kinematic constraints of the system. As a result, the operation δ is defined at all points of the configuration space.

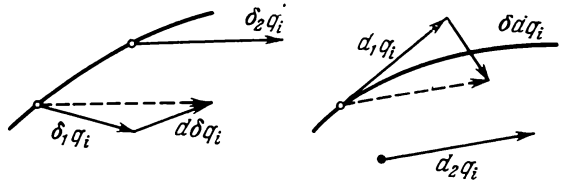

FIGURE 3.9

It follows from what we have said that at the point of an arbitrary (actual or kinematically possible) trajectory of the motion, the operation $d\delta$ is defined, but the operation δd is not defined. The situation is illustrated geometrically in Figure 3.9, from which it can be seen that the operation $d\delta q_i$ has a unique meaning, since all three operations d, δ_1, and δ_2 are well defined. In the second example shown in the figure, only the first two of the operations d_1, δ, and d_2 are defined whereas the operation d_2 is meaningless, since the vector corresponding to it is not situated on the curve of the motion. It follows that the operation $\delta dq_i = d_2 q_i - d_1 q_i$ is not defined. The definition of the operation d must

be supplemented in such a way that the operation δd acquires a meaning. In considering this definition, it is essential to note that the operations d and δ can be defined quite arbitrarily off the curve of the motion $q_i = q_i(t)$, but, on the curve, they must coincide with the operations of differentiation with respect to the time and virtual variation, respectively, since the d'Alembert-Lagrange equations (1.12) must remain valid.

Among the innumerable set of conceivable methods of supplementing the definition of the operations d and δ that preserve their ordinary meaning on the curve of the motion, we give the two following possibilities.

First method. In the neighborhood of the motion $q_i = q_i(t)$, we introduce a curvilinear system of coordinates $q_i = (u_1, u_2, \cdots, u_n)$ such that: a) The line $u_2 = u_3 = \cdots = u_n = 0$ is the curve of the motion; b) the parameter u_1 on this line coincides with the time t; c) the planes tangent to the surface $u_{m+1} = u_{m+2} = \cdots = u_n = 0$ at the points for which $u_2 = u_3 = \cdots = u_n = 0$ are the planes of the virtual displacements of the system.[9] To be specific, we shall assume that the equations of the kinematic constraints are linear and homogeneous. In the neighborhood of the curve $q_i = q_i(t)$ we define dq_i as $(\partial q_1/\partial u_1)\,du_1$ and δq_i as $\sum_{\alpha=1}^{m} (\partial q_i/\partial u_\alpha)\,\delta u_\alpha$.

For the values $u_2 = u_3 = \cdots = u_n = 0$, the operation d coincides with differentiation with respect to the time and the operation δ with virtual variation.

It is clear that, with this definition, the operations d and δ commute, i.e., equations (6.4) hold for all generalized coordinates. In this case, however, $dq_1, dq_2, \cdots, dq_n$ and $\delta q_1, \delta q_2, \cdots, \delta q_n$ are displacements that are compatible with the constraints only for $u_2 = u_3 = \cdots = u_n = 0$, i.e., only on the curve of the motion. In this connection, we note that Volterra's error in deriving his equations of motion was that he assumed that $dq_1, \cdots, dq_n$ and $\delta q_1, \cdots, \delta q_n$ satisfy the constraints of the system not only on the curve of the motion, but also off it.

Second method. Let $q_i = q_i(u_1, u_2, \cdots, u_n)$ be a curvilinear system of coordinates in the neighborhood of the motion $q_i = q_i(t)$ such that $q_i(u_1, 0, \cdots, 0) = q_i(u_1)$. Let us assume that the equations of the kinematic constraints can be written in the form

---

[9] In the case of linear inhomogeneous constraint equations, the surface $u_{m+1} = u_{m+2} = \cdots = u_n = 0$ must be replaced by the surface $u_1 = \text{const}$, $u_{m+2} = \cdots = u_n = 0$ and δq_i must be defined, accordingly, as $\sum_{\alpha=2}^{m+1} (\partial q_i/\partial u_\alpha)\,\delta u_\alpha$.

(6.6) $\dot{q}_j = a_{js}\dot{q}_s \quad (j = m+1,\ m+2,\ldots,\ n;\ s = 1,\ 2,\ldots,\ m).$

We now supplement the definition of the operations d and δ in the neighborhood the curve of the motion as follows:

$$dq_\tau = \begin{cases} \dfrac{\partial q_\tau}{\partial u_1} du_1 & (\tau = 1, 2, \ldots, m), \\ a_{\tau s} dq_s & (\tau = m+1, \ldots, n), \end{cases}$$

(6.7)

$$\delta q_\tau = \begin{cases} \dfrac{\partial q_\tau}{\partial u_\alpha} \delta u_\alpha & (\tau = 1, 2, \ldots, m), \\ a_{\tau s} \delta q_s & (\tau = m+1, \ldots, n). \end{cases}$$

In this definition, the displacements $dq_1, \cdots, dq_n$ and $\delta q_1, \cdots, \delta q_n$ are obviously compatible with the constraints of the system in the whole neighborhood of the motion and not only on the curve itself. For the first m generalized coordinates $q_1, q_2, \cdots, q_m$, the operations d and δ commute. For the remaining $n - m$ coordinates $q_{m+1}, q_{m+2}, \cdots, q_n$, we obtain from (6.6)

$$d\,\delta q_j - \delta\, dq_j = \left(\frac{\partial a_{js}}{\partial q_r} - \frac{\partial a_{jr}}{\partial q_s} + \frac{\partial a_{js}}{\partial q_{m+l}} a_{m+l,\,r} - \frac{\partial a_{jr}}{\partial q_{m+l}} a_{m+l,\,s}\right) dq_r \delta q_s$$

(6.8)

$$(l = 1,\ 2,\ldots,\ n - m;\ j = m+1,\ldots,\ n;\ r,\ s = 1,\ 2,\ldots,\ m),$$

the vanishing of all these expressions implying, by virtue of the theorem of Frobenius (see §7, Chapter I), complete integrability of the kinematic constraint equations of the system.

It follows that the second method of supplementing the definitions corresponds to the point of view of Suslov, Levi-Civita, and Amaldi, who assumed that the operations d and δ commute only for the independent coordinates. It also follows that these are not the only possible methods and that others could be adopted, leading to new forms of the equations of motion. As an example, let us consider one such case, corresponding, in the language of the previous examples, to an "intermediate" point of view regarding the transpositional relations. Suppose that we have equations of kinematic constraints of the form (6.6) and we introduce in the neighborhood of the curve of the motion $q_i = q_i(t)$ a curvilinear coordinate system $q_i = q_i(u_1, \cdots, u_n)$ such that: a) The line $u_2 = u_3 = \cdots = u_n = 0$ is the curve of the actual motion; b) the parameter u_1 on this line coincides with the time t; c) the planes tangent to the surface $u_{m+1} = \cdots = u_n = 0$ at the points $u_2 = u_3 = \cdots = u_n = 0$ are the planes of the virtual displacements of the system. Let the operations d and δ be defined by the following expressions:

$$
(6.9)\qquad
dq_\tau = \begin{cases} \dfrac{\partial q_\tau}{\partial u_1}\, du_1 & (\tau = 1, 2, \ldots, m+k), \\ a_{\tau s} dq_s & (\tau = m+k+1, \ldots, n), \end{cases}
$$

$$
\delta q_\tau = \begin{cases} \dfrac{\partial q_\tau}{\partial u_\alpha}\, \delta u_\alpha & (\tau = 1, 2, \ldots, m+k), \\ a_{\tau s} \delta q_s & (\tau = m+k \mp 1, \ldots, n), \end{cases}
$$

where k is an integer ($0 \leqq k \leqq n - m$). These differ from (6.7) in that the subscript τ runs through different sets of values. According to the definition (6.9), the operations d and δ commute only for $\tau = 1, 2, \cdots, m + k$ and the displacements dq_τ and δq_τ are compatible with the constraints in the neighborhood of the curve of the motion $q_i = q_i(u_1, \cdots, u_n)$ for all $\tau = 1, 2, \cdots, n$ apart from $\tau = m + 1, m + 2, \cdots, m + k$.

In accordance with the equations of the kinematic constraints (6.6), we introduce the quasi-coordinates $\pi_1, \pi_2, \cdots, \pi_{m+k}$ by means of the relations

$$
(6.10)\qquad
\dot{\pi}_r = a_{rs}\dot{q}_s, \quad \dot{\pi}_\rho = a_{\rho s}\dot{q}_s - \dot{q}_\rho
$$
$$
(r,\ s = 1,\ 2,\ldots,\ m;\quad \rho = m+1,\ m+2,\ldots,\ m+k).
$$

Then, by virtue of (6.9), (6.10), and (6.6), we obtain the following transpositional relations:

$$
(6.11)\qquad
d\,\delta q_\lambda - \delta\, dq_\lambda = 0,\ d\,\delta\pi_\lambda - \delta\, d\pi_\lambda = \gamma_{\nu\lambda\mu} d\pi_\mu\, \delta\pi_\nu,
$$
$$
d\,\delta q_\sigma - \delta\, dq_\sigma = B^\sigma_{rs} dq_r \delta q_s,
$$

where

$$
(6.12)\qquad
\gamma_{\nu\lambda\mu} = b_{\alpha\nu} b_{\beta\mu}\left(\frac{\partial a_{\lambda\alpha}}{\partial q_\beta} - \frac{\partial a_{\lambda\beta}}{\partial q_\alpha}\right) \quad (b_{\alpha\lambda} a_{\lambda\beta} = \delta_{\alpha\beta}),
$$
$$
B^\sigma_{rs} = \frac{\partial a_{\sigma s}}{\partial q_r} - \frac{\partial a_{\sigma r}}{\partial q_s} + \frac{\partial a_{\sigma s}}{\partial q_j} a_{jr} - \frac{\partial a_{\sigma r}}{\partial q_j} a_{js}
$$

($r, s = 1, 2, \cdots, m$; $\rho = m + 1, \cdots, m + k$; $\alpha, \beta, \lambda, \mu, \nu = 1, 2, \cdots, m + k$; $\sigma = m + k + 1, m + k + 2, \cdots, n$; and $\delta_{\alpha\beta}$ is the Kronecker delta).

Comparing the expressions (6.9) and the preceding expressions, one sees that the two original points of view concerning the transpositional relations correspond to the extreme values of the number k, namely, $k = 0$ and $k = n - m$.

Using the transpositional relations (6.11), we shall now derive a form of the equations of motion of nonholonomic systems from which, in particular, one can obtain both the equations in quasi-coordinates of Boltzmann and Hamel as well as the equations in true coordinates of Voronec and Čaplygin. For this purpose, we use the d'Alembert-Lagrange equation in the form (5.30) and substitute into it the first and third

of the relations (6.11). We then obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right) - \delta T - \frac{\partial T}{\partial \dot{q}_\sigma} B^\sigma_{rs}\dot{q}_r\delta q_s = Q_i\delta q_i \tag{6.13}$$

$$(i = 1, 2,\ldots, n;\ \sigma = m + k + 1,\ldots, n;\ r, s = 1, 2,\ldots, m).$$

By virtue of (6.6) and (6.10), the variations of the coordinates and quasi-coordinates are related as follows:

$$\delta q_\lambda = b_{\lambda\nu}\delta\pi_\nu;\ \delta q_\sigma = a_{\sigma s}\delta q_s = a_{\sigma s}b_{s\nu}\delta\pi_\nu \tag{6.14}$$

$$(\lambda, \mu, \nu = 1, 2,\ldots, m + k;\ \sigma = m + k + 1,\ldots, n;\ r, s = 1, 2,\ldots, m).$$

We introduce the function

$$T^*(q_i, \dot{\pi}_\nu) = T(q_i, b_{\lambda\nu}\dot{\pi}_\nu, a_{\sigma s}b_{s\nu}\dot{\pi}_\nu), \tag{6.15}$$

which can be obtained from the expression for the kinetic energy $T(q_1, \dot{q}_i)$ by eliminating the generalized velocities $\dot{q}_{m+k+1}, \cdots, \dot{q}_n$ by means of the relations (6.6) and by replacing $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{m+k}$ by $\dot{\pi}_1$, $\dot{\pi}_2, \cdots, \dot{\pi}_{m+k}$ from the relations (6.10). Going over in equation (6.13) to quasi-coordinates, we obtain

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\pi}_\nu}\delta\pi_\nu\right) - \delta T^* - \frac{\partial T}{\partial \dot{q}_\sigma} B^\sigma_{rs}\dot{q}_r b_{s\nu}\delta\pi_\nu = \Pi_\nu\delta\pi$$

or, after transformations using the second transpositional relation of (6.11),

$$\left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\nu} - \frac{\partial T^*}{\partial q_\lambda}b_{\lambda\nu} - \frac{\partial T^*}{\partial q_\sigma}a_{\sigma s}b_{s\nu} + \gamma_{\nu\lambda\mu}\frac{\partial T^*}{\partial \dot{\pi}_\lambda}\dot{\pi}_\mu - \frac{\partial T}{\partial \dot{q}_\sigma}B^\sigma_{rs}\dot{q}_r b_{s\nu} - \Pi_\nu\right)\delta\pi_\nu = 0. \tag{6.16}$$

According to the conditions of the nonholonomic constraints, we have $\delta\pi_{m+1} = \delta\pi_{m+2} = \cdots = \delta_{m+k} = 0$, and, consequently, the corresponding terms in the sum (6.16) vanish. By virtue of the independence of the variations $\delta\pi_l$ $(l = 1, 2, \cdots, m)$, the remaining sum can be split up into m equations of the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_l} - \frac{\partial T^*}{\partial \pi_l} + \gamma_{l\lambda\mu}\frac{\partial T^*}{\partial \dot{\pi}_\lambda}\dot{\pi}_\mu - B^\sigma_{rs}\frac{\partial T}{\partial \dot{q}_\sigma}\dot{q}_r b_{sl} = \Pi_l \tag{6.17}$$

$$(l, r, s = 1, 2,\ldots, m;\ \sigma = m + k + 1,\ldots, n;\ \lambda, \mu = 1, 2,\ldots, m + k),$$

where

$$\frac{\partial T^*}{\partial \pi_l} = \frac{\partial T^*}{\partial q_\lambda}b_{\lambda l} + \frac{\partial T^*}{\partial q_\sigma}a_{\sigma s}b_{sl}.$$

Equations (6.17) are a form of the equations of motion of nonholonomic systems intermediate between the Boltzmann-Hamel equations in quasi-coordinates and the equations of Čaplygin and Voronec. For, if $k = n - m$, the terms

$$B_{rs}^{\sigma} \frac{\partial T}{\partial q_{\sigma}} \dot{q}_s b_{sl}$$

disappear and equations (6.17) are identical with equations (5.7) in quasi-coordinates. For $k = 0$, equations (6.17) are Voronec's equations. These equations can be written down in true coordinates. One must therefore set $a_{sl} = b_{sl} = \delta_{sl}$ in the relations (6.14). It therefore follows, in conjunction with (6.12), that $\gamma_{l\lambda\mu} = 0$. In addition, if $k = 0$, then, by (6.15), the function T^* is transformed into the function $T^0(q_i, \dot{q}_i)$, which is usually used in writing down Voronec's equations. Thus, we arrive at the equations

$$\frac{d}{dt} \frac{\partial T^0}{\partial \dot{q}_l} - \frac{\partial T^0}{\partial q_l} - \frac{\partial T^0}{\partial q_j} a_{jl} - B_{rl}^{j} \frac{\partial T}{\partial \dot{q}_j} \dot{q}_r = \Pi_l, \tag{6.18}$$

which are identical with Voronec's equations (3.38). For Čaplygin systems, equations (6.18) are identical with Čaplygin's equations (3.33).

§7. Canonical form of the equations of motion of nonholonomic systems

It was pointed out in 1906 by Čaplygin and Quangel that the equations of motion of nonholonomic systems can be written in a form resembling that of the ordinary canonical equations for holonomic systems.

The value of such a form of the equations of motion of nonholonomic systems is that it enables one to generalize to nonholonomic systems certain theorems and assertions from the theory of integration of the ordinary canonical equations considered in the mechanics of holonomic systems.

One can arrive at canonical equations by starting from different forms of the equations of motion of nonholonomic systems. We shall assume that the external forces acting on a nonholonomic system are potential forces and, hence, that we can write down an expression for the Lagrange function $L = T - V$ without taking into account the nonholonomic constraints

$$\alpha_{si} \dot{q}_i + \alpha_s = 0 \ (s = 1, 2, \ldots, n - m; \ i = 1, 2, \ldots, n). \tag{7.1}$$

In this case, the equations of motion of the nonholonomic system with Lagrangian multipliers have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \alpha_{si}\lambda_s \quad (i = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n - m). \tag{7.2}$$

In order to pass from these equations to canonical equations, we introduce the function

$$H = p_i \dot{q}_i - L \quad (i = 1, 2, \ldots, n), \tag{7.3}$$

where, as usual, the generalized momenta p_i are defined by the expressions

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i = 1, 2, \ldots, n). \tag{7.4}$$

Varying the expression (7.3), we obtain

$$\delta H = \dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i}\delta q_i. \tag{7.5}$$

On the other hand, if the variables $\dot{q}_i$ on the right-hand side of (7.3) are eliminated by means of the relations (7.4), we obtain $H = H(q, p, t)$ and, consequently,

$$\delta H = \frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i \quad (i = 1, 2, \ldots, n). \tag{7.6}$$

Equating the coefficients of the same variations on the right-hand sides of (7.5) and (7.6), we obtain the relations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i},$$

which, in conjunction with (7.2) and (7.4), enable one to write down the equations of motion in the form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \lambda_s \alpha_{si}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad (i = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n - m). \tag{7.7}$$

These equations are the canonical equations of the nonholonomic system in the case under consideration. Together with the equations of the nonholonomic constraints (7.1), equations (7.7) form a complete system of $3n - m$ differential equations for the determination of the quantities q_i, p_i, and λ_s $(i = 1, 2, \cdots, n;\ s = 1, 2, \cdots, n - m)$ as functions of the time t.

Now suppose that the original equations are the equations of motion of a nonholonomic system (5.15) in quasi-coordinates. If the forces have a potential, these equations take the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_k} - \frac{\partial L^*}{\partial \pi_k} + \gamma_{kij}\frac{\partial L^*}{\partial \dot{\pi}_i}\dot{\pi}_j = 0 \tag{7.8}$$
$$(j, k = 1, 2, \ldots, m;\ i = 1, 2, \ldots, n).$$

In equations (7.8), the Lagrange function L^* depends, in general, on all the quasi-velocities $\dot{\pi}_i$ $(i = 1, 2, \cdots, n)$, which are related to the generalized velocities $\dot{q}_i$ $(i = 1, 2, \cdots, n)$ by means of the relations (5.1), the quasi-coordinates π_i being chosen such that the last $n - m$ quasi-velocities $\dot{\pi}_{m+1}, \dot{\pi}_{m+2}, \cdots, \dot{\pi}_n$ in the relations (5.1) vanish if the variables q_i and $\dot{q}_i$ $(i = 1, 2, \cdots, n)$ satisfy the equations of the nonholonomic constraints (7.1). Thus, equations (7.8) must be augmented by the condition

$$\dot{\pi}_{m+\nu} = 0 \quad (\nu = 1, 2, \ldots, n - m) \tag{7.9}$$

and the relations (5.1), which, with allowance for (7.9), can be written in the form

$$\dot{q}_i = b_{ik}\dot{\pi}_k \quad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m). \tag{7.10}$$

Equations (7.8) and (7.10), in which we have used the condition (7.9), form a complete system of $n + m$ differential equations for the determination of the quantities q_i and $\dot{\pi}_k$ $(i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, m)$ as functions of the time t.

We introduce n generalized momenta

$$p_i = \frac{\partial L^*}{\partial \dot{\pi}_i} \quad (i = 1, 2, \ldots, n) \tag{7.11}$$

and a function K by means of the expression

$$K = p_i\dot{\pi}_i - L^* \quad (i = 1, 2, \ldots, n). \tag{7.12}$$

Varying this expression, we obtain

$$\begin{aligned}\delta K &= \dot{\pi}_i\delta p_i + p_i\delta\dot{\pi}_i - \frac{\partial L^*}{\partial q_i}\delta q_i - \frac{\partial L^*}{\partial \dot{\pi}_i}\delta\dot{\pi}_i \\ &= \dot{\pi}_i\delta p_i - \frac{\partial L^*}{\partial q_i}b_{il}\delta\pi_l = \dot{\pi}_i\delta p_i - \frac{\partial L^*}{\partial \pi_l}\delta\pi_l \quad (i, l = 1, 2, \ldots, n).\end{aligned} \tag{7.13}$$

Here, we have used the relations (5.1) and have introduced the notation

$$\frac{\partial T^*}{\partial \pi_l} = \frac{\partial T^*}{\partial q_i}b_{il} \quad (i, l = 1, 2, \ldots, n), \tag{7.14}$$

since the canonical variables in this case are the quantities π_i and p_i

$(i = 1, 2, \cdots, n)$. We now vary the function $K = K(q, p, t)$, which is obtained after the quantities π_i $(i = 1, 2, \cdots, n)$ have been eliminated from the right-hand side of (7.12) by means of the relations (7.11):

$$\delta K = \frac{\partial K}{\partial q_i}\delta q_i + \frac{\partial K}{\partial p_l}\delta p_l = \frac{\partial K}{\partial \pi_l}\delta \pi_l + \frac{\partial K}{\partial p_l}\delta p_l. \tag{7.15}$$

Equating coefficients of the same variations on the right-hand sides of (7.13) and (7.15), we obtain the relations

$$\dot{\pi}_l = \frac{\partial K}{\partial p_l}, \quad -\frac{\partial L^*}{\partial \pi_l} = \frac{\partial K}{\partial \pi_l} \qquad (l = 1, 2, \ldots, n). \tag{7.16}$$

Using these relations, we write the equations of motion (7.8) in the form

$$\dot{\pi}_h = \frac{\partial K}{\partial p_h}, \quad \dot{p}_h = -\frac{\partial K}{\partial \pi_h} - \gamma_{hij} p_i \frac{\partial K}{\partial p_j} \qquad (j, k = 1, 2, \ldots, m;\ i = 1, 2, \ldots, n). \tag{7.17}$$

Equations (7.17) are the canonical equations in the given case. To these equations, we must add the relations

$$\frac{\partial K}{\partial p_{m+\nu}} = 0 \qquad (\nu = 1, 2, \ldots, n - m), \tag{7.18}$$

which can be obtained from (7.16) by using the condition (7.9), and also the relations (7.10), written in the form

$$\dot{q}_i = b_{ik}\frac{\partial K}{\partial p_k} \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m). \tag{7.19}$$

Equations (7.17) – (7.19) form a complete system of $2n + m$ equations for the determination of the quantities q_i, p_i, and π_k $(i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, m)$ as functions of the time.

Finally, we consider the case when the original equations of motion of the nonholonomic system are Čaplygin's equations (3.19), in which the function $L = L(q_1, q_2, \cdots, q_m, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n)$ and the coefficients $b_{s,m+k}$ of the equations of the nonholonomic constraints (3.1) do not depend on the last $n - m$ coordinates $q_{m+1}, q_{m+2}, \cdots, q_n$. Since $L^* = L^*(q_1, \cdots, q_m, \dot{q}_1, \cdots, \dot{q}_m)$ and the symbol $(\partial L/\partial \dot{q}_{m+k})^*$ stands for the expression which is obtained after the generalized velocities $\dot{q}_{m+1}$, $\dot{q}_{m+2}, \cdots, \dot{q}_n$ have been eliminated from $\partial L/\partial \dot{q}_{m+k}$ by means of the equations of the nonholonomic constraints (3.1), Čaplygin's equations (3.19) form a closed system of m equations for the determination of the quantities $q_1, q_2, \cdots, q_m$ as functions of the time t.

In order to write down these equations in a canonical form, it is sufficient to introduce m generalized momenta p_s $(s = 1, 2, \cdots, m)$ by

means of the relations

$$p_s = \frac{\partial L^*}{\partial \dot{q}_s} \quad (s = 1, 2, \ldots, m) \tag{7.20}$$

and form the function

$$H^* = p_s \dot{q}_s - L^* \quad (s = 1, 2, \ldots, m). \tag{7.21}$$

After calculations similar to those above, we obtain the relations

$$\dot{q}_s = \frac{\partial H^*}{\partial p_s}, \quad \frac{\partial H^*}{\partial q_s} = -\frac{\partial L^*}{\partial q_s} \quad (s = 1, 2, \ldots, m),$$

which enable us to write equations (3.18) in the canonical form

$$\begin{aligned} \dot{q}_s &= \frac{\partial H^*}{\partial p_s} \quad (s, r = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n - m), \\ \dot{p}_s &= -\frac{\partial H^*}{\partial p_s} - \left[\frac{\partial L}{\partial \dot{q}_{m+k}}\right]^* \left(\frac{\partial b_{r,\,m+k}}{\partial q_s} - \frac{\partial b_{s,\,m+k}}{\partial q_r}\right) \frac{\partial H^*}{\partial p_r}, \end{aligned} \tag{7.22}$$

where the symbol $[\partial L/\partial \dot{q}_{m+k}]^*$ means that all the generalized velocities in the expression $\partial L/\partial \dot{q}_{m+k}$ have been eliminated by means of the equations of the nonholonomic constraints (3.1) and the relations (7.20).

Equations (7.22) form a complete system of $2m$ equations for the determination of the quantities q_s and p_s $(s = 1, 2, \cdots, m)$ as functions of the time t.

§8. Appell's equations

In his paper [13], Appell proposed a new form of the equations of motion of holonomic and nonholonomic systems. In doing so, he introduced a new function S analogous to the kinetic energy in Lagrange's equations; this function subsequently came to be called the acceleration function or acceleration energy. The function S by itself completely characterizes the dynamics of a nonholonomic system in the same way as the kinetic energy T characterizes a holonomic system. Although the form of Appell's equations is very simple, in concrete examples it is usually much harder to write down the acceleration function S than the expression for the kinetic energy T.

We shall now derive Appell's equations.

In the d'Alembert-Lagrange equation written in the form

$$\sum_{\nu=1}^{N} m_\nu \boldsymbol{w}_\nu \delta \boldsymbol{r}_\nu = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu \delta \boldsymbol{r}_\nu, \tag{8.1}$$

we express the variations of the radius vectors of the points of the system in terms of the variations of the quasi-coordinates $\delta\pi_1, \delta\pi_2, \cdots, \delta\pi_m$. Suppose that the velocities of the points of the system are related to the quasi-velocities by the equations

$$\dot{\boldsymbol{r}}_\nu = \boldsymbol{a}_{\nu s}\dot{\pi}_s + \boldsymbol{b}_\nu \quad (\nu = 1, 2, \ldots, N;\ s = 1, 2, \ldots, m), \tag{8.2}$$

where $\boldsymbol{a}_{\nu s}$ and $\boldsymbol{b}_s$ are vectors that depend on the configuration of the system and, possibly, the time t. Then, omitting the summation sign, we have

$$m_\nu \boldsymbol{w}_\nu \boldsymbol{a}_{\nu s}\delta\pi_s = \boldsymbol{F}_\nu \boldsymbol{a}_{\nu s}\delta\pi_s.$$

The equations derived from these equations

$$m_\nu \boldsymbol{w}_\nu \boldsymbol{a}_{\nu s} = \Pi_s \quad (\nu = 1, 2, \ldots, N;\ s = 1, 2, \ldots, m), \tag{8.3}$$

where Π_s is the generalized force corresponding to the quasi-coordinate π_s, can be transformed to an elegant form. Differentiating (8.2), we find

$$\boldsymbol{w}_\nu = \ddot{\boldsymbol{r}}_\nu = \boldsymbol{a}_{\nu s}\ddot{\pi}_s + \dot{\boldsymbol{a}}_{\nu s}\dot{\pi}_s + \dot{\boldsymbol{b}}_\nu \quad (\nu = 1, 2, \ldots, N;\ s = 1, 2, \ldots, m),$$

and, therefore,

$$\frac{\partial \boldsymbol{w}_\nu}{\partial \ddot{\pi}_s} = \boldsymbol{a}_{\nu s}. \tag{8.4}$$

The acceleration w_ν is a function of the coordinates (or the generalized coordinates) and the quantities $\dot{\pi}_s$ and $\ddot{\pi}_s$. Using the relations (8.4), we write equations (8.3) in the form

$$m_\nu \boldsymbol{w}_\nu \frac{\partial \boldsymbol{w}_\nu}{\partial \ddot{\pi}_s} = \Pi_s \quad (\nu = 1, 2, \ldots, N;\ s = 1, 2, \ldots, m)$$

or

$$\frac{\partial}{\partial \ddot{\pi}_s}\left(\frac{m_\nu w_\nu^2}{2}\right) = \Pi_s.$$

We now introduce the function S defined by the expression[10]

$$S = \frac{1}{2}\, m_\nu w_\nu^2 = \frac{1}{2}\, m_\nu \ddot{\boldsymbol{r}}_\nu^2, \tag{8.5}$$

and, regarding it as a function of the coordinates and the quantities $\dot{\pi}_s$ and $\ddot{\pi}_s$, we arrive at Appell's equations

$$\frac{\partial S}{\partial \ddot{\pi}_s} = \Pi_s \quad (s = 1, 2, \ldots, m), \tag{8.6}$$

---

[10] The function S is composed of the accelerations w_ν of the points of the system in the same way as the kinetic energy is composed of the velocities v_ν of the same points.

the number of these equations being equal to the number of degrees of freedom of the system. Equations (8.6) together with the equations of the kinematic constraints (8.2) constitute a complete system of $3N + m$ equations for the determination of the quantities $x_1, y_1, z_1, \cdots, x_N, y_N, z_N$; $\dot{\pi}_1, \dot{\pi}_2, \cdots, \dot{\pi}_m$ as functions of the time t.

We mention that the acceleration energy completely characterizes the dynamics of a nonholonomic system in the sense that, possessing the expression of only the function S and having no further information about the system at our disposal (in particular, not knowing anything about the constraints imposed on the system), we can write down the equations of motion. Thus, for nonholonomic systems, the acceleration function S plays the same role as the kinetic energy T for holonomic systems. Hence, it also follows that knowledge of only the function T or T^* is insufficient for the study of the behavior of nonholonomic systems. In other words, if we know only the expression for the kinetic energy T or T^*, we cannot yet say anything about the dynamics of a nonholonomic system. To prove this assertion, it is sufficient to find two different dynamical systems for which the expression T is the same but the function S differs. An example of two such systems with the same function T but different functions S was put forward by Appell [25]. The first system is a disc of radius k with moments of inertia A, B, and C which rolls on a rough plane. The second system is a figure

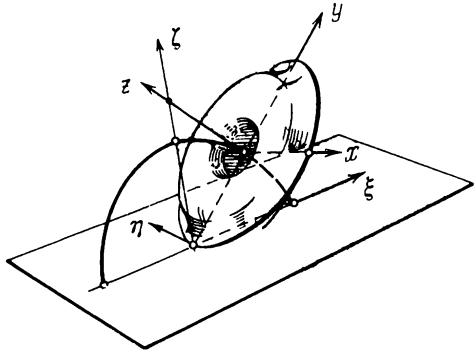

FIGURE 3.10

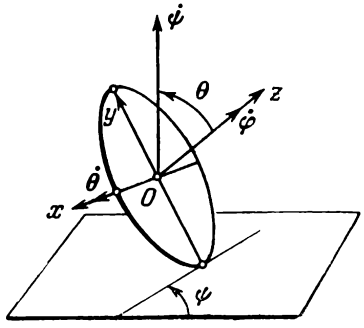

FIGURE 3.11

of revolution of radius a with a mass distribution such that $A_1 = A$ and $C_1 = ma^2$. The motion of the second system is restricted as follows: a) The center of the body slides on a vertical circle $\eta = 0$, $\xi^2 + \zeta^2 = a^2$ (Figure 3.10); b) the body slides on a plane without friction, the coordinate of the center of gravity being $\xi = a \sin\theta$.

Let us write down the expressions for the kinetic energy and Appell's

function for these two systems. We shall assume that the disc and the figure of revolution have unit mass.

In the case of the first system, we choose a moving system of coordinates $Oxyz$ (the Ox axis is always horizontal) (Figure 3.11); the projections of the instantaneous angular velocity are then

$$p = \dot{\theta},\ q = \dot{\psi}\sin\theta,\ r = \dot{\psi}\cos\theta + \dot{\varphi}.$$

The velocity of the center of gravity is

$$v_c^2 = a^2(p^2 + r^2).$$

We have

$$2T = a^2(p^2 + r^2) + A(p^2 + q^2) + Cr^2,\ V = ga\sin\theta.$$

The Lagrange function is

$$L = \tfrac{1}{2}(A + a^2)p^2 + \tfrac{1}{2}Aq^2 + \tfrac{1}{2}(C + a^2)r^2 - ga\sin\theta.$$

In the general case, the acceleration function for a rotational motion of the body about a fixed point (or the center of gravity) can be written as follows:[11]

$$S_\omega = \tfrac{1}{2}(A\dot{p}^2 + B\dot{q}^2 + C\dot{r}^2) - [(C - B)qr + A(r\Omega_2 - q\Omega_3)]\dot{p} + [(A - C)pr + B(p\Omega_3 - r\Omega_1)]\dot{q} + [(B - A)pq + C(q\Omega_1 - p\Omega_2)]\dot{r} + \ldots,$$

where the moving axes coincide with the directions of the principal axes of the ellipsoid of inertia. We denote by A, B, and C the principal moments of inertia. In our case, $A = B$, $\Omega_1 = p$, $\Omega_2 = q$, and $\Omega_3 = q\cot\theta$; therefore,

$$S_\omega = \tfrac{1}{2}[A(\dot{p}^2 + \dot{q}^2) + C\dot{r}^2] + (Aq\cot\theta - Cr)(p\dot{q} - q\dot{p}) + \ldots.$$

The projections of the absolute acceleration of the center of gravity onto the moving axes expressed in terms of the projections of the velocities onto these axes are

$$\dot{u} + \Omega_2 w - \Omega_3 v,\ \dot{v} + \Omega_3 u - \Omega_1 w,\ \dot{w} + \Omega_1 v - \Omega_2 u.$$

From the nonholonomic constraints, we have

$$u = -ar,\quad v = 0,\quad w = ap.$$

---

[11] See Example 1 below.

After substituting these values and the expressions for Ω_i, we obtain

$$S_c = \frac{1}{2} w_c^2 = \frac{1}{2} a^2 (\dot{p}^2 + \dot{r}^2) - a^2 q (\dot{p}r - r\dot{p}) + \ldots.$$

Appell's function for the first system is

$$S = \frac{1}{2} [(A + a^2) \dot{p}^2 + A\dot{q}^2 + (C + a^2) \dot{r}^2] + \\ + (Aq \cot \theta - Cr)(p\dot{q} - q\dot{p}) - a^2 q (\dot{p}r - r\dot{p}) + \ldots.$$

We now consider the second system. From the constraint conditions a) and b) we obtain $\xi = a \cos\theta$. The Lagrange function can then be written in the form

$$L_1 = \frac{1}{2} (A + a^2) p^2 + \frac{1}{2} Aq^2 + \frac{1}{2} (C + a^2) r^2 - ga \sin \theta.$$

Appell's function is

$$S_1 = S'_\omega + S'_c,$$

where

$$S'_\omega = \frac{1}{2} [A (\dot{p}^2 + \dot{q}^2) + C\dot{r}^2] + (Aq \cot \theta - Cr)(p\dot{q} - q\dot{p}) + \ldots.$$

The projections of the acceleration of the center of gravity onto the moving axes are, by virtue of the imposed constraints,

$$-apq, \quad ap^2, \quad -a\dot{p}.$$

Therefore

$$S'_c = \frac{1}{2} w'^2_c = a^2 [p^2 (p^2 + q^2) + \dot{p}^2],$$

$$S_1 = \frac{1}{2} [(A + a^2) \dot{p}^2 + A\dot{q}^2 + C\dot{r}^2] + (Aq \cot \theta - Cr)(p\dot{q} - q\dot{p}) + \ldots.$$

Comparison of the results shows that the Lagrange functions of these systems are identical but the equations of motion are different, since the acceleration functions of the two systems are different.

One can easily show that equations (8.6) can also be obtained in the case when the d'Alembert-Lagrange equation is written in the form

$$\left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k \right) \delta q_k = 0 \quad (k = 1, 2, \ldots, n). \tag{8.7}$$

For the equations of the kinematic constraints

$$\dot{q}_k = a_{ks} \dot{\pi}_s + a_k \quad (k = 1, 2, \ldots, n; \; s = 1, 2, \ldots, m) \tag{8.8}$$

yield the relations

$$(8.9)\quad \delta q_k = a_{ks}\delta\pi_s,\ a_{ks} = \frac{\partial \dot{q}_k}{\partial \dot{\pi}_s} = \frac{\partial \ddot{q}_k}{\partial \ddot{\pi}_s}\ (k = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m).$$

Substituting δq_k expressed in terms of the variations $\delta\pi_s$ by means of (8.9) into (8.7), we obtain, by virtue of the independence of $\delta\pi_s$, the equations

$$(8.10)\quad \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}\right) a_{ks} = \Pi_s \quad (k = 1, 2, \ldots, n;\ s = 1, 2, \ldots m).$$

We now show that the expression in the brackets is equal to $\partial S/\partial \ddot{q}_k$, where

$$S = \frac{1}{2}\sum_{i=1}^{3N} m_i \ddot{x}_i^2$$

is regarded as a function of the variables q_k, $\dot{q}_k$, $\ddot{q}_k$, and t $(k = 1, 2, \cdots, n)$, since the following relations hold:

$$x_i = f_i(q_k, t)\ (i = 1, 2, \ldots, 3N;\ k = 1, 2, \ldots, n).$$

Differentiating these equations with respect to the time, we find

$$\dot{x}_i = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial q_k}\dot{q}_k \quad (i = 1, 2, \ldots, 3N;\ k = 1, 2, \ldots, n),$$

from which it follows that

$$(8.11)\quad \frac{\partial \dot{x}_i}{\partial q_k} = \frac{\partial \ddot{x}_i}{\partial \dot{q}_k} = \frac{\partial \dddot{x}_i}{\partial \ddot{q}_k}.$$

Using this equation, we find

$$\frac{\partial S}{\partial \ddot{q}_k} = m_i \ddot{x}_i \frac{\partial \dddot{x}_i}{\partial \ddot{q}_k} = m_i \ddot{x}_i \frac{\partial \ddot{x}_i}{\partial \dot{q}_k} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - m_i \dot{x}_i \frac{d}{dt}\frac{\partial \dot{x}_i}{\partial \dot{q}_k}$$

$$= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}.$$

Equation (8.6) can now be written in the form

$$\frac{\partial S}{\partial \ddot{q}_k} a_{ks} = \Pi_s,$$

after which, using (8.9), we arrive at Appell's equations (8.6). In calculating the function S, one can exploit a theorem similar to Koenig's well-known theorem. For let $\boldsymbol{r}_\nu$ be the radius vector of the νth point of the system; $\boldsymbol{r}_0$, the radius vector of the center of mass; and $\boldsymbol{r}'_\nu$, the radius vector from the center of mass to the νth point of the system. Then, obviously,

$$\boldsymbol{r}_\nu = \boldsymbol{r}_0 + \boldsymbol{r}'_\nu$$

and hence

$$\boldsymbol{w}_\nu = \boldsymbol{w}_0 + \boldsymbol{w}'_\nu.$$

Substituting this equation into the expression (8.5) for the acceleration function, we have

$$S = \frac{1}{2} m_\nu w_\nu^2 = \frac{1}{2} m w_0^2 + \boldsymbol{w}_0 \sum_{\nu=1}^{N} m_\nu \boldsymbol{w}'_\nu + \frac{1}{2} m_\nu w_\nu'^2 \quad (\nu = 1, 2, \ldots, N),$$

where $m = \sum_{\nu=1}^{N} m_\nu$ is the mass of the complete system. We note that

$$m_\nu \boldsymbol{w}'_\nu = m_\nu \ddot{\boldsymbol{r}}'_\nu = \frac{d^2}{dt^2}(m_\nu \boldsymbol{r}'_\nu) = 0,$$

since, by the definition of the center of mass,

$$\sum_{\nu=1}^{N} m_\nu \boldsymbol{r}'_\nu = 0.$$

Thus,

$$S = \frac{1}{2} m w_0^2 + \frac{1}{2} m_\nu w_\nu'^2 = S_0 + S', \tag{8.12}$$

which is the analog of Koenig's theorem.

Let us consider examples of the determination of the acceleration function S in the case of the motion of an absolutely rigid body.

EXAMPLE 1. *Rigid body with a fixed point.* We shall now determine the acceleration function of a rigid body rotating about a fixed point O in a system of coordinates $Ox'y'z'$ that moves in the space $Oxyz$ and also relative to the body. Let $\boldsymbol{\omega}(p, q, r)$ be the angular velocity of the body and $\Omega(P, Q, R)$ be the angular velocity of the system of coordinates $Ox'y'z'$ expressed in terms of the projections onto the axes Ox', Oy', Oz'.

By definition,

$$S' = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu (w_{\nu x'}^2 + w_{\nu y'}^2 + w_{\nu z'}^2), \tag{8.13}$$

where N is the number of material points composing the body. The acceleration of the νth point can be expressed in the form

$$\boldsymbol{w}_\nu = \frac{d\boldsymbol{v}_\nu}{dt} = \frac{\tilde{d}}{dt}(\boldsymbol{\omega} \times \boldsymbol{r}_\nu) + \Omega \times (\boldsymbol{\omega} \times \boldsymbol{r}_\nu),$$

where we have used formula (0.1) of Chapter II, which connects the absolute and relative derivatives.

We perform an auxiliary calculation

$$\frac{\tilde{d}}{dt}(\boldsymbol{\omega}\times\boldsymbol{r}_\nu)=\frac{\tilde{d}\boldsymbol{\omega}}{dt}\times\boldsymbol{r}_\nu+\boldsymbol{\omega}\times\frac{\tilde{d}\boldsymbol{r}_\nu}{dt}=\frac{\tilde{d}\boldsymbol{\omega}}{dt}\times\boldsymbol{r}_\nu+\boldsymbol{\omega}\times[(\boldsymbol{\omega}-\boldsymbol{\Omega})\times\boldsymbol{r}_\nu],$$

after which we obtain

$$\boldsymbol{w}_\nu=\frac{\tilde{d}\boldsymbol{\omega}}{dt}\times\boldsymbol{r}_\nu+\boldsymbol{\omega}\times[(\boldsymbol{\omega}-\boldsymbol{\Omega})\times\boldsymbol{r}_\nu]+\boldsymbol{\Omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_\nu),$$

and, hence,

$$\begin{aligned}
w_{\nu x'} &= -x'_\nu(q^2+r^2)+y'_\nu[q(p-P)+pQ-\dot{r}]\\
&\quad +z'_\nu[r(p-P)+pR+\dot{q}],\\
w_{\nu y'} &= -y'_\nu(r^2+p^2)+z'_\nu[r(q-Q)+qR-\dot{p}]\\
&\quad +x'_\nu[p(q-Q)+qP+\dot{r}],\\
w_{\nu z'} &= -z'_\nu(p^2+q^2)+x'_\nu[p(r-R)+rP-\dot{q}]\\
&\quad +y'_\nu[q(r-R)+rQ+\dot{p}].
\end{aligned}\tag{8.14}$$

Substituting these projections into (8.13), we obtain the desired expression for S. We note that only the terms containing the derivatives $\dot{p}$, $\dot{q}$, and $\dot{r}$ should be retained on the right-hand side of (8.13).

We now write down an expression for the function S for two special cases.

1. The axes of the moving system $Ox'y'z'$ are attached to the body and coincide with the axes of the ellipsoid of inertia. In this case, $p=P$, $q=Q$, $r=R$ and, taking A, B, and C as the principal moments of inertia, we obtain

$$\begin{aligned}
S'=\frac{1}{2}(A\dot{p}^2+B\dot{q}^2+C\dot{r}^2)+(C-B)\dot{p}qr+(A-C)\dot{q}rp\\
+(B-A)\dot{r}pq.
\end{aligned}\tag{8.15}$$

Using this function when writing down Appell's equations, we arrive at Euler's equations of motion of a rigid body with a fixed point.

2. The system possesses axial symmetry and the Oz' axis of the moving system coincides with the symmetry axis of the body.[12] In this case, $p=P$, $q=Q$, but $r\neq R$. All the products of inertia vanish and, in addition, $A=B\neq C$. Using this fact, we find

$$S'=\frac{1}{2}[A(\dot{p}^2+\dot{q}^2)+C\dot{r}^2]+(Cr-AR)(\dot{p}q-\dot{q}p).\tag{8.16}$$

In this case, Appell's equations have the form

[12] Such axes are widely used in the study of gyroscopes (see [212]).

$$\left.\begin{aligned} A\dot{p} + (Cr - AR)\,q &= M_{x'}, \\ A\dot{q} + (AR - Cr)\,p &= M_{y'}, \\ C\dot{r} &= M_{z'}, \end{aligned}\right\} \tag{8.17}$$

where R is assumed to be a known function.

EXAMPLE 2. *Homogeneous sphere on a rotating plane.* We now write down Appell's function for the problem of the sphere considered in §5. Appell's function (the acceleration energy) has the form

$$S = \frac{m}{2} w_0^2 + S',$$

where w_0 is the acceleration of the center of mass. The function S' can easily be obtained by using the expressions (8.14), in which, in the case of a sphere, $A = B = C = mk^2$. In addition, for a system of coordinates in translational motion, $P = Q = R = 0$. We then have $(m = 1)$

$$S' = \frac{1}{2} k^2(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2).$$

From the condition of rolling without sliding (5.43)

$$\dot{x} = a\omega_2 - \Omega y, \quad \dot{y} = -a\omega_1 + \Omega x$$

we find

$$\ddot{x} = a\dot{\omega}_2 + \Omega\,(a\omega_1 - \Omega x),$$
$$\ddot{y} = -a\dot{\omega}_1 + \Omega\,(a\omega_2 - \Omega y),$$

and, hence,

$$w_0^2 = a^2\,(\dot{\omega}_1^2 + \dot{\omega}_2^2) + 2a\Omega\,[\dot{\omega}_2\,(a\omega_1 - \Omega x) - \dot{\omega}_1\,(a\omega_2 - \Omega y)] + \ldots.$$

Thus,

$$S = \frac{1}{2}[(a^2 + k^2)(\dot{\omega}_1^2 + \dot{\omega}_2^2) + k^2\dot{\omega}_3^2] + a\Omega\,[\dot{\omega}_2\,(a\omega_1 - \Omega x) - \dot{\omega}_1\,(a\omega_2 - \Omega y)].$$

Writing down Appell's equations, we obtain equations of motion identical with (5.42).

EXAMPLE 3. *Rolling without sliding of a homogeneous heavy disc on a horizontal plane.* Let the disc have unit mass, $m = 1$. We shall denote by w_0 the acceleration of the center of mass G and choose the directions of the axes $Gx'y'z'$ with origin at the center of the disc in such a way that Gx' is directed toward the point

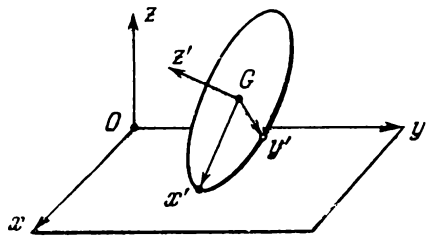

FIGURE 3.12

of contact of the disc and the plane and Gz' is perpendicular to the plane of the disc (Figure 3.12). In accordance with (8.12), the acceleration energy of the disc is

$$S = \frac{1}{2} w_0^2 + S',$$

where S' is given by (8.16). It remains to calculate w_0^2. Denoting by u', v', and z' the projections of the velocity of the center of the disc onto the axes $Gx'y'z'$ and by a the radius of the disc, we write down the condition for rolling without sliding

$$u' = 0, \quad v' + ar = 0, \quad w' = aq. \tag{8.18}$$

The projections of $\boldsymbol{w}_0$ onto the moving axes $Gx'y'z'$ are given by the expressions

$$\dot{u}' + qw' - Rv', \quad \dot{v}' + Ru' - pw', \quad \dot{w}' + pv' - qu'$$

or, with allowance for (8.18),

$$a(q^2 + rR), \quad -a(\dot{r} + pq), \quad a(\dot{q} - pr).$$

Hence,

$$w_0^2 = a^2[\dot{r}^2 + \dot{q}^2 + 2p(q\dot{r} - r\dot{q})] + \ldots.$$

Thus, we have the following expression for the acceleration energy of the disc:

$$S = \frac{1}{2}[A\dot{p}^2 + (A + a^2)\dot{q}^2 + (C + a^2)\dot{r}^2] + (AR - Cr)(p\dot{q} - \dot{q}p) + a^2p(q\dot{r} - r\dot{q}). \tag{8.19}$$

EXAMPLE 4. *Dynamics of free soaring of a disc without allowance for the parachute effect.* Suppose that a thin uniform disc of unit mass falls under the influence of the force of gravity. The absence of the parachute effect is manifested by the fact that the vector of the velocity of the center of disc always lies in the plane of the disc. We take the Oz axis of a fixed system of coordinates $Oxyz$ along the vector of the angular momentum $\boldsymbol{K}$, whose direction and magnitude remains constant because of the vanishing of the moment of the external forces (Figure 3.13). We shall take Ox horizontal. We denote the angle between the vector $\boldsymbol{K}$ and the vertical

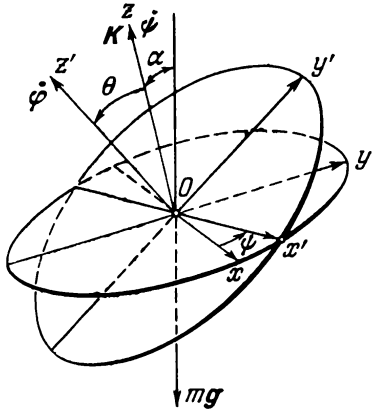

FIGURE 3.13

by α. The position of the system is determined by six parameters: x, y, and z, the coordinates of the center of gravity of the disc, and θ, ψ, and φ, the Eulerian angles of rotation. The configuration space consists of the topological product of a three-dimensional projective space and a three-dimensional Euclidean space. A single constraint is imposed on the virtual displacements of the system:

$$\sin\theta \sin\psi\delta x - \sin\theta\cos\psi\delta y + \cos\theta\delta z = 0, \tag{8.20}$$

which is nonholonomic. Thus, the system has five degrees of freedom.

We introduce a moving system of coordinates $Ox'y'z'$ such that the plane $x'Oy'$ coincides with the plane of the disc and the Ox' axis is directed along the "line of nodes" (see Figure 3.13). We now write down the equations of motion of the disc in terms of the projections onto the axes of the coordinate system $Ox'y'z'$. Let $v_C(u,v,0)$ be the velocity of the center of gravity of the disc; let

$$\boldsymbol{F}[-g\sin\alpha\sin\psi, -g(\cos\alpha\sin\theta + \sin\alpha\cos\theta\cos\psi), \\ g(\sin\alpha\sin\theta\cos\psi - \cos\alpha\cos\theta)]$$

be the force of gravity; $\omega(p,q,r)$, the instantaneous angular velocity of the disc; and $\Omega(p,q,q\cot\theta)$, the instantaneous angular velocity of the trihedron $Ox'y'z'$. We shall use Appell's equations (8.6). The acceleration energy of the disc is

$$S = \frac{1}{2}[\dot{u}^2 + \dot{v}^2 + A(\dot{p}^2 + \dot{q}^2) + C\dot{r}^2] + q\cot\theta(\dot{v}u - \dot{u}v) \\ + (Aq\cot\theta - Cr)(p\dot{q} - q\dot{p}) + \ldots.$$

The generalized forces are determined by the expression

$$\Pi_x' = -g\sin\alpha\sin\psi, \quad \Pi_{\cdot}' = -g(\sin\alpha\cos\theta\cos\psi + \cos\alpha\sin\theta).$$

Hence, we find the equations of motion of the disc

$$\begin{aligned} &\dot{u} - qv\cot\theta = -g\sin\alpha\sin\psi, \\ &\dot{v} + qu\cot\theta = -g.(\sin\alpha\cos\theta\cos\psi + \cos\alpha\sin\theta), \\ &A\dot{p} - q(aq\cot\theta - Cr) = 0, \\ &A\dot{q} + p(aq\cot\theta - Cr) = 0, \\ &C\dot{r} = 0. \end{aligned} \tag{8.21}$$

The equations (8.21) can be integrated without difficulty. From the last equation, we obtain

$$r = r_0 = \text{const}.$$

Since the vector $\boldsymbol{K}$ and also its projection onto the Oz' axis is constant,

we have $\theta = \theta_0 = \text{const}$. From the kinematic formulas

$$p = \dot{\theta}, \quad q = \dot{\psi} \sin \theta, \quad r = \dot{\varphi} + \dot{\psi} \cos \theta$$

we find $p = 0$ and, in accordance with the third equation of (8.21),

$$q = q_0 = \frac{Cr_0}{A \cot \theta_0}.$$

Further, $\psi = \omega_1 t$, where $\omega_1 = q_0 \sin^{-1}\theta_0$ and we have assumed $\psi_0 = 0$. From the last kinematic relation, we obtain $\varphi = \omega_2 t$, where $\omega_2 = -(C - A)r_0/A$, and we have assumed $\varphi_0 = 0$. Thus, the disc executes regular retrograde precession (with respect to the vector $\boldsymbol{K}$). We now find the motion of the center of the disc. The two first equations of (8.21) now take the form

$$\dot{u} - q_0 v \cot \theta_0 = -g \cos \alpha \sin \omega_1 t,$$
$$\dot{v} + q_0 u \cot \theta_0 = -g (\cos \alpha \sin \theta_0 + \sin \alpha \cos \theta_0 \cos \omega_1 t),$$

from which we obtain the following solution:

$$u = B \sin (\omega t - \beta) - A_1 \cos \omega_1 t - A_2,$$
$$v = B \cos (\omega t - \beta) + A_3 \sin \omega_1 t,$$

where B and β are arbitrary constants and we have introduced the notation

$$\omega = q_0 \cot \theta_0, \quad A_1 = \frac{g}{\omega^2 - \omega_1^2} \left(q_0 \sin \alpha \frac{\cos^2 \theta_0}{\sin \theta_0} + \omega_1 \cos \alpha \right),$$

$$A_2 = \frac{q_0 g}{\omega^2} \cos \theta_0 \cos \alpha, \quad A_3 = \frac{1}{\omega} (A_1 \omega_1 + g \cos \alpha).$$

The trajectory of the center of the disc can be found from the relations

$$\dot{x} = u \cos \omega_1 t - v \cos \theta_0 \sin \omega_1 t,$$
$$\dot{y} = u \sin \omega_1 t + v \cos \theta_0 \cos \omega_1 t,$$
$$\dot{z} = v \sin \theta_0.$$

Integrating these relations, we find

$$\begin{aligned} x = {} & C_1 \cos [(\omega + \omega_1)\, t - \beta] - C_2 \cos [(\omega - \omega_1)\, t - \beta] \\ & + \frac{1}{4} (A_3 \cos \theta_0 - A_1) \sin 2\omega_1 t - \frac{A_2}{\omega_1} \sin \omega_1 t \\ & - \frac{1}{2} (A_3 \cos \theta_0 + A_1)\, t, \end{aligned}$$

(8.22)

$$\begin{aligned} y = {} & C_1 \sin [(\omega + \omega_1)\, t - \beta] + C_2 \sin [(\omega - \omega_1) t - \beta] \\ & - \frac{1}{4} (A_3 \cos \theta_0 - A_1) \cos 2\omega_1 t + \frac{A_2}{\omega_1} \cos \omega_1 t, \\ z = {} & C_3 \sin (\omega t - \beta) - \frac{A_3 \cos \theta_0}{\omega_1} \cos \omega_1 t, \end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants. It follows from the expressions (8.22) that the center of the disc executes complex rotational motions about a point which moves in the horizontal direction along the Ox axis with a constant velocity $\frac{1}{2}(A_3\cos\theta_0 + A_1)$. The presence of the nonholonomic constraints (8.20) and the absence of dissipation leads to a motion which, on the average, takes place at the same height, i.e., the force of gravity does not, on the average, appear to influence the motion of the disc.

§9. Impulsive motion of nonholonomic systems

As in the dynamics of ordinary systems, an impulsive motion of a nonholonomic system arises when it is subjected to the influence of impact forces $\boldsymbol{F}^{\text{im}}$ (which may be external or internal). These are forces of very large magnitude, but they act for only a very short interval of time $t_1 - t_0 \to 0$, in such a way that the quantity

$$\hat{\boldsymbol{F}} = \lim_{\substack{F\to\infty \\ t_1 - t_0 \to 0}} \int_{t_0}^{t_1} \boldsymbol{F}^{\text{im}}\, dt, \tag{9.1}$$

which is called the impulse, remains finite. We can readily see from general considerations that nonholonomy introduces nothing that is fundamentally new in the case of impulsive motion. For at the moment of impact the coordinates and time may be assumed fixed and the coefficients of the kinematic characteristics in the equations of the nonholonomic constraints are effectively constant quantities. The nonholonomy effect is therefore not manifested in an impulsive motion. Consequently, the form of the equations of impulsive motions of nonholonomic systems must be the same as in the case of holonomic systems, despite the fact that the equations differ radically in the case of ordinary (nonimpulsive) motions.

We can convince ourselves of the truth of this assertion directly. In order to obtain the equations of impulsive motions of a nonholonomic system, we shall consider two cases.

1. The impulses acting on a nonholonomic system are given, i.e., they are known quantities.

2. Nonholonomic constraints are applied instantaneously to a system, i.e., the resulting impulsive forces are not known in advance.

The general case of impulsive motion is obviously found by combining these special cases.

A. *The case of given impulsive forces.* Let the position of a nonholonomic system be determined by n coordinates $q_1, q_2, \cdots, q_n$ that are connected by the nonintegrable relations

$$\dot{q}_{m+k} = a_{ki}\dot{q}_i + a_k \qquad (k = 1, 2, \ldots, n - m;\ i = 1, 2, \ldots, m). \tag{9.2}$$

If an impulsive force Q_s^{im} begins to act at the instant of time t_0, then the velocities $(\dot{q}_1)_0, \cdots, (\dot{q}_m)_0, (\dot{q}_{m+1})_0, \cdots, (\dot{q}_n)_0$ will have changed their values at the end of the interval $t_1 - t_0$ by a finite amount, taking the new values $(\dot{q}_1)_1, \cdots, (\dot{q}_m)_1, (\dot{q}_{m+1})_1, \cdots, (\dot{q}_n)_1$, but, to a first approximation, the coordinates themselves $q_1, q_2, \cdots, q_n$ remain constant during the impact.

We now write down the general equation of dynamics in Cartesian coordinates:

$$\sum_{i=1}^{3N} (m_i\ddot{x}_i - X_i)\,\delta x_i = 0,$$

where N is the number of material points composing the given system. By means of the well-known transformation, this equation can be reduced to the form

$$\sum_{k=1}^{m} \left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_k} - R_k - Q_k \right) \delta q_k = 0, \tag{9.3}$$

where T^* is the kinetic energy written down with allowance for the relations (9.2); Q_k is the generalized force; and R_k is, in general, a function of all the coordinates $q_1, q_2, \cdots, q_n$ and velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_m, \cdots, \dot{q}_n$, and contains the nonholonomy terms. If the system is holonomic, i.e., the relations (9.2) are integrable, then the function R_k satisfies

$$R_k = \frac{\partial T^*}{\partial q_k}.$$

By virtue of the independence of the variations δq_k, the sum (9.3) can be split up into m equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_k} - R_k = Q_k, \tag{9.4}$$

which describe the motion of the nonholonomic system at any instant of time, including times t in the interval $t_0 < t < t_1$, when the infinitely large impulsive forces Q_k^{im} are applied together with the finite generalized forces Q_k. In view of the fact that, in the limit, the impulsive forces act instantaneously, we are only interested in the integrated effects during the impact time $t_1 - t_0$. We integrate the equations of motion (9.4) from t_0 to t_1:

$$\frac{\partial T^*}{\partial \dot{q}_k}\Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} (R_k + Q_k)\,dt = \int_{t_0}^{t_1} Q_k^{\text{im}}\,dt.$$

Since the velocities and their variations during the time of the impact are finite, the integrand is also a finite quantity. Therefore, by the mean value theorem, the first integral tends to zero as $(t_1 - t_0) \to 0$. As regards the second integral, it is, in accordance with (9.1), the generalized impulse

$$\hat{Q}_k = \lim_{\substack{t_1 \to t_0 \\ Q_k^{\text{im}} \to \infty}} \int_{t_0}^{t_1} Q_k^{\text{im}}\, dt,$$

which, by hypothesis, is a known finite quantity.

Thus, we finally obtain

$$\left(\frac{\partial T^*}{\partial \dot{q}_k}\right)_1 - \left(\frac{\partial T^*}{\partial \dot{q}_k}\right)_0 = \hat{Q}_k \quad (k = 1, 2, \ldots, m), \tag{9.5}$$

where the subscript 1 refers to the post-impact state and the subscript 0 to the pre-impact state. These equations for the impulsive motions of nonholonomic systems have exactly the same form as the analogous equations for holonomic systems. The only difference is that the number of equations for nonholonomic systems is reduced by the number of nonholonomic constraints and the kinetic energy T^* contains only independent parameters, i.e., it is written down with allowance for the nonholonomic constraints. In equations (9.5), all the pre-impact velocities $(\dot{q}_1)_0, (\dot{q}_2)_0, \cdots, (\dot{q}_n)_0$ are known as well as $\hat{Q}_k$. The post-impact velocities $(\dot{q}_1)_1, (\dot{q}_2)_1, \cdots, (\dot{q}_m)_1$ can be determined from (9.5) and the remaining $(\dot{q}_{m+1})_1, (\dot{q}_{m+2})_1, \cdots, (\dot{q}_n)_1$ from the relations

$$(\dot{q}_{m+k})_1 - (\dot{q}_{m+k})_0 = \sum_{i=1}^{m} a_{ki}\,[(\dot{q}_i)_1 - (\dot{q}_i)_0] \quad (k = 1, \ldots, n - m), \tag{9.6}$$

which can be obtained from the condition that the nonholonomic constraints (9.2) are not violated. The impulsive forces of the reaction of the constraints which arise when the given impulses are applied to the system do not enter the equations of impulsive motion (9.5), since we have assumed from the very beginning that all constraints imposed on the system are ideal and that the constraints are maintained during the impact.

B. *The case of instantaneous application of nonholonomic constraints.* Suppose that prior to the application of the constraints a system has m degrees of freedom and that, at the time t_0, $m - l$ ideal nonholonomic constraints of the form

$$\dot{q}_{l+i} = a_{is}\dot{q}_s + a_i \quad (i = 1, 2, \ldots, m - l;\ s = 1, 2, \ldots, l) \tag{9.7}$$

are imposed on the system, which is not subjected to external impulses.

Then, during the impact, the variations of the coordinates are restricted by the additional relations

$$\delta q_{l+i} = a_{is}\delta q_s \quad (i = 1, 2, \ldots, m - l;\ s = 1, 2, \ldots, l), \tag{9.8}$$

so that the number of independent virtual displacements is now l $(l < m)$. Substituting the expressions (9.8) into the general equation of dynamics (9.3), we obtain

$$\left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_s} + a_{is}\frac{\partial T^*}{\partial \dot{q}_{l+i}}\right) - R_s - Q_s - (R_{l+i} + Q_{l+i})\,a_{is} - \frac{\partial T^*}{\partial \dot{q}_{l+1}}\frac{da_{is}}{dt}\right]\delta q_s = 0$$

$$(s = 1, 2, \ldots, l;\ \ i = 1, 2, \ldots, m - l).$$

By virtue of the independence of the variations δq_s, this sum can be split up into l equations, which, after integration from t_0 to t_1 as above, take the form

$$\left(\frac{\partial T^*}{\partial \dot{q}_s} + a_{is}\frac{\partial T^*}{\partial \dot{q}_{l+i}}\right)_1 = \left(\frac{\partial T^*}{\partial \dot{q}_s} + a_{is}\frac{\partial T^*}{\partial \dot{q}_{l+i}}\right)_0 \quad \begin{pmatrix} i = 1, 2, \ldots, m - l; \\ s = 1, 2, \ldots, l \end{pmatrix}. \tag{9.9}$$

These are the desired equations for the case when nonholonomic constraints are applied instantaneously. If we denote by $\widetilde{T}^*$ the function T^* which is a function of only $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_l$ and depends on the latter not only explicitly but also implicitly through $\dot{q}_{l+1}, \cdots, \dot{q}_m$ by virtue of the relations (9.7), equations (9.9) can be written in the more compact form

$$\left(\frac{\partial \widetilde{T}^*}{\partial \dot{q}_s}\right)_1 = \left(\frac{\partial \widetilde{T}^*}{\partial \dot{q}_s}\right)_0 \quad (s = 1, 2, \ldots, l). \tag{9.10}$$

Thus, in the case when nonholonomic constraints are applied instantaneously, the m pre-impact and m post-impact velocities are connected by the system of l equations (9.10).

It should be noted that although equations (9.9) may be written formally in the form (9.10), it would be incorrect to eliminate all the velocities $\dot{q}_{l+1}, \dot{q}_{l+2}, \cdots, \dot{q}_m$ at the start in the expression for T^* by means of the relations (9.7) and then differentiate the resulting function with respect to the velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_l$. For this case differs from the preceding in that the relations (9.7) do not hold prior to the impact and it would therefore be incorrect to use them to eliminate the velocities $\dot{q}_{l+1}, \dot{q}_{l+2}, \cdots, \dot{q}_m$ from the expressions $(\partial \widetilde{T}^*/\partial \dot{q}_s)_0$. As regards the

question of whether one can use the relations (9.7) to eliminate the velocities $\dot{q}_{l+1}, \dot{q}_{l+2}, \cdots, \dot{q}_m$ from the expressions $(\partial \widetilde{T}^*/\partial \dot{q}_s)_1$, which characterize the post-impact state, two subcases must be distinguished: 1) The instantaneously applied nonholonomic constraints continue to act; 2) the instantaneously applied constraints are lifted immediately after the completion of the impact.

Physically, the first subcase corresponds to an absolutely inelastic impact, when the so-called coefficient of restitution vanishes, $\kappa = 0$, and the second subcase corresponds to an absolutely elastic impact ($\kappa = 1$) and also all intermediate cases when the coefficient of restitution lies in the interval $0 < \kappa < 1$.

Obviously, in the first subcase, the constraint equations (9.7) are valid not only during the impact but also remain in force for the post-impact state. Therefore, the relations (9.7) may be used to eliminate $\dot{q}_{l+1}, \dot{q}_{l+2}, \cdots, \dot{q}_m$ from the expressions $(\partial \widetilde{T}^*/\partial \dot{q}_s)_1$, after which (9.10) is a system of l algebraic equations for the determination of the l independent post-impact velocities $(\dot{q}_1)_1, (\dot{q}_2)_1, \cdots, (\dot{q}_l)_1$, provided the pre-impact velocities $(\dot{q}_1)_0, (\dot{q}_2)_0, \cdots, (\dot{q}_l)_0$, $(\dot{q}_{l+1})_0, \cdots, (\dot{q}_m)_0$ are known, i.e. the problem is well defined.

In the second subcase, the relations (9.7) hold neither before nor after the impact and, consequently, they cannot be used to eliminate the velocities $\dot{q}_{l+1}, \dot{q}_{l+2}, \cdots, \dot{q}_m$ from $(\partial \widetilde{T}^*/\partial \dot{q}_s)_1$ or from $(\partial \widetilde{T}^*/\partial \dot{q}_s)_0$. Thus, in the subcase $0 < \kappa < 1$, we have the l equations (9.10) for the determination of the m $(m > l)$ post-impact velocities $(\dot{q}_1)_1, (\dot{q}_2)_1, \cdots, (\dot{q}_l)_1, (\dot{q}_{l+1})_1, \cdots, (\dot{q}_m)_1$. In order to be able to solve the problem, one must have additional information on the nature of the impact or the imposed constraints which enables one to write down a further $m - l$ relations, and thus obtain a closed system of equations.

The foregoing theory of the impulsive motion of nonholonomic systems applies to systems with ideal constraints. We shall now consider the problem of impulsive motion of nonholonomic systems in the general case when the constraints need not be ideal.

In cases when the impulsive forces are given and also when nonholonomic constraints are applied instantaneously and continue to apply after the impact, one can write down equations of impulsive motion containing impulsive reactions whose number is equal to the number of degrees of freedom of the system in the pre-impact state. Thus, if n and m are, respectively, the number of degrees of freedom of the system before and after the instantaneous application of the constraints, we have n equations for $2n - (n - m) = n + m$ unknowns.

In order to obtain a complete system of equations, we require a further m equations, which may be obtained either from the condition that the constraints are ideal or from other additional conditions which permit us to write down these m equations.

In the case when nonholonomic constraints are applied instantaneously to a system and then lifted immediately after the impact ($0 < \kappa \leqq 1$), one can, introducing the impulses of the reactions of the constraints, write down n equations for $2n$ unknowns (the n post-impact velocities and n impulsive forces of the reactions of the constraint). The assumption of ideal constraints does not permit one in this case to write down a complete system of equations. For if $(n - m)$ is the number of constraints imposed instantaneously, the assumption that they are ideal yields only $n - (n - m) = m$ equations and, consequently, $n - m$ unknowns remain undetermined. Therefore, in order to solve the problem completely in the given case, one must have additional information about the character of the imposed constraints which permits one to write down a further $n - m$ equations.

To conclude this section, we shall consider some examples of impulsive motion.

EXAMPLE 1. An impulse is applied to the front of a sleigh on a horizontal plane in a direction perpendicular to the plane of the runner. Find the impulsive motion.

Let us consider the simplest model of a sleigh (Figure 3.14) where A is the point of contact of the runner and the plane, C is the position of the center of mass, and B is the point of application of the impulse $\hat{F}$.

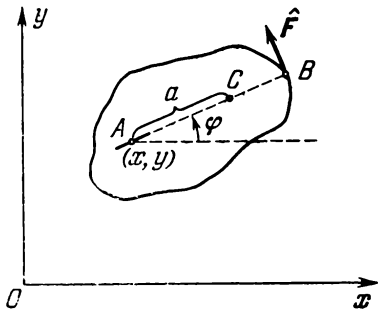

FIGURE 3.14

The kinetic energy of the system without allowance for the nonholonomic constraint $dy - \tan\varphi\, dx = 0$ has the form

$$T = \frac{1}{2} m [(\dot{x} - a\dot{\varphi}\sin\varphi)^2 + (\dot{y} + a\dot{\varphi}\cos\varphi)^2 + k^2\dot{\varphi}^2],$$

where m is the mass of the sleigh and k is the radius of gyration about the center of mass. Using the equation of the nonholonomic constraint, we find

$$T^* = \frac{1}{2} m \left[\frac{\dot{x}^2}{\cos^2\varphi} + (a^2 + k^2)\dot{\varphi}^2 \right]. \tag{9.11}$$

We can find the generalized impulses corresponding to $\hat{F}$ by the general rules, using the expression for the virtual work

$$\delta A = -\hat{F} \sin\varphi\, \delta x + \hat{F} \cos\varphi\, \delta y + 2a\hat{F}\, \delta\varphi,$$

in which, in accordance with the equation of the nonholonomic constraint, only a single term remains. Hence, $\hat{Q}_\varphi = 2a\hat{F}$ and $\hat{Q}_x = 0$. Now, writing down equations (9.5) and bearing in mind that the sleigh is at rest prior to the application of the impulse, we obtain

$$(\dot{x})_1 = 0,\ m\,(a^2 + k^2)(\dot{\varphi})_1 = 2a\hat{F}.$$

EXAMPLE 2. A homogeneous circular disc rotates with angular velocity ω about one of its diameters. A point P on the circumference of the disc is suddenly stopped. We shall show that the velocity of the center immediately after the stopping of the point is 1/5 of the velocity of the point P before it is stopped.

Let m be the mass of the disc and α the angle between the radius of the point P and the fixed diameter about which the disc rotates prior to the stopping of the point P. The initial velocity of the point P is $r\omega\sin\alpha$, where r is the radius of the disc. The initial angular momentum of the body about the axis passing through the point P parallel to the original axis of rotation is $mr^2\omega/4$ and remains the same after the point P has been stopped. Therefore, the angular momentum of the disc about the tangent to the point P after the latter has been stopped has the value $\frac{1}{4}mr^2\omega\sin\alpha$. Since the moment of inertia of the disc about this tangent is $5mr^2/4$, the angular velocity of rotation of the disc about the tangent is $(1/5)\omega\sin\alpha$. Consequently, the center of the disc has the velocity $(1/5)r\omega\sin\alpha$, which is 1/5 of the original velocity of the point P.

EXAMPLE 3. A parallelogram is formed by two pairs of uniform rods which are connected to each other by ideal hinges. The lengths of the rods are $2a$ and $2b$; their masses, m and m_1; and their radii of gyration, k and k_1. The parallelogram is in translational motion along one of its diagonals with a velocity v. It strikes a smooth rigid wall perpendicular to the diagonal, the wall forming angles θ and φ with the sides. What is the magnitude of the impulse if the corner of the parallelogram which strikes the wall is brought

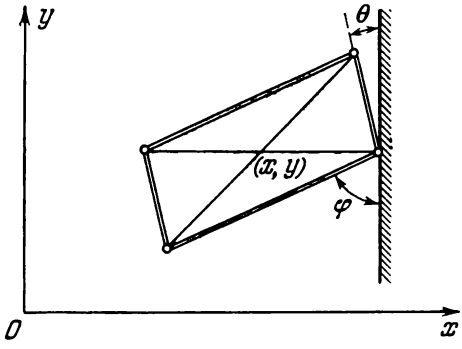

FIGURE 3.15

instantaneously to rest when the impact occurs?

Let x and y be the coordinates of the center of the parallelogram, where x is measured in the direction perpendicular to the wall (Figure 3.15). The kinetic energy of the system is

$$T = (m + m_1)(\dot{x}^2 + \dot{y}^2) + (mk^2 + m_1a^2)\,\dot{\theta}^2 + (mb^2 + m_1k_1^2)\dot{\varphi}^2.$$

At the instant of the impact the abscissa of the corner that strikes the wall is $x + a\sin\theta + b\sin\varphi$. Therefore, in an arbitrary displacement $(\delta x, \delta y, \delta\theta, \delta\varphi)$, its displacement along the x axis is $\delta x + a\cos\theta\delta\theta + b\cos\varphi\delta\varphi$. Hence, denoting the magnitude of the impulse by I, we obtain the equations of motion in the form

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 - \left(\frac{\partial T}{\partial \dot{x}}\right)_0 = -I,\quad \left(\frac{\partial T}{\partial \dot{\theta}}\right)_1 - \left(\frac{\partial T}{\partial \dot{\theta}}\right)_0 = -aI\cos\theta,$$
$$\left(\frac{\partial T}{\partial \dot{\varphi}}\right)_1 - \left(\frac{\partial T}{\partial \dot{\varphi}}\right)_0 = -bI\cos\varphi,$$

or

$$2\,(m + m_1)[(\dot{x})_1 - v] = -I,\quad 2\,(mk^2 + m_1a^2)\,(\dot{\theta})_1 = -aI\cos\theta,$$
$$2(mk^2 + m_1k_1^2)(\dot{\varphi})_1 = -bI\cos\varphi.$$

In addition, the condition that the point of contact is brought to rest yields

$$(\dot{x})_1 + a\,(\theta)_1\cos\theta + b\,(\varphi)_1\cos\varphi = 0.$$

Eliminating $(\dot{x})_1$, $(\dot{\theta})_1$, and $(\dot{\varphi})_1$ from these equations, we obtain

$$I = v\left[\frac{1}{2\,(m + m_1)} + \frac{a^2\cos^2\theta}{2\,(mk^2 + m_1a^2)} + \frac{b^2\cos^2\varphi}{2\,(mb^2 + m_1k_1^2)}\right]^{-1}.$$

EXAMPLE 4. A homogeneous sphere of radius a moves in an arbitrary manner in space. At the instant of time t_0 it strikes a rough horizontal plane. After the impact, it remains on the plane and is able to spin and roll on it without sliding. Determine the post-impact velocities.

We take the origin of a Cartesian system of coordinates $Oxyz$ on the plane with the Oz axis vertically upward. The kinetic energy of the sphere is

$$2T = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2),$$

where k is the radius of gyration about any diameter and the sphere is assumed to have unit mass.

Using the expressions for ω_x, ω_y, and ω_z in terms of the Eulerian angles θ, φ, and ψ

$$\begin{aligned}\omega_x &= \dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi,\\ \omega_y &= \dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi,\\ \omega_z &= \dot\varphi\cos\theta + \dot\psi,\end{aligned} \tag{9.12}$$

we transform the expression for the kinetic energy to the form

$$2T = \dot x^2 + \dot y^2 + \dot z^2 + k^2(\dot\theta^2 + \dot\varphi^2 + \dot\psi^2 + 2\dot\varphi\,\dot\psi\cos\theta). \tag{9.13}$$

When the sphere strikes the horizontal plane, the constraints

$$\dot x - a\omega_y = 0;\ \dot y + a\omega_x = 0;\ \dot z = 0,$$

are imposed instantaneously. They express the vanishing of the velocity of the point of contact of the sphere and the plane. In accordance with (9.12), these relations have the form

$$\begin{aligned}\dot x &= a(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi),\\ \dot y &= -a(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi),\\ \dot z &= 0,\end{aligned} \tag{9.14}$$

of which the first two are nonholonomic constraints. We now write down the equations of the impulsive motion (9.10). We differentiate (9.13), taking $\dot\theta$, $\dot\varphi$, and $\dot\psi$ as the independent velocities:

$$\begin{aligned}\frac{\partial T^*}{\partial\dot\theta} &= \frac{\partial T}{\partial\dot\theta} + \frac{\partial T}{\partial\dot x}\frac{\partial\dot x}{\partial\dot\theta} + \frac{\partial T}{\partial\dot y}\frac{\partial\dot y}{\partial\dot\theta} = k^2\dot\theta + a\dot x\sin\psi - a\dot y\cos\psi,\\ \frac{\partial T^*}{\partial\dot\varphi} &= k^2(\dot\varphi + \dot\psi\cos\theta) - a\dot x\sin\theta\cos\psi - a\dot y\sin\theta\sin\psi,\\ \frac{\partial T^*}{\partial\dot\psi} &= k^2(\dot\psi + \dot\varphi\cos\theta).\end{aligned}$$

Eliminating $(\dot x)_1$, $(\dot y)_1$, and $(\dot z)_1$ from the expressions $(\partial T^*/\partial\dot\theta)_1$, $(\partial T^*/\partial\dot\varphi)_1$, and $(\partial T^*/\partial\dot\psi)_1$, we finally obtain the following equations of the impulsive motion:[13]

$$\begin{aligned}k^2(\dot\theta)_0 + a(\dot x)_0\sin\psi - a(\dot y)_0\cos\psi &= (k^2 + a^2)(\dot\theta)_1,\\ k^2[(\dot\varphi)_0 + (\dot\psi)_0\cos\theta] - a(\dot x)_0\sin\theta\cos\psi - a(\dot y)_0\sin\theta\sin\psi &=\\ = (k^2 + a^2\sin^2\theta)(\dot\varphi)_1 + k^2(\dot\psi)_1\cos\theta,&\\ (\dot\psi)_0 + (\dot\varphi)_0\cos\theta &= (\dot\psi)_1 + (\dot\varphi)_1\cos\theta.\end{aligned}$$

[13] A "plane" variant of this problem, solved analogously, may be stated as follows. A sphere moves arbitrarily on an absolutely smooth horizontal plane. Find the motion of the sphere after it crosses the boundary into an absolutely rough part of the surface.

Hence, the post-impact angular velocities of the sphere can be expressed by the formulas:

$$(\dot\theta)_1 = \frac{1}{k^2+a^2}\,[k^2(\dot\theta)_0 + a(\dot x)_0 \sin\psi - a(\dot y)_0\cos\psi],$$

$$(\dot\varphi)_1 = \frac{1}{(k^2+a^2)\sin\theta}\,[k^2(\dot\varphi)_0\sin\theta - a(\dot x)_0\cos\psi - a(\dot y)_0\sin\psi],$$

$$(\dot\psi)_1 = \frac{1}{(k^2+a^2)\sin\theta}\,[(k^2+a^2)(\dot\psi)_0\sin\theta + a^2(\dot\varphi)_0\sin\theta\cos\theta + a(\dot x)_0\cos\theta\cos\psi + a(\dot y)_0\cos\theta\sin\psi].$$

The post-impact velocity of the center of mass has, by (9.14), the components

$$(\dot x_c)_1 = \frac{a}{a^2+k^2}\,[k^2(\omega_y)_0 + a(\dot x)_0],$$

$$(\dot y_c)_1 = \frac{a}{a^2+k^2}\,[a(\dot y)_0 - k^2(\omega_x)_0], \quad (\dot z_c)_1 = 0.$$

EXAMPLE 5. A homogeneous sphere rolls without sliding on a rough horizontal plane. At the instant of time t_0, it strikes a rough elastic vertical wall. Find the post-impact velocities (Figure 3.16).

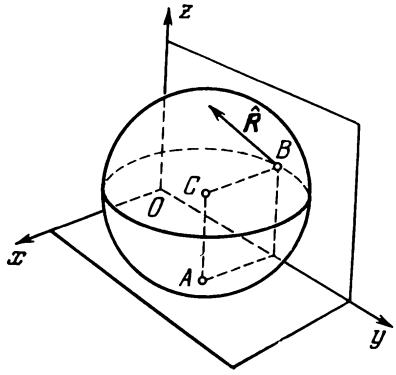

FIGURE 3.16

In this example, nonholonomic constraints are imposed instantaneously and then lifted immediately after the impact. Apart from these constraints, the system is subjected to the constantly acting nonholonomic constraints (9.14). Using the latter, we can obtain from (9.13) the following expression for the kinetic energy T^*:

$$2T^* = (a^2+k^2)\dot\theta^2 + (a^2\sin^2\theta + k^2)\dot\varphi^2 + k^2(\dot\psi^2 + 2\dot\varphi\dot\psi\cos\theta). \tag{9.15}$$

Let the vertical wall with which the sphere collides be parallel to the yOz plane. Then, when the sphere strikes the wall, the following additional constraints are instantaneously imposed on the system:

$$\dot x = 0, \quad \dot y + a\omega_z = 0, \quad \omega_y = 0.$$

In accordance with (9.14), these can be reduced to the two equations

$$\begin{aligned} &\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi = 0,\\ &\dot\theta\cos\psi + \dot\varphi(\sin\theta\sin\psi - \cos\theta) - \dot\psi = 0. \end{aligned} \tag{9.16}$$

Choosing $\dot{\varphi}$ as the independent velocity, we obtain from these relations

$$\dot{\psi} = \dot{\varphi}\left(\frac{\sin\theta}{\sin\psi} - \cos\theta\right), \dot{\theta} = \dot{\varphi}\sin\theta\cot\psi.$$

By virtue of (9.15) and (9.16), we have

$$\frac{\partial \widetilde{T}^*}{\partial \dot{\varphi}} = \left[(a^2 + k^2)\sin\theta + k^2\frac{\cos\theta}{\sin\psi}\right]\dot{\varphi}\sin\theta + k^2\dot{\psi}\frac{\sin\theta}{\sin\psi} + \\ + (a^2 + k^2)\dot{\theta}\sin\theta\cot\psi,$$

and the equations of the impulsive motion reduce to a single relation between the pre-impact and post-impact velocities

$$[(a^2 + k^2)\sin\theta\sin\psi + k^2\cos\theta]\,\Delta\dot{\varphi} + k^2\Delta\dot{\psi} + (a^2 + k^2)\,\Delta\dot{\theta}\cos\psi = 0, \tag{9.17}$$

where $\Delta\dot{\varphi} = (\dot{\varphi})_1 - (\dot{\varphi})_0$, $\Delta\dot{\psi} = (\dot{\psi})_1 - (\dot{\psi})_0$, $\Delta\dot{\theta} = (\dot{\theta})_1 - (\dot{\theta})_0$. In order to obtain a further two equations, we must make certain definite additional assumptions about the impact. If we wished to determine accurately the post-impact state of the sphere, we should have to take into account the deformability of both the sphere and the wall and, in an even more complete treatment, the finite velocity of propagation of the deformation waves in the body of the sphere. All this greatly complicates the problem and removes it from the field of the analytical mechanics of systems with a finite number of degrees of freedom. Therefore, without going into the mechanics of the impact process, we shall assume that the velocity (v_x, v_y, v_z) of the point of contact of the sphere and the wall remains equal to zero until the end of the impact. This condition is in complete agreement with the assumption (in accordance with the conditions of the problem) that the constant nonholonomic constraints imposed on the system are not violated. Such an assumption means that the relations (9.16) remain in force for the post-impact state as well. Thus, we have three equations (9.16) and (9.17) for the determination of the post-impact velocities $(\dot{\varphi})_1$, $(\dot{\psi})_1$, and $(\dot{\theta})_1$; hence

$$(a^2 + 2k^2)\,(\dot{\varphi})_1\frac{\sin\theta}{\sin\psi} = [(a^2 + k^2)\sin\theta\sin\psi + k^2\cos\theta](\dot{\varphi})_0 \\ + k^2(\dot{\psi})_0 + (a^2 + k^2)\,(\dot{\theta})_0\cos\psi,$$

$$(\dot{\psi})_1 = (\dot{\varphi})_1\left(\frac{\sin\theta}{\sin\psi} - \cos\theta\right), \quad (\dot{\theta})_1 = (\dot{\varphi})_1\frac{\sin\theta\cos\psi}{\sin\psi}.$$

The physical reason why we must make additional assumptions in

order to obtain a complete description of the effect of the impact on the rough wall in the case of rolling on a rough plane is connected with the fact that two points of the sphere are fixed at the instant of the impact. This leads us to the problem of an impact of a body with two fixed points. It is well known that this problem is undetermined in the framework of the idealization of an absolutely rigid body. In order to obtain a well-defined problem, it is customary to assume that one of the points of the body is partially freed (by assuming, for example, that a journal bearing is replaced by a sliding bearing). A similar freeing of one of the points can be made in our problem with a sphere.

As an example, we shall solve this problem under the assumption that the impact occurs on an absolutely smooth wall, the coefficient of restitution being κ. In this case, only the single constraint $\dot{x} = 0$ is imposed at the instant of the impact. Using the relation (9.14), we can express this constraint by the equation

$$\dot{\theta} \sin \psi - \dot{\varphi} \sin \theta \cos \psi = 0.$$

Using this relation and (9.15), we obtain the following equations of motion:

$$\left.\begin{aligned} (a^2 + k^2)\Delta\dot{\theta} \sin\theta \cos\theta + (a^2 \sin^2\theta + k^2)\,\Delta\dot{\varphi} \sin\psi \\ + k^2 \Delta\dot{\psi} \sin\psi \cos\theta = 0, \\ \Delta\dot{\psi} + \Delta\dot{\varphi} \cos\theta = 0, \end{aligned}\right\} \tag{9.18}$$

where

$$\Delta\dot{\theta} = (\dot{\theta})_1 - (\dot{\theta})_0, \quad \Delta\dot{\varphi} = (\dot{\varphi})_1 - (\dot{\varphi})_0, \quad \Delta\dot{\psi} = (\dot{\psi})_1 - (\dot{\psi})_0.$$

The third equation yields the well-known Carnot theorem, which holds in the case of smooth constraints

$$T_1 - T_0 = -\frac{1-\kappa}{1+\kappa} T_l, \tag{9.19}$$

where T_l is the so-called kinetic energy of the lost velocities. According to (9.15) and (9.19), we have

$$\begin{aligned} &(a^2 + k^2)\,[(\dot{\theta})_1^2 - (\dot{\theta})_0^2] + (a^2 \sin^2\theta + k^2)\,[(\dot{\varphi})_1^2 - (\dot{\varphi})_0^2] + k^2\,[(\dot{\psi})_1^2 \\ &\quad - (\dot{\psi})_0^2 + 2\,(\dot{\psi})_1\,(\dot{\varphi})_1 \cos\theta - 2\,(\dot{\psi})_0\,(\dot{\varphi})_1 \cos\theta] \\ &\quad = -\frac{1-\kappa}{1+\kappa}\{(a^2 + k^2)\,(\Delta\dot{\theta})^2 + (a^2 \sin^2\theta + k^2)\,(\Delta\dot{\varphi})^2 \\ &\qquad + k^2\,[(\Delta\dot{\psi})^2 - 2\Delta\dot{\psi}\Delta\dot{\varphi} \cos\theta]\}. \end{aligned} \tag{9.20}$$

The system of equations (9.18) and (9.20) solve the problem in the case of an impact on an absolutely smooth wall.

As another example of the "freeing" of one of the fixed points, we shall now consider a more "realistic" formulation. Suppose we have a heavy homogeneous sphere which rolls without sliding because of Coulomb forces of friction. As before, we wish to determine the velocity of the sphere after it has collided with a rough vertical wall. Let us suppose that the sphere rolls without sliding on the xOy plane and collides with the yOz plane (Figure 3.16). We shall assume that the impulsive force of friction, as in the case of finite forces, cannot exceed the μ-fold value of the normal pressure, where μ is the coefficient of dry friction. We should mention that when the sphere rolls up to the wall and collides with it, the impulse of the reaction of the wall at the point B always has a positive vertical component. As a result, the impulsive reaction at the point of contact A of the sphere and the horizontal plane is equal to zero. Consequently, the finite forces of friction at the point A during the time of the impact have no influence at all on the kinematics of the sphere, i.e., this constraint is effectively lifted during the impact.

We introduce some notation: $\hat{\boldsymbol{R}}(\hat{R}_x, \hat{R}_y, \hat{R}_z)$ is the impulse at the point B (see Figure 3.16); $\boldsymbol{V}(-u_0, v_0, 0)$ is the pre-impact velocity of the center of the sphere ($u_0 > 0$; $v_0 > 0$); $\boldsymbol{\omega}_0(\omega_{0x}, \omega_{0y}, \omega_{0z})$ is the pre-impact angular velocity, which, by virtue of the condition of rolling without sliding, is related to $\boldsymbol{V}$ by the equations

$$u_0 + a\omega_{0y} = 0, \quad v_0 + a\omega_{0x} = 0. \tag{9.21}$$

Thus, the pre-impact state can be characterized by three quantities, which we shall take to be u_0, v_0, and ω_{0z}. The equations of the impulsive motion are

$$\begin{aligned} &m(u_1 + u_0) = \hat{R}_x, \quad m(v_1 - v_0) = \hat{R}_y, \quad mw_1 = \hat{R}_z, \\ &mk^2(\omega_{x1} - \omega_{x0}) = 0, \quad mk^2(\omega_{y1} - \omega_{y0}) = a\hat{R}_z, \\ &mk^2(\omega_{z1} - \omega_{z0}) = -a\hat{R}_y, \end{aligned} \tag{9.22}$$

where m is the mass of the sphere and k its radius of gyration.

The form of the remaining three equations depends on the impact conditions and, in particular, on the values of the coefficient of restitution κ and the coefficient of friction μ. We shall assume that the normal component of the velocity satisfies Newton's well-known kinematic relation

$$u = \kappa u_0. \tag{9.23}$$

In order to obtain the conditions imposed on the tangential component of the velocity of the sphere at the point B, we consider two possible

cases. In the first case, the vector $\hat{\boldsymbol{R}}$ lies within the friction cone, i.e.,

$$|\hat{R}_y| < \mu |\hat{R}_x|, \quad |\hat{R}_z| < \mu |\hat{R}_x|. \tag{9.24}$$

At the same time, the point B of the sphere is not displaced tangentially before the end of the impact. Hence,

$$v_1 - a\omega_{z1} = 0, \quad w_1 + a\omega_{y1} = 0.$$

Thus, we have nine equations for the nine unknowns u_1, v_1, w_1, ω_{x1}, ω_{y1}, ω_{z1}, $\hat{R}_x$, $\hat{R}_y$, and $\hat{R}_z$. Solving these equations, we obtain

$$\left.\begin{aligned} &u_1 = \varkappa u_0, \quad v_1 = \frac{a(k^2\omega_{z0} + av_0)}{a^2 + k^2}, \quad w_1 = \frac{k^2 u_0}{a^2 + k^2}, \\ &\omega_{x1} = -\frac{v_0}{a}, \quad \omega_{y1} = -\frac{u_0 k^2}{a(a^2 + k^2)}, \quad \omega_{z1} = \frac{k^2\omega_{z0} + av_0}{a^2 + k^2}, \\ &\hat{R}_x = mu_0(1 + \varkappa), \quad \hat{R}_y = \frac{mk^2(a\omega_{z0} - v_0)}{a^2 + k^2}, \quad \hat{R}_z = \frac{mu_0 k^2}{a^2 + k^2}. \end{aligned}\right\} \tag{9.25}$$

This result shows that after the impact the sphere leaves the horizontal plane with the velocity

$$V = \frac{1}{a^2 + k^2}\{[\varkappa^2(a^2 + k^2)^2 + k^2]\, u_0^2 + a^2(k^2\omega_{z0} + av_0)^2\}^{1/2}.$$

This situation obtains if the following inequalities, obtained by substituting (9.25) into (9.24), are satisfied:

$$\mu(1 + \varkappa)(a^2 + k^2) > k^2,$$
$$\mu(1 + \varkappa)u_0(a^2 + k^2) > k^2 |a\omega_{z0} - v_0|.$$

Ignoring the intermediate cases, let us consider the conditions under which none of these three inequalities is satisfied. In this second case, the impulses of the forces of friction at the point B are connected to the normal impulse by the relations

$$\hat{R}_y = \mu\hat{R}_x, \quad \hat{R}_z = \mu\hat{R}_x,$$

which, together with (9.22) and (9.23), form a complete system of equations for the determination of the post-impact velocities and impulses of the reaction forces. Solving these equations for the unknowns, we find

$$\begin{aligned} &u_1 = \varkappa u_0, && \omega_{x1} = -\frac{v_0}{a}, \\ &v_1 = v_0 + \mu u_0(1 + \varkappa), && \omega_{y1} = -\frac{u_0}{a} + \frac{au_0\mu}{k^2}(1 + \varkappa), \\ &w_1 = \mu u_0(1 + \varkappa), && \omega_{z1} = \omega_{z0} - \frac{\mu a u_0}{k^2}(1 + \varkappa), \end{aligned}$$

$$\hat{R}_x = mu_0(1 + \varkappa),$$
$$\hat{R}_y = \mu mu_0(1 + \varkappa),$$
$$\hat{R}_z = \mu mu_0(1 + \varkappa).$$

Comparing these results with (9.25), we can readily see that, in the second case, the vertical component of the velocity of the sphere is less than in the first case, i.e., the sphere does not bounce as high.

EXAMPLE 6. A homogeneous disc of radius a moves in an arbitrary manner in space. At the instant of time t_0, its sharp edge strikes a horizontal plane. Determine the post-impact velocities of the disc if the coefficient of restitution is κ.

We take the Ox and Oy axes of a Cartesian system of coordinates in the horizontal plane with the Oz axis upward. If the disc has unit mass, its kinetic energy is

$$T = \frac{1}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + A(p^2 + q^2) + Cr^2], \tag{9.26}$$

where x, y, and z are the coordinates of the center of the disc; A and C are the transverse and axial moments of inertia of the disc; p, q, and r are the projections of the instantaneous angular velocity of the disc onto the axes $O_1\xi\eta\zeta$, where $O_1\zeta$ is perpendicular to the plane of the disc and the $O_1\xi$ axis is always horizontal. In accordance with Figure 3.17,

$$p = -\dot{\theta}, \quad q = \dot{\psi}\cos\theta, \quad r = \dot{\varphi} + \dot{\psi}\sin\theta. \tag{9.27}$$

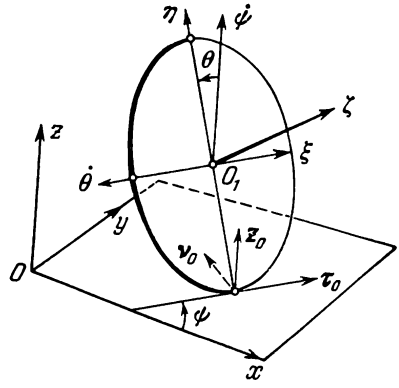

FIGURE 3.17

Substituting these relations into (9.26), we obtain

$$2T = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + A(\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta) + C(\dot{\varphi} + \dot{\psi}\sin\theta)^2. \tag{9.26'}$$

At the instant $t = t_0$, when the disc strikes the plane, the equation $v + \omega - r = 0$ holds, where r is the radius vector from the venter of the disk to its point of contact with the plane. Projecting this equation onto the $Oxyz$ axes, we obtain

$$\left.\begin{aligned} \dot{x} + r_z\omega_y - r_y\omega_z &= 0, \\ \dot{y} + r_x\omega_z - r_z\omega_x &= 0, \\ \dot{z} + r_y\omega_x - r_x\omega_y &= 0. \end{aligned}\right\} \tag{9.28}$$

The projections of the vector ω onto the Cartesian coordinate axes can be expressed by the relations

$$\begin{aligned}\omega_x &= \dot{\varphi}\cos\theta\sin\psi - \dot{\theta}\cos\psi,\\ \omega_y &= -\dot{\varphi}\cos\theta\cos\psi - \dot{\theta}\sin\psi,\\ \omega_z &= \dot{\varphi}\sin\theta + \dot{\psi}\end{aligned}$$

and, in addition, we have $\boldsymbol{r}\{a\sin\theta\sin\psi, -a\sin\theta\cos\psi, -a\cos\theta\}$. Therefore, the relations (9.28) take the form

$$\begin{aligned}&\dot{x} + a(\dot{\varphi}\cos\psi + \dot{\theta}\cos\theta\sin\psi + \dot{\psi}\sin\theta\cos\psi) = 0,\\ &\dot{y} + a(\dot{\varphi}\sin\psi - \dot{\theta}\cos\theta\cos\psi + \dot{\psi}\sin\theta\sin\psi) = 0,\\ &\dot{z} + a\dot{\theta}\sin\theta = 0.\end{aligned} \tag{9.29}$$

Taking $\dot{\theta}$, $\dot{\varphi}$, and $\dot{\psi}$ as the independent variables, we obtain, by virtue of (9.9), the following expressions:

$$\begin{aligned}\frac{\partial T^*}{\partial\dot{\theta}} &= A\dot{\theta} - a\dot{x}\cos\theta\sin\psi + a\dot{y}\cos\theta\cos\psi - a\dot{z}\sin\theta,\\ \frac{\partial T^*}{\partial\dot{\varphi}} &= C(\dot{\varphi} + \dot{\psi}\sin\theta) - a\dot{x}\cos\psi - a\dot{y}\sin\psi,\\ \frac{\partial T^*}{\partial\dot{\psi}} &= A\dot{\psi}\cos^2\theta + C(\dot{\varphi} + \dot{\psi}\sin\theta)\sin\theta - a\dot{x}\sin\theta\cos\psi - a\dot{y}\sin\theta\sin\psi.\end{aligned}$$

The equations of the impulsive motion have the form

$$\begin{aligned}&A\Delta\dot{\theta} - a\Delta\dot{x}\cos\theta\sin\psi - a\Delta\dot{y}\cos\theta\cos\psi - a\Delta\dot{z}\sin\theta = 0,\\ &C(\Delta\dot{\varphi} + \Delta\dot{\psi}\sin\theta) - a\Delta\dot{x}\cos\psi - a\Delta\dot{y}\sin\psi = 0,\\ &(A\cos^2\theta + C\sin^2\theta)\Delta\dot{\psi} + C\Delta\dot{\varphi}\sin\theta - a\Delta\dot{x}\sin\theta\cos\psi - \\ &\qquad - a\Delta\dot{y}\sin\theta\sin\psi = 0.\end{aligned} \tag{9.30}$$

At the point of impact of the disc on the plane, the following relations are satisfied:

$$v_{a1} = -\kappa x_{z0}, \quad v_{\tau 1} = v_{\tau 0}, \quad v_{\nu 1} = 0, \tag{9.31}$$

where v_z is the vertical component of the velocity of the point of the disc which touches the plane; v_τ is the component of the velocity along the line of intersection of the plane of the disc and the horizontal plane; and v_ν is the third orthogonal component. In accordance with Figure 3.17,

$$\begin{Vmatrix} v_\tau \\ v_\nu \\ v_z \end{Vmatrix} = \begin{Vmatrix} \dot{x}\cos\psi + \dot{y}\sin\psi \\ -\dot{x}\sin\psi + \dot{y}\cos\psi \\ \dot{z} \end{Vmatrix} + \begin{vmatrix} \tau_0 & \nu_0 & z_0 \\ -\dot{\theta} & -\dot{\varphi}\cos\theta & \dot{\psi} + \dot{\varphi}\sin\theta \\ 0 & -a\sin\theta & -a\cos\theta \end{vmatrix}.$$

Substituting this into (9.31), we obtain

$$\begin{aligned} &\Delta\dot{x}\cos\psi + \Delta\dot{y}\sin\psi + a\Delta\dot{\varphi} - a\Delta\dot{\psi}\sin\theta = 0,\\ &(\dot{y})_1\cos\psi - (\dot{x})_1\sin\psi - a\,(\dot{\theta})_1\cos\theta = 0,\\ &(\dot{z})_1 - \varkappa(\dot{z})_0 + a\,(\dot{\theta})_1\sin\theta - a\varkappa\,(\dot{\theta})_0\sin\theta = 0. \end{aligned} \tag{9.32}$$

Equations (9.30) and (9.32) form a complete system for the solution of our problem. In the case of an absolutely inelastic impact of the disc on the horizontal plane, the first equation of (9.31) is replaced by

$$v_{z1} = 0, \quad \text{i.e.} \quad (\dot{z})_1 + a(\dot{\theta})_1\sin\theta = 0.$$

If, at the same time, the plane is absolutely rough, the second equation of (9.31) is replaced by the equation $(v_\tau)_1 = 0$, i.e., in this case, the post-impact velocities must satisfy the system of equations (9.29).

§10. Variational principles in the mechanics of nonholonomic systems

One of the most general integral variational principles is the principle of stationary action, according to which the actual motion of a system is such that

$$\int_{t_0}^{t_1}(\delta T + \delta A)\,dt = 0, \tag{10.1}$$

where δT is a variation of the kinetic energy and δA is the virtual work of the forces applied to the system.

The expression (10.1), which is applicable in the general case to a nonconservative dynamical system, can be simplified if the applied forces have a potential. Then $\delta A = -\delta V$, where δV is the variation of the potential energy. Introducing the Lagrange function $L = T - V$, one can transform (10.1) to the form

$$\int_{t_0}^{t_1}\delta L\,dt = 0. \tag{10.2}$$

From the formal point of view, the principle of stationary action in the form (10.2) is equivalent to a problem of variational calculus. However, despite the superficial similarity, they differ essentially. Namely, in mechanics the symbol δ stands for a virtual variation, i.e., not an arbitrary infinitesimally small variation but a displacement compatible with the constraints imposed upon the system. Thus, it is only in the case of a holonomic system, for which the number of degrees of freedom is equal to the number of generalized coordinates,

that the virtual variations are arbitrary and the principle of stationary action (10.2) is completely equivalent to the corresponding problem in variational calculus. An important difference arises for systems with nonholonomic constraints, when the variations of the generalized coordinates are connected by subsidiary relations

$$\delta q_{m+l} = a_{m+l,\,s}\,\delta q_s \qquad (l = 1, 2, \ldots, n-m;\ s = 1, 2, \ldots, m). \tag{10.3}$$

For a nonholonomic system, a curve obtained from the curve of the actual motion by means of an infinitesimally small virtual variation is not, in general, a kinematically possible trajectory. This circumstance prompted the conclusion that the principle of stationary action cannot be applied to nonholonomic systems, at least not in the form (10.1) or (10.2), in which it is usually employed for holonomic systems. However, other authors, for example, Hamel [**193**], have assumed that this restricted understanding of the principle of stationary action is due to an incorrect interpretation and that, if it is properly understood, the principle may be used to derive the equations of motion of all dynamical systems, including nonholonomic systems.

Although Hamel [**193**] showed the way in which correct equations of motion of a nonholonomic system can be derived from the principle of stationary action in the form (10.1), the essence of the problem of the applicability of this principle remained unclarified.

In order to clarify the situation, it is sufficient to note that the question of the applicability of the principle of stationary action to nonholonomic systems is intimately related to the question of transpositional relations, which we studied in §6 of this chapter.

The point is that the derivation of the principle of stationary action in the form (10.1) presupposes that the operations d and δ commute for all generalized coordinates $q_1, q_2, \cdots, q_n$.

To see this, we write down the d'Alembert-Lagrange equation in the form (3.12):

$$\left(-\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} + \frac{\partial T}{\partial q_k} + Q_k\right)\delta q_k = 0 \qquad (k = 1, 2, \ldots, n).$$

Performing identity transformations, we obtain

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right) + \frac{\partial T}{\partial \dot{q}_k}\left(\frac{d}{dt}\delta q_k - \delta \dot{q}_k\right) + \frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial T}{\partial q_k}\delta q_k + \delta A = 0.$$

We now integrate this expression in the limits between the initial and final position of the system

$$\left(\frac{\partial T}{\partial \dot{q}_k}\,\delta q_k\right)\Bigg|_{t_0}^{t_1} = \int_{t_0}^{t_1}\left[\delta T + \delta A + \frac{\partial T}{\partial \dot{q}_k}\left(\frac{d}{dt}\,\delta q_k - \delta \dot{q}_k\right)\right]dt$$

and, as is required by the principle of stationary action, we assume that the variations vanish at the ends of the interval of integration. We then obtain

$$\int_{t_0}^{t_1}\left[\delta T + \delta A + \frac{\partial T}{\partial \dot{q}_k}\left(\frac{d}{dt}\,\delta q_k - \delta \dot{q}_k\right)\right]dt = 0. \tag{10.4}$$

This expression may be regarded as the most general mathematical formulation of the principle of stationary action, suitable for both holonomic and nonholonomic systems. As regards holonomic systems, the relation $d\delta q_k/dt - \delta \dot{q}_k = 0$ always holds for them and, hence, the expression (10.4) is identical with (10.1). The latter is therefore seen to be a special case of (10.4), obtained under the assumption that the operations d and δ commute for all the true coordinates $q_1, q_2, \cdots, q_n$.

In the case of a nonholonomic system, the special form of the principle of stationary action obtained from (10.4) will depend on the point of view adopted with regard to the transpositional relations. Ultimately, it will depend, as we have seen in §6, on the choice of the local system of coordinates in the neighborhood of the curve of the actual motion. In particular, this system of coordinates may be chosen in such a way that the operations d and δ commute for all true coordinates. In this case, the principle of stationary action can be written in the form (10.2) for nonholonomic system as well. However, the conditions of the nonholonomic constraints will then no longer be satisfied in the neighborhood of the curve of the actual motion and, consequently, in performing the operation δT, one cannot then utilize the conditions (10.3). As we have explained in §6, it is expedient in this case to introduce quasi-coordinates such that the last $n - m$ quasi-coordinates π_{m+l} ($l = 1, 2, \cdots, n - m$) are connected to the true coordinates by the relations

$$\dot{\pi}_{m+l} = a_{m+l,s}\,\dot{q}_s - \dot{q}_{m+l}, \tag{10.5}$$

and the first m quasi-coordinates are arbitrary linear combinations of the coordinates (in particular, they may also be true coordinates). Then, regarding T as a function of the coordinates $q_1, q_2, \cdots, q_n$ and the derivatives of the quasi-coordinates $\dot{\pi}_1, \dot{\pi}_2, \cdots, \dot{\pi}_n$, we have

$$\delta T = \frac{\partial T}{\partial \dot{\pi}_k}\,\delta \dot{\pi}_k - \frac{\partial T}{\partial q_i}\,b_{ik}\delta \pi_k = \frac{\partial T}{\partial \dot{\pi}_k}\,\delta \dot{\pi}_k + \frac{\partial T}{\partial \pi_k}\,\delta \pi_k,$$

where b_{ik} are the coefficients in the relations connecting $\dot{q}_i$ and $\dot{\pi}_k$. We now perform some simple transformations in the first sum:

$$\begin{aligned}\frac{\partial T}{\partial \dot{\pi}_k}\delta\dot{\pi}_k &= \frac{\partial T}{\partial \dot{\pi}_k}\delta\dot{\pi}_k - \frac{\partial T}{\partial \dot{\pi}_k}\frac{d}{dt}\delta\pi_k + \frac{\partial T}{\partial \dot{\pi}_k}\frac{d}{dt}\delta\pi_k \\ &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_k}\delta\pi_k\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_k}\right)\delta\pi_k - \frac{\partial T}{\partial \dot{\pi}_i}\left(\frac{d}{dt}\delta\pi_i - \delta\dot{\pi}_i\right) \\ &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_k}\delta\pi_k\right) - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_k} + \gamma_{kij}\frac{\partial T}{\partial \dot{\pi}_i}\dot{\pi}_j\right)\delta\pi_k.\end{aligned}$$

Hence,

$$\delta T = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_k}\delta\pi_k\right) - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_k} - \frac{\partial T}{\partial \pi_k} + \gamma_{kij}\frac{\partial T}{\partial \dot{\pi}_i}\dot{\pi}_j\right)\delta\pi_k.$$

Substituting this into the expression of the principle of stationary action (10.1) and writing the virtual work in the form $\delta A = \Pi_k \delta\pi_k$, we obtain, after integration of the first term, the following integral:

$$\int_{t_0}^{t_1}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_k} + \gamma_{kij}\frac{\partial T}{\partial \dot{\pi}_i}\dot{\pi}_j - \frac{\partial T}{\partial \pi_k} - \Pi_k\right)\delta\pi_k\, dt = 0.$$

Now, the last $n - m$ terms of the sum in the integrand vanish by virtue of (10.5) and the conditions of the nonholonomic constraints (10.3), and the remaining sum breaks up into m equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_k} + \gamma_{kij}\frac{\partial T}{\partial \dot{\pi}_i}\dot{\pi}_j - \frac{\partial T}{\partial \pi_k} = \Pi_k \quad (j, k = 1, 2, \ldots, m;\ i = 1, 2, \ldots, n), \tag{10.6}$$

by virtue of the independence of the variations $\delta\pi_1, \delta\pi_2, \cdots, \delta\pi_m$. Thus, starting from the principle of stationary action in the form (10.1), we have obtained the equations (10.6) of motion of a nonholonomic system in quasi-coordinates.[14]

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[14] In his paper [472], Voronec proposed a method for obtaining equations of motion of conservative nonholonomic systems, taking as his starting point a variational principle (in the notation adopted in this book) in the form

$$\text{(A)} \qquad \int_{t_0}^{t_1}\left[\delta(T^* + U) + \frac{\partial T}{\partial \dot{q}_{m+l}}\delta(\dot{q}_{m+l} - a_{m+l,k}\dot{q}_k)\right]dt = 0$$

in which it is assumed that the operations d and δ commute for all true coordinates. This method is actually identical with our method if true coordinates are chosen as the

If the local system of coordinates is chosen differently, the principle of stationary action cannot be used in the form (10.1) for nonholonomic systems.

For suppose the operations d and δ in the neighborhood of the trajectory of the actual motion are defined by the relations (6.7). Under these conditions, the operations d and δ commute only for the first m coordinates; for the remaining coordinates

$$d\delta q_{m+l} - \delta dq_{m+l} = \beta_{rs}^{m+l} dq_r \delta q_s.$$

Substituting this into (10.4), we arrive at the following mathematical formulation of the principle of stationary action in this second case:

$$\int_{t_0}^{t_1} \left(\delta T^0 + \delta A + \beta_{rs}^{m+l} \frac{\partial T}{\partial \dot{q}_{m+l}} \dot{q}_r \delta q_s\right) dt = 0 \tag{10.7}$$

$$(r, s = 1, 2, \ldots, m;\ l = 1, 2, \ldots, n - m),$$

where the kinetic energy T^0 is a function of $q_1, \cdots, q_n$, $\dot{q}_1, \cdots, \dot{q}_m$, obtained from T by means of the equations of the nonholonomic constraints (10.3). The use of the latter is completely justified in the given case, since, by virtue of the relations (6.11), which reflect the special choice of the local system of coordinates, the displacements $dq_1, \cdots, dq_n$ and $\delta q_1, \delta q_2, \cdots, \delta q_n$ are compatible with the nonholonomic constraints imposed on the system in the neighborhood of the trajectory of the actual motion as well as on the trajectory itself. It is readily seen that the principle of stationary action in the form (10.7) leads to Voronec's equations (6.3) or, if the integrand does not contain the coordinates $q_{m+1}, q_{m+2}, \cdots, q_n$, to Čaplygin's equations (3.17). To prove this, we transform the expression for δT^0 in (10.7):

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first m quasi-coordinates. For in this case we have

$$\delta T = \frac{\partial T}{\partial \dot{\pi}_k} \delta \dot{\pi}_k + \frac{\partial T}{\partial \dot{\pi}_{m+l}} \delta \dot{\pi}_{m+l} + \frac{\partial T}{\partial q_i} \delta q_i = \delta T^* + \frac{\partial T}{\partial \dot{q}_{m+l}} \delta (\dot{q}_{m+l} - a_{m+l,\,k} \dot{q}_k), \tag{B}$$

where T^* is the function obtained from T by eliminating $\dot{q}_{m+1}, \dot{q}_{m+2}, \cdots, \dot{q}_n$ by means of the equations of the nonholonomic constraints (10.3). Substituting (B) into (10.1), we obtain Voronec's expression (A).

$$\delta T^0 = \frac{\partial T^0}{\partial \dot{q}_k}\delta\dot{q}_k + \left(\frac{\partial T^0}{\partial q_s} + \frac{\partial T^0}{\partial q_{m+l}}a_{m+l,\,s}\right)\delta q_s$$

$$= \frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{q}_k}\delta q_k\right)^{\cdot} - \left(\frac{d}{dt}\frac{\partial T^0}{\partial \dot{q}_s} - \frac{\partial T^0}{\partial q_s} - \frac{\partial T^0}{\partial \dot{q}_{m+l}}a_{m+l,s}\right)\delta q_s$$

$$(k, s = 1, 2, \ldots, m;\ \ l = 1, 2, \ldots, n - m).$$

Substituting the expressions found into (10.7), we obtain

$$(10.8)\qquad \int_{t_0}^{t_1}\left(\frac{d}{dt}\frac{\partial T^0}{\partial \dot{q}_s} - \frac{\partial T^0}{\partial q_s} - \frac{\partial T^0}{\partial q_{m+l}}a_{m+l,\,s} - \beta_{rs}^{m+l}\frac{\partial T}{\partial \dot{q}_{m+l}}\dot{q}_r - Q_s\right)\delta q_s dt = 0.$$

From this result we obtain Voronec's equations (6.3).

It is also possible to obtain the "intermediate" form (6.16) of the equations of motion of nonholonomic systems from the general expression of the principle of stationary action (10.4).

Another variational principle which can be used to obtain the equations of motion of nonholonomic systems is Gauss's principle of least constraint.

This principle, like the preceding, reduces the problem of the derivation of the equations of motion of a dynamical system to the determination of the minimum of a certain function, which is a polynomial of second degree. This principle, which is fairly general and comparatively simple in application, can be applied without any restrictions to both holonomic and nonholonomic systems. The essence of this principle can be obtained as follows. Suppose that, at a certain instant of time t, a system of N material points is in a state in which the coordinates of the points are specified by the values x_i and their velocities by $\dot{x}_i$ $(i = 1, 2, \cdots, 3N)$. After a small interval of time dt, the coordinates of the points will have changed, being equal, to within infinitesimally small quantities of second order, to

$$x_i + \dot{x}_i\,dt + \frac{1}{2}\ddot{x}_i(dt)^2.$$

On the other hand, if the system is subjected to the same forces X_i but the constraints imposed on it are removed, the position to which the system moves from the same initial state is given by

$$x_i + \dot{x}_i\,dt + \frac{1}{2}\frac{X_i}{m_i}(dt)^2,$$

where m_i is the mass of the corresponding material point. The distance between these two positions is given by the modulus of the vector $\boldsymbol{a}$ with components

$$\frac{1}{2}\left(\ddot{x}_i - \frac{X_i}{m_i}\right)(dt)^2$$

which tells us the deviation of the system when it executes its actual motion from the position it would have occupied if moving freely. As a measure of this deviation, Gauss adopted the so-called constraint Z_i, i.e., the quantity

$$Z_i = \frac{2m_i a^2}{(dt)^4},$$

which is proportional to the square of the deviation a, the constraint of the system being understood as the sum of the constraints of the particles composing the system. Gauss's principle states that at any instant of time the constraint associated with the actual motion is smaller than that associated with any other kinematically possible motion. It follows that the first variation of the constraint must vanish; at the same time, it is clear from what we have said that only the accelerations must be varied. Thus, we obtain the following mathematical formulation of the principle of least constraint:

$$\delta Z = 0, \tag{10.9}$$

where the function

$$Z = \frac{1}{2}\sum_{i=1}^{3N} m_i\left(\ddot{x}_i - \frac{X_i}{m_i}\right)^2 \tag{10.10}$$

is a polynomial of second degree in the accelerations.

In order to obtain the equations of motion from Gauss's principle of least constraint, we multiply out the brackets in the expression (10.10) and, in accordance with the meaning of the principle, retain only the terms containing the accelerations

$$Z = \frac{1}{2}\sum_{i=1}^{3N} m_i\ddot{x}_i^2 - \sum_{i=1}^{3N} X_i\ddot{x}_i + \ldots. \tag{10.11}$$

It is readily seen that the first sum is the acceleration function

$$S = \frac{1}{2}\sum_{i=\mathrm{I}}^{3N} m_i\ddot{x}_i^2,$$

which must be written down with allowance for all the constraints imposed on the system. If the number of independent accelerations $\ddot{q}_1, \ddot{q}_2, \cdots, \ddot{q}_m$ is m, then, transforming the second sum in (10.11), we obtain

$$\sum_{i=1}^{3N} X_i \ddot{x}_i = \sum_{k=1}^{m} Q_k \ddot{q}_k,$$

so that (10.11) can be written in the form

$$Z = S - Q_k \ddot{q}_k \qquad (k = 1, 2, \cdots, m). \tag{10.12}$$

Equation (10.9) now leads directly to equations that are identical with Appell's equation:

$$\frac{\partial S}{\partial \ddot{q}_k} = Q_k \qquad (k = 1, 2, \ldots, m),$$

their number being equal to the number of degrees of freedom. As we have seen in §8, Appell's equations can indeed be applied to both holonomic and nonholonomic systems.

To conclude this section, we give an example of the derivation of the equations of motion by means of the principle of stationary action.

EXAMPLE. *Equations of motion of a heavy rigid body rolling without sliding on a horizontal plane.* We take the center of mass of the body as the origin O of coordinate axes $O\xi\eta\zeta$ that coincide with the principal axes of the ellipsoid of inertia of the body, denoting by $\dot{w}_1$, $\dot{w}_2$, $\dot{w}_3$ and $\dot{\varphi}_1$, $\dot{\varphi}_2$, $\dot{\varphi}_3$ the projections onto these axes of the velocity V of the point O and the instantaneous angular velocity ω of the body. These quantities are derivatives of quasi-coordinates, since, as one can easily see, they are related to the true coordinates by nonintegrable equations. The condition of rolling without sliding yields

$$\begin{aligned} \dot{w}_1 - \eta\dot{\varphi}_3 + \zeta\dot{\varphi}_2 &= 0, \\ \dot{w}_2 - \zeta\dot{\varphi}_1 + \xi\dot{\varphi}_3 &= 0, \\ \dot{w}_3 - \xi\dot{\varphi}_2 + \eta\dot{\varphi}_1 &= 0, \end{aligned} \tag{10.13}$$

which are nonholonomic constraints. Thus, the system has three degrees of freedom. As independent parameters in terms of which we shall write down the equations of motion, we choose u, v, and n, where the family $u = \text{const}$ and $v = \text{const}$ form a coordinate mesh of lines on the surface S that bounds the body and $\overline{n}$ is a quasi-coordinate whose derivative is the projection of the instantaneous angular velocity ω onto the vertical direction. Let α_1, α_2, and α_3 be the cosines of the angles of the outer normal to the surface S at the point of contact of the body and the plane. Together with the coordinates ξ, η, and ζ of the point of contact, they are then known functions of the variables u and v.

Since one of the parameters we have chosen is a quasi-coordinate,

it is most convenient to use the principle of least action in the form (10.2) in order to derive the equations of motion. In this case, the function L is to be regarded as a function of the quasi-coordinates $\pi_1, \pi_2, \cdots, \pi_6$, given by

$$\begin{aligned} &\dot{\pi}_1 = \dot{u}, \quad \dot{\pi}_2 = \dot{v}, \quad \dot{\pi}_3 \equiv \dot{n} = \alpha_1\dot{\varphi}_1 + \alpha_2\dot{\varphi}_2 + \alpha_3\dot{\varphi}_3, \\ &\dot{\pi}_4 = \dot{\omega}_1 - \eta\dot{\varphi}_3 + \zeta\dot{\varphi}_2, \quad \dot{\pi}_5 = \dot{\omega}_2 - \zeta\dot{\varphi}_1 + \xi\dot{\varphi}_3, \\ &\dot{\pi}_6 = \dot{\omega}_3 - \xi\dot{\varphi}_2 + \eta\dot{\varphi}_1. \end{aligned} \tag{10.14}$$

Our problem consists of transforming the expression δL and separating the terms that contain the variations δu, δv, and δn as factors; setting the coefficients of these factors equal to zero, we shall obtain the desired equations of motion.

We have

$$\delta L = \delta T - \delta \Pi,$$

where the kinetic energy is

$$T = \frac{1}{2} m(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2) + \frac{1}{2}(A\dot{\varphi}_1^2 + B\dot{\varphi}_2^2 + C\dot{\varphi}_3^2) \tag{10.15}$$

and the potential energy is

$$\Pi = mg\,(\alpha_1\xi + \alpha_2\eta + \alpha_3\zeta),$$

m is the mass of the body, and A, B and C are the principal moments of inertia.

We now perform the necessary transformations. We must first express T in terms of the quasi-coordinates. By virtue of the well-known formulas of kinematics, we have

$$\begin{aligned} \dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial u}\dot{u} + \frac{\partial \alpha_1}{\partial v}\dot{v} = \alpha_2\dot{\varphi}_3 - \alpha_3\dot{\varphi}_2, \\ \dot{\alpha}_2 &= \frac{\partial \alpha_2}{\partial u}\dot{u} + \frac{\partial \alpha_2}{\partial v}\dot{v} = \alpha_3\dot{\varphi}_1 - \alpha_1\dot{\varphi}_3, \\ \dot{\alpha}_3 &= \frac{\partial \alpha_3}{\partial u}\dot{u} + \frac{\partial \alpha_3}{\partial v}\dot{v} = \alpha_1\dot{\varphi}_2 - \alpha_2\dot{\varphi}_1. \end{aligned} \tag{10.16}$$

We now solve any two of these equations together with the third equation of (10.14) for $\dot{\varphi}_1$, $\dot{\varphi}_2$, and $\dot{\varphi}_3$, obtaining

$$\begin{aligned} \dot{\varphi}_1 &= \alpha_1\dot{n} + \left(\alpha_3\frac{\partial \alpha_2}{\partial u} - \alpha_2\frac{\partial \alpha_3}{\partial u}\right)\dot{u} + \left(\alpha_3\frac{\partial \alpha_2}{\partial v} - \alpha_2\frac{\partial \alpha_3}{\partial v}\right)\dot{v}, \\ \dot{\varphi}_2 &= \alpha_2\dot{n} + \left(\alpha_1\frac{\partial \alpha_3}{\partial u} - \alpha_3\frac{\partial \alpha_1}{\partial u}\right)\dot{u} + \left(\alpha_1\frac{\partial \alpha_3}{\partial v} - \alpha_3\frac{\partial \alpha_1}{\partial v}\right)\dot{v}, \\ \dot{\varphi}_3 &= \alpha_3\dot{n} + \left(\alpha_2\frac{\partial \alpha_1}{\partial u} - \alpha_1\frac{\partial \alpha_2}{\partial u}\right)\dot{u} + \left(\alpha_2\frac{\partial \alpha_1}{\partial v} - \alpha_1\frac{\partial \alpha_2}{\partial v}\right)\dot{v}, \end{aligned} \tag{10.17}$$

where, in accordance with (10.14), $\dot{u} = \dot{\pi}_1$, $\dot{v} = \dot{\pi}_2$, $\dot{n} = \dot{\pi}_3$. Substituting the expressions (10.17) into (10.14), we find expressions for $\dot{w}_1$, $\dot{w}_2$, $\dot{w}_3$ in terms of the quasi-coordinates. Then, substituting these together with (10.17) into (10.15), we obtain the function $T = T(u, v, \pi_1, \pi_2, \cdots, \pi_6)$.

For the sake of brevity, we shall now adopt the vector notation. We introduce the vectors with components

$$\boldsymbol{\alpha}\ (\alpha_1, \alpha_2, \alpha_3), \quad \boldsymbol{\varphi}\ (\varphi_1, \varphi_2, \varphi_3), \quad \boldsymbol{w}\ (w_1, w_2, w_3),$$
$$\boldsymbol{\pi}\ (\pi_4, \pi_5, \pi_6), \quad \boldsymbol{\rho}\ (\xi, \eta, \zeta).$$

The relations (10.16) and (10.17) then take the form

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \dot{\boldsymbol{\varphi}}, \tag{10.16'}$$

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\alpha}\dot{n} + \left(\frac{\partial \boldsymbol{\alpha}}{\partial u} \times \boldsymbol{\alpha}\right)\dot{u} + \left(\frac{\partial \boldsymbol{\alpha}}{\partial v} \times \boldsymbol{\alpha}\right)\dot{v} = \boldsymbol{\alpha}\dot{n} + \dot{\boldsymbol{\alpha}} \times \boldsymbol{\alpha}. \tag{10.17'}$$

The variation of the kinetic energy is

$$\delta T = \frac{\partial T}{\partial \dot{\pi}_k}\delta\dot{\pi}_k + \frac{\partial T}{\partial \pi_k}\delta\pi_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_k}\delta\pi_k\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\pi}_k}\right)\delta\pi_k + \frac{\partial T}{\partial \dot{\pi}_k}\delta\pi_k - \frac{\partial T}{\partial \dot{\pi}_k}\left(\frac{d}{dt}\delta\pi_k - \delta\dot{\pi}_k\right).$$

Having performed the operation of differentiation with respect to the quasi-coordinates $\dot{\pi}_4$, $\dot{\pi}_5$, and $\dot{\pi}_6$, we must set $\dot{\pi}_4 = \dot{\pi}_5 = \dot{\pi}_6 = 0$ in the resulting expressions in accordance with the equations of the nonholonomic constraints. Consequently, in order to obtain the final expressions for $\partial T/\partial \dot{\pi}_4$, $\partial T/\partial \dot{\pi}_5$, $\partial T/\partial \dot{\pi}_6$, we must use (10.13) and (10.17):

$$\begin{aligned}\dot{w}_1 &= \eta\dot{\varphi}_3 - \zeta\dot{\varphi}_2 \\ &= (\eta\alpha_3 - \zeta\alpha_2)\dot{n} + \left[\eta\left(\alpha_2\frac{\partial\alpha_1}{\partial u} - \alpha_1\frac{\partial\alpha_2}{\partial u}\right) - \zeta\left(\alpha_1\frac{\partial\alpha_3}{\partial u} - \alpha_3\frac{\partial\alpha_1}{\partial u}\right)\right]\dot{u} \\ &\quad + \left[\eta\left(\alpha_2\frac{\partial\alpha_1}{\partial v} - \alpha_1\frac{\partial\alpha_2}{\partial v}\right) - \zeta\left(\alpha_1\frac{\partial\alpha_3}{\partial v} - \alpha_3\frac{\partial\alpha_1}{\partial v}\right)\right]\dot{v}\end{aligned}$$

(similar expressions for $\dot{w}_2$ and $\dot{w}_3$ can be obtained by cyclic permutation of the variables) or, in a vector form,

$$\dot{\boldsymbol{w}} = (\boldsymbol{\rho} \times \boldsymbol{\alpha})\dot{n} + \boldsymbol{\rho} \times (\dot{\boldsymbol{\alpha}} \times \boldsymbol{\alpha}). \tag{10.18}$$

Thus, in the expression for δT we must calculate the scalar product of the following vectors:

$$m\dot{\boldsymbol{w}}\left(\frac{d}{dt}\delta\boldsymbol{\pi} - \delta\dot{\boldsymbol{\pi}}\right), \tag{10.19}$$

where the components of the vectors $\dot{\pi}$ and $\delta\dot{\pi}$ are given by the last three relations of (10.14).

We shall first evaluate the bilinear form $d\delta\pi_3/dt - \delta\dot{\pi}_3$. Using the formulas

$$\frac{d}{dt}(\alpha\cdot\delta\varphi) = \dot{\alpha}\delta\varphi + \alpha\frac{d}{dt}\delta\varphi, \quad \delta(\alpha\cdot\dot{\varphi}) = \delta\alpha\cdot\dot{\varphi} + \alpha\cdot\delta\dot{\varphi},$$

we find

$$\frac{d}{dt}\delta\pi_3 - \delta\dot{\pi}_3 = \alpha\left(\frac{d}{dt}\delta\varphi - \delta\dot{\varphi}\right) + \dot{\alpha}\cdot\delta\varphi - \delta\alpha\cdot\dot{\varphi}.$$

However,

$$\frac{d}{dt}\delta\varphi - \delta\dot{\varphi} = \delta\varphi\times\dot{\varphi}, \quad \dot{\alpha} = \alpha\times\dot{\varphi}.$$

Hence and by (10.17′)

$$\begin{aligned}\frac{d}{dt}\delta\pi_3 - \delta\dot{\pi}_3 &= \alpha\cdot(\delta\varphi\times\dot{\varphi}) + \dot{\alpha}\cdot\delta\varphi - \delta\alpha\cdot\dot{\varphi}\\ &= -\delta\alpha\,(\alpha\dot{n} + \dot{\alpha}\times\alpha) = \alpha\cdot(\dot{\alpha}\times\delta\alpha) = N(\dot{u}\,\delta v - \dot{v}\,du),\end{aligned} \tag{10.20}$$

where

$$N = \alpha\left(\frac{\partial\alpha}{\partial u}\times\frac{\partial\alpha}{\partial v}\right).$$

The last three relations of (10.14) can be expressed by a single vector relation

$$\dot{\pi} = \varpi\dot{w} + \dot{\varphi}\times\rho.$$

Hence

$$\delta\pi = \delta\varpi w + \delta\varphi\times\rho$$

and, consequently,

$$\frac{d}{dt}\delta\pi - \delta\dot{\pi} = \frac{d}{dt}\delta\varpi w - \delta\dot{\varpi}\dot{w} + \left(\frac{d}{dt}\delta\varphi - \delta\dot{\varphi}\right)\times\rho + \delta\varphi\times\dot{\rho} - \dot{\varphi}\times\delta\rho.$$

However,

$$\frac{d}{dt}\delta\dot{\varpi}w - \delta\dot{\varpi}\dot{w} = \delta\varphi\times\dot{\varpi}w - \dot{\varphi}\times\delta\varpi w, \quad \frac{d}{dt}\delta\varphi - \delta\dot{\varphi} = \delta\varphi\times\dot{\varphi}.$$

Hence, using (10.13), we obtain

$$\begin{aligned}\frac{d}{dt}\delta\pi - \delta\dot{\pi} &= \delta\varphi\times(\rho\times\dot{\varphi}) - \dot{\varphi}\times(\rho\times\delta\varphi) + (\delta\varphi\times\dot{\varphi})\times\rho\\ &\quad + \delta\varphi\times\dot{\rho} - \dot{\varphi}\times\delta\rho = \delta\varphi\times\dot{\rho} - \dot{\varphi}\times\delta\rho.\end{aligned}$$

In accordance with (10.17), we find

$$\delta\varphi \times \dot{\rho} = (\alpha \times \dot{\rho})\,\delta n + (\delta\alpha \times \alpha) \times \dot{\rho} = \left(\alpha \times \frac{\partial \varrho}{\partial u}\right) \dot{u}\,\delta n$$
$$+ \left[\left(\frac{\partial \alpha}{\partial u} \times \alpha\right) \times \frac{\delta \varrho}{\partial u}\right] \dot{u}\,\delta u + \left[\left(\frac{\partial \alpha}{\partial v} \times \alpha\right) \times \frac{\partial \varrho}{\partial u}\right] \dot{u}\,\delta v$$
$$+ \left[\left(\frac{\partial \alpha}{\partial u} \times \alpha\right) \times \frac{\partial \varrho}{\partial v}\right] \dot{v}\,\delta u + \left[\left(\frac{\partial \alpha}{\partial v} \times \alpha\right) \times \frac{\partial \varrho}{\partial v}\right] \dot{v}\,\delta v$$

and a similar expression for $\dot{\varphi} \times \delta\rho$. Finally,

$$\frac{d}{dt}\delta\pi - \delta\dot{\pi} = \left(\alpha \times \frac{\partial \varrho}{\partial u}\right)(\dot{u}\,\delta n - \dot{n}\,\delta u) + \left(\alpha \times \frac{\partial \varrho}{\partial v}\right)(\dot{v}\,\delta n - \dot{n}\,\delta v),$$

since

$$\frac{\partial \varrho}{\partial u} \cdot \alpha = \frac{\partial \varrho}{\partial v} \cdot \alpha = 0.$$

Substituting this expression together with (10.18) into (10.19) and writing

$$U = m\dot{w}\left(\alpha \times \frac{\partial \varrho}{\partial u}\right), \quad V = m\dot{w}\left(\alpha \times \frac{\partial \varrho}{\partial v}\right), \tag{10.21}$$

we obtain

$$m\dot{w}\left(\frac{d}{dt}\delta\pi - \delta\dot{\pi}\right) = (U\dot{u} + V\dot{v})\,\delta n - U\dot{n}\,\delta u - V\dot{n}\,\delta v.$$

Finally, we shall express the remaining part of the variation of the kinetic energy T and the variation of the potential energy Π as follows

$$\frac{\partial T}{\partial \pi_k}\delta\pi_k = \frac{\partial T}{\partial u}\delta u + \frac{\partial T}{\partial v}\delta v, \qquad \delta\Pi = \frac{\partial \Pi}{\partial \pi_k}\delta\pi_k = \frac{\partial \Pi}{\partial u}\delta u + \frac{\partial \Pi}{\partial v}\delta v.$$

These transformations having been completed, we can write the principle of stationary action (10.2) in the form

$$\int_{t_0}^{t_1} \left\{\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{n}} + U\dot{u} + V\dot{v}\right]\delta n + \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{u}} - \frac{\partial}{\partial u}(T - \Pi) - \frac{\partial T}{\partial \dot{n}}N\dot{v} - U\dot{n}\right]\delta u\right.$$
$$\left. + \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{v}} - \frac{\partial}{\partial v}(T - \Pi) + \frac{\partial T}{\partial \dot{n}}N\dot{u} - V\dot{n}\right]\delta v\right\} dt = \frac{\partial T}{\partial \dot{\pi}_k}\delta\pi_k\Big|_{t_0}^{t_1}.$$

Bearing in mind that the variations vanish at the ends of the interval of integration, we obtain three differential equations by virtue of the independence of the variations δn, δu, and δv in the integrand

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{n}} = -(U\dot{u} + V\dot{v}), \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{u}} - \frac{\partial}{\partial u}(T - \Pi) = \frac{\partial T}{\partial \dot{n}}N\dot{v} + U\dot{n};$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{v}} - \frac{\partial}{\partial v}(T - \Pi) = -\frac{\partial T}{\partial \dot{n}}N\dot{u} + V\dot{n}. \tag{10.22}$$

Denoting by E, F, G, D, D', and D'' the coefficients of the first two quadratic forms of the surface S that bounds the body:

$$E\,(du)^2 + 2F\,du\,dv + G\,(dv)^2,\ D\,(du)^2 + 2D'du\,dv + D''(dv)^2,$$

we can show that the quantities N, U, and V in the equations (10.22) can be expressed in terms of these coefficients as follows:

$$U = \frac{mh}{H}\left[(DF - D'E)\,\dot{u} + (D'F - D''E)\,\dot{v}\right] - m\rho\dot{n}\frac{\partial\rho}{\partial u},$$

$$V = \frac{mh}{H}\left[(DG - D'F')\,\dot{u} + (D'G - D''F)\,\dot{v}\right] - m\rho\dot{n}\frac{\partial\rho}{\partial v}, \tag{10.23}$$

$$N = \frac{1}{H}(DD'' - D'^2),$$

where $H = (EG - F^2)^{1/2}$; $h = \boldsymbol{\rho}\cdot\boldsymbol{\alpha}$ is the height of the center of mass of the body above the horizontal plane; and $\rho = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$ is the distance from the center of mass to the point of contact of the body and the plane.

Substituting the expressions (10.23) and (10.22), we obtain the desired equations of motion, which are identical with the equations (24) derived by Voronec in 1902 [**475**]:

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{n}} = m\rho\left(\frac{\partial\rho}{\partial u}\,\dot{u} + \frac{\partial\rho}{\partial v}\,\dot{v}\right)\dot{n} - \frac{mh}{H}\left[(DF - D'E)\,\dot{u}^2\right.$$
$$\left. + (DG - D''E)\,\dot{u}\dot{v} + (D'G - D''F)\,\dot{v}^2\right],$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{u}} - \frac{\partial}{\partial u}(T - \Pi) = \frac{DD'' - D'^2}{H}\frac{\partial T}{\partial\dot{n}}\,\dot{v} - m\rho\frac{\partial\rho}{\partial u}\dot{n}^2$$
$$+ \frac{mh}{H}\left[(DF - D'E)\,\dot{u} + (D'F - D''E)\,\dot{v}\right]\dot{n},$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{v}} - \frac{\partial}{\partial v}(T - \Pi) = -\frac{DD'' - D'^2}{H}\frac{\partial T}{\partial\dot{n}}\,\dot{u} - m\rho\frac{\partial\rho}{\partial v}\dot{n}^2$$
$$+ \frac{mh}{H}\left[(DG - D'F)\,\dot{u} + (D'G - D''F)\,\dot{v}\right]\dot{n}.$$

§11. First integrals of the equations of motion of nonholonomic systems

Let us consider a dynamical system whose motion is described by differential equations of nth order

$$\frac{dx_k}{dt} = X_k(x_1, x_2, \ldots, x_n, t) \qquad (k = 1, 2, \ldots, n). \tag{11.1}$$

Suppose that a certain function $f(x_1, x_2, \cdots, x_n, t)$ possesses the property that $df/dt \equiv 0$ if its arguments $x_1, x_2, \cdots, x_n$ are replaced by any functions of the variables t that satisfy equations (11.1). The relation

$$f(x_1, x_2, \ldots, x_n, t) = C, \tag{11.2}$$

where T is an arbitrary constant, is then called a first integral of the motion. It follows from what we have said that the identity

$$\frac{\partial f}{\partial t} + \sum_{k=1}^{n} X_k \frac{\partial f}{\partial x_k} \equiv 0 \tag{11.5}$$

is a sufficient condition for the relation (11.2) to be a first integral of the motion.

Among the first integrals of the motion of holonomic systems, particular importance attaches to cyclic integrals and the generalized energy integral. These integrals frequently have a simple physical meaning and can be readily found in a chosen system of coordinates.

The motion of a conservative holonomic system with n degrees of freedom is determined by the differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \quad (k = 1, 2, \ldots, n), \tag{11.4}$$

where $q_1, q_2, \cdots, q_n$ are generalized coordinates and L is the Lagrange function. It may happen that a certain coordinate, say q_1, does not occur explicitly in the expression for the Lagrange function (which, however, does contain $\dot{q}_1$). Such a coordinate q_1 is called a cyclic coordinate. The Lagrange equation of the second kind (11.4) corresponding to the coordinate q_1 is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0, \tag{11.5}$$

which can be integrated to give

$$\frac{\partial L}{\partial \dot{q}_1} = C_1, \tag{11.6}$$

where C_1 is an arbitrary constant.

Equation (11.6), which is a first integral of the motion, is called a cyclic integral.

It should be noted that the presence or absence of cyclic integrals depends on the choice of the generalized coordinates.

In order to elucidate the physical meaning of cyclic integrals, we shall consider individual types of motion. Suppose first that a cyclic coordinate q_1 has the property that a variation of it by an amount l, the values of the remaining coordinates $q_2, q_3, \cdots, q_n$ remaining constant, corresponds to a translational displacement of the system by a distance

l. Choosing the direction of this displacement as the Ox axis, we have

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{q}_1} = \sum_{\nu=1}^{N} m_\nu \left(\dot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_1} + \dot{y}_\nu \frac{\partial \dot{y}_\nu}{\partial \dot{q}_1} + \dot{z}_\nu \frac{\partial \dot{z}_\nu}{\partial \dot{q}_1} \right) = \sum_{\nu=1}^{N} m_\nu \dot{x}_\nu = C_1.$$

Thus, if the change of a cyclic coordinate corresponds to translational displacement of the system in a certain direction, the corresponding cyclic integral reflects the conservation of the projection of the momentum of the system onto this direction.

Let us now suppose that the change of a cyclic coordinate by a quantity α corresponds to a rotation of the complete system through an angle α about some straight line fixed in space. Taking this straight line as the Oz axis and assuming

$$x_\nu = r_\nu \cos \varphi_\nu, \quad y_\nu = r_\nu \sin \varphi_\nu \quad (\nu = 1, 2, \ldots, N),$$

we find that

$$\frac{\partial x_\nu}{\partial q_1} = \frac{\partial x_\nu}{\partial \varphi_\nu} = -r_\nu \sin \varphi_\nu, \quad \frac{\partial y_\nu}{\partial q_1} = \frac{\partial y_\nu}{\partial \varphi_\nu} = r_\nu \cos \varphi_\nu, \quad \frac{\partial z_\nu}{\partial q_1} = 0,$$

and, hence,

$$\frac{\partial L}{\partial \dot{q}_1} = \sum_{\nu=1}^{N} m_\nu (x_\nu \dot{y}_\nu - \dot{x}_\nu y_\nu) = C_1.$$

The last expression is the angular momentum of the system about the Oz axis.

Thus, if the change of a cyclic coordinate q_1 corresponds to a rotation of the complete system about a certain fixed straight line, the corresponding cyclic integral expresses the conservation of the angular momentum of the system about this straight line.

Finally, in an investigation of the motion of an electromechanical system, one of the generalized coordinates is frequently taken to be the amount of electricity passing through the transverse section of a conductor during the time t:

$$q_1 = \int_0^t i \, dt.$$

If the electric circuit containing this conductor has no capacitors, the coordinate q_1 is cyclic. Maxwell [**295**] showed that in this case the electric moment is conserved. This law of conservation is also reflected by a cyclic integral.

We now turn to the derivation of the generalized energy integral, which plays a very important role in dynamical investigations and in many questions of physics. Suppose that the constraints of a conservative dynamical system do not depend on the time and, consequently, the Lagrange function L does not contain t explicitly. The total derivative of the Lagrange function with respect to the time can be reduced in this case to the form

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k = \frac{d}{dt}\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k.$$

Hence, we obtain the differential equation

$$\frac{d}{dt}\left(\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = 0,$$

which can be integrated:

$$\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L = h, \tag{11.7}$$

where h is an arbitrary constant. This equation, which is a first integral of the motion, is called the generalized energy integral. In the special case when the Lagrange function L can be represented in the form $L = T - V$, where the kinetic energy T is a homogeneous quadratic form in the generalized velocities and the potential energy V depends only on the generalized coordinates, the generalized energy integral takes the form

$$\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L = 2T - T + V = T + V = h. \tag{11.8}$$

Thus, in this special case the integral (11.7) is identical with the usual energy integral, which, physically, expresses the conservation of the total mechanical energy. In the general case, when

$$T = T_2 + T_1 + T_0,$$

where T_2 is a quadratic form; T_i is a linear form in the generalized velocities; and T_0 is a function that does not depend on the generalized velocities,[15] the expression for the generalized energy integral (11.7) differs from (11.8). Thus, in the general case we have

[15] The appearance of the terms T_1 and T_0 in the expression for the kinetic energy T is connected with the presence of gyroscopic forces, with the choice of a rotating frame of reference, etc.

$$\sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L = 2T_2 + T_1 - T_2 - T_1 - T_0 + V = T_2 - T_0 + V = h$$

and the total mechanical energy is not, in general, conserved.

First integrals of motion, especially cyclic integrals and the generalized energy integral, are widely used in the dynamics of holonomic systems. The present section is devoted to an extension of this theory of integration to nonholonomic systems.

Cyclic integral. In the case of a system with nonintegrable constraints, the presence of a cyclic coordinate does not always lead to a first integral of the form (11.6). This is because the equations of motion of a nonholonomic system do not, in general, reduce to the form (11.4). Hence, the absence of the term $\partial L/\partial q_k$ (if q_k is a cyclic coordinate) does not give rise to a total differential of a certain expression. A cyclic integral for a nonholonomic system is obtained only if, besides the cyclic coordinate q_k, the additional nonholonomy terms are absent in the corresponding equation. In other words, it must be possible to write the equation of motion for the kth generalized coordinate in the form of a Lagrange equation of the second kind. This can only be done if a definite condition is satisfied.

Suppose that the motion of a given holonomic system is described by Čaplygin's equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} - \sum_{l=1}^{n-m} \frac{\partial L}{\partial \dot{q}_{m+l}} \left(\frac{\partial b_{m+l,\,r}}{\partial q_s} - \frac{\partial b_{m+l,\,s}}{\partial q_r} \right) \dot{q}_r = 0 \ (s, r = 1, 2, \ldots, m), \tag{11.9}$$

where L is the Lagrange function written down without allowance for the nonholonomic constraints

$$\dot{q}_{m+l} = b_{m+l,s}\dot{q}_s \quad (l = 1, 2, \ldots, n - m;\ s = 1, 2, \ldots, m), \tag{11.10}$$

imposed on the system and L^* is the function obtained from L by eliminating the dependent velocities $\dot{q}_{m+1}, \dot{q}_{m+2}, \cdots, \dot{q}_n$ by means of the relations (11.10). A necessary and sufficient condition for equation (11.9) to take the form of a Lagrange equation of the second kind for $s = 1$ is the fulfilment of the equations

$$\frac{\partial b_{m+l,\,r}}{\partial q_1} = \frac{\partial b_{m+l,\,1}}{\partial q_r} \quad (l = 1, 2, \ldots, n - m;\ r = 1, 2, \ldots, m). \tag{11.11}$$

If these equations are satisfied, one can introduce the function u_{m+l}

$= u_{m+l}(q_1, \cdots, q_n)$ by means of the expressions

$$u_{m+l} = \int_{q_{10}}^{q_1} b_{m+l,\,1}\, dq_1,$$

where q_{10} is an arbitrary constant. Hence, using (11.11), we find

$$\frac{\partial u_{m+l}}{\partial q_r} = \int_{q_{10}}^{q_1} \frac{\partial b_{m+l,\,1}}{\partial q_r} dq_1 = \int_{q_{10}}^{q_1} \frac{\partial b_{m+l,\,r}}{\partial q_1} dq_1$$
$$= b_{m+l,\,r}(q_1, \ldots, q_n) - b_{m+l,\,r}(q_{10}, \ldots, q_n).$$

or

$$b_{m+l,\,r} = \frac{\partial u_{m+l}}{\partial q_r} + b_{m+l,\,r}(q_{10}, q_2, \ldots, q_n).$$

Substituting these expressions into (11.10), we obtain the following differential relations:

$$dq_{m+l} = du_{m+l} + \sum_{r=2}^{m} b_{m+l,\,r}(q_{10}, q_2, \ldots, q_n)\, dq_r, \tag{11.12}$$

which are the desired conditions.

Thus, an equation of motion of a nonholonomic system can be written down for the coordinate q_1 in the form of a Lagrange equation of the second kind if a total differential of a certain function can be separated in the equations of the nonholonomic constraints in such a way that the remaining differential expression does not contain the coordinate q_1.

EXAMPLE 1. *Disc on a rough plane* (*Figure* 3.17). The condition of rolling of the disc without sliding can be expressed in terms of the two differential equations

$$dx = a\cos\psi d\varphi, \quad dy = a\sin\psi d\varphi. \tag{11.13}$$

The disc has three degrees of freedom. Taking θ, ψ, and φ as the independent coordinates, we can readily see from the form of equations (11.13) that the differential expressions that remain after the total differentials (dx and dy) have been separated do not contain the coordinate θ. Hence, the equation of motion of the disc for the coordinate θ can be written in the form of a Lagrange equation of the second kind. The Lagrange function L^* written down with allowance for the equations of the nonholonomic constraints (11.13) has the form

$$L^* = \frac{1}{2}[(A + ma^2)\dot\theta^2 + (C + ma^2)(\dot\varphi - \dot\psi\sin\theta)^2 + A\dot\psi^2\cos^2\theta]$$
$$- mga\cos\theta. \tag{11.14}$$

For the coordinate θ, we write down the equation

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot\theta} - \frac{\partial L^*}{\partial \theta} = 0, \tag{11.15}$$

obtaining

$$(A + ma^2)\ddot\theta + (C + ma^2)(\dot\varphi - \dot\psi \sin\theta)\dot\psi \cos\theta + A\dot\psi^2 \sin\theta \cos\theta - mga \sin\theta = 0.$$

Writing down Čaplygin's equation for this coordinate, we see that nonholonomy terms are not present in these equations. For, taking

$$q_1 = \theta,\ q_2 = \varphi,\ q_3 = \psi,\ q_4 = x,\ q_5 = y,$$

we obtain, in accordance with (11.10) and (11.13),

$$b_{42} = a\cos\psi, \quad b_{52} = a\sin\psi,$$

the remaining $b_{m+l,r}$ vanishing identically. Evaluating the partial derivatives, we find

$$\frac{\partial b_{42}}{\partial q_1} = \frac{\partial}{\partial\theta}(a\cos\psi) = 0, \quad \frac{\partial b_{52}}{\partial q_1} = \frac{\partial}{\partial\theta}(a\sin\psi) = 0, \quad \frac{\partial b_{m+l,1}}{\partial q_s} \equiv 0,$$

and, consequently, the relations (11.11) are satisfied; as is required for the vanishing of the nonholonomy terms.

However, equations of motion of the disc cannot be written down in the form (11.15) for the coordinates ψ and φ. In this example, there is no cyclic integral, since the coordinate θ is contained explicitly in the expression for the Lagrange function (11.14).

Now suppose that the motion of a nonholonomic system with the constraints

$$a_{m+l,i}\dot q_i = 0 \quad (i = 1, 2, \ldots, n;\ l = 1, 2, \ldots, n - m), \tag{11.16}$$

can be described by the equations in quasi-coordinates (5.15), which, in the case of potential forces, have the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot\pi_k} - \frac{\partial L^*}{\partial \pi_k} + \gamma_{kij}\frac{\partial L^*}{\partial \dot\pi_i}\dot\pi_j = 0 \ (j, k = 1, 2, \ldots, m;\ i = 1, 2, \ldots, n), \tag{11.17}$$

where L^* is a function of the coordinates $q_1, q_2, \cdots, q_n$ and, in general, all n quasi-velocities $\dot\pi_1, \dot\pi_2, \cdots, \dot\pi_n$. The coefficients γ_{kij} are determined by the expression

$$\gamma_{kij} = b_{\alpha k}b_{\beta j}\left(\frac{\partial a_{i\alpha}}{\partial q_\beta} - \frac{\partial a_{i\beta}}{\partial q_\alpha}\right) \quad (i, \alpha, \beta = 1, 2, \ldots, n;\ k, j = 1, 2, \ldots, m), \tag{11.18}$$

where we have introduced the notation

$$\frac{\partial L^*}{\partial \pi_k} = b_{ki} \frac{\partial L^*}{\partial q_i} \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m), \tag{11.19}$$

and $\| b_{\alpha i} \|$ is the inverse of the matrix $\| a_{i\beta} \|$ composed of the coefficients of the linear relations

$$\dot{\pi}_i = a_{i\beta} \dot{q}_\beta \qquad (i, \beta = 1, 2, \cdots, n). \tag{11.20}$$

It follows from equations (11.17) that a cyclic integral for the quasi-coordinate π_1 exists if (11.17) can be represented in the form

$$\frac{d}{dt} \frac{\partial L^*}{\partial \dot{\pi}_1} = 0, \tag{11.21}$$

i.e., if the following conditions hold:

$$b_{\alpha 1} b_{\beta j} \left(\frac{\partial a_{i\alpha}}{\partial q_\beta} - \frac{\partial a_{i\beta}}{\partial q_\alpha} \right) \frac{\partial L^*}{\partial \dot{\pi}_i} \dot{\pi}_j = 0, \tag{11.22}$$

$$b_{1i} \frac{\partial L^*}{\partial q_i} = 0 \qquad (\alpha, \beta, i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \tag{11.23}$$

The second condition (11.23) is satisfied, if, for example, T^* does not depend explicitly on the generalized coordinates. The condition (11.22) is satisfied for $\gamma_{1ij} = 0$, but this condition is not necessary, as will be seen from the following example.

EXAMPLE 2. *Free motion of a homogeneous sphere on a horizontal rough plane.* The kinetic energy T, written down without allowance for the nonholonomic constraints

$$\dot{x} - a\omega_y = 0, \quad \dot{y} + a\omega_x = 0,$$

where a is the radius of the sphere, has the form (k is the radius of gyration and the sphere has unit mass, $m = 1$)

$$T = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k^2 (\omega_x^2 + \omega_y^2 + \omega_z^2).$$

We introduce quasi-coordinates by means of the relations

$$\begin{aligned}
\dot{\pi}_1 &\equiv \omega_x = \dot{\theta} \cos\psi + \dot{\varphi} \sin\theta \sin\psi, \\
\dot{\pi}_2 &\equiv \omega_y = \dot{\theta} \sin\psi - \dot{\varphi} \sin\theta \cos\psi, \\
\dot{\pi}_3 &\equiv \omega_z = \dot{\varphi} \cos\theta + \dot{\psi}, \\
\dot{\pi}_4 &= a\dot{\theta} \sin\psi - a\dot{\varphi} \sin\theta \cos\psi - \dot{x}, \\
\dot{\pi}_5 &= a\dot{\theta} \cos\psi + a\dot{\varphi} \sin\theta \sin\psi + \dot{y}.
\end{aligned}$$

Solving these equations for the generalized velocities, we obtain

$$\dot{\theta} = \dot{\pi}_1 \cos\psi + \dot{\pi}_2 \sin\psi, \quad \dot{\varphi} = \dot{\pi}_1 \frac{\sin\psi}{\sin\theta} - \dot{\pi}_2 \frac{\cos\psi}{\sin\theta},$$
$$\dot{\psi} = -\dot{\pi}_1 \sin\psi \cot\theta + \dot{\pi}_2 \cos\psi \cot\theta + \dot{\pi}_3,$$
$$\dot{x} = a\dot{\pi}_2 - \dot{\pi}_4, \quad \dot{y} = -a\dot{\pi}_1 + \dot{\pi}_5.$$

Substituting these relations into the expression for T, we obtain

$$T^* = \frac{1}{2}[(a\dot{\pi}_2 - \dot{\pi}_4)^2 + (a\dot{\pi}_1 - \dot{\pi}_5)^2] + \frac{1}{2}k^2(\dot{\pi}_1^2 + \dot{\pi}_2^2 + \dot{\pi}_3^2). \tag{11.24}$$

Since the coefficients in the expression for the kinetic energy T^* are constants, the condition (11.23) is satisfied. Calculating the coefficients γ_{kij}, we find

$$\gamma_{213} = -\gamma_{312} = -1, \quad \gamma_{123} = -\gamma_{321} = 1,$$
$$\gamma_{231} = -\gamma_{132} = 1, \gamma_{341} \equiv -\gamma_{143} = a, \quad \gamma_{352} = -\gamma_{253} = a.$$

The remaining coefficients vanish identically, $\gamma_{kij} \equiv 0$. It follows from the expression (11.24) that

$$\frac{\partial T^*}{\partial \dot{\pi}_1} = (a^2 + k^2)\dot{\pi}_1 - a\dot{\pi}_5, \quad \frac{\partial T^*}{\partial \dot{\pi}_2} = (a^2 + k^2)\dot{\pi}_2 + a\dot{\pi}_4,$$
$$\frac{\partial T^*}{\partial \dot{\pi}_3} = k^2\dot{\pi}_3, \quad \frac{\partial T^*}{\partial \dot{\pi}_4} = a\dot{\pi}_2 + \dot{\pi}_4, \quad \frac{\partial T^*}{\partial \dot{\pi}_5} = -a\dot{\pi}_1 + \dot{\pi}_5.$$

Hence, it is clear that although not all the coefficients γ_{kij} vanish, the condition (11.22) is also satisfied. For direct calculation shows that

$$\gamma_{1ij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j = \left(\frac{\partial T^*}{\partial \dot{\pi}_2} - a\frac{\partial T^*}{\partial \dot{\pi}_4}\right)\dot{\pi}_3 - \frac{\partial T^*}{\partial \dot{\pi}_3}\dot{\pi}_2 = k^2\dot{\pi}_2\dot{\pi}_3 - k^2\dot{\pi}_3\dot{\pi}_2 = 0,$$

$$\gamma_{2ij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j = -\left(\frac{\partial T^*}{\partial \dot{\pi}_1} + a\frac{\partial T^*}{\partial \dot{\pi}_5}\right)\dot{\pi}_3 + \frac{\partial T^*}{\partial \dot{\pi}_3}\dot{\pi}_1 = -k^2\dot{\pi}_1\dot{\pi}_3 + k^2\dot{\pi}_3\dot{\pi}_1 = 0,$$

$$\gamma_{3ij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j = \left(\frac{\partial T^*}{\partial \dot{\pi}_1} + a\frac{\partial T^*}{\partial \dot{\pi}_5}\right)\dot{\pi}_2 + \left(-\frac{\partial T^*}{\partial \dot{\pi}_2} + a\frac{\partial T^*}{\partial \dot{\pi}_4}\right)\dot{\pi}_1$$
$$= k^2\dot{\pi}_1\dot{\pi}_2 - k^2\dot{\pi}_2\dot{\pi}_1 = 0.$$

Thus, equations (11.17) take the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_1} = 0, \quad \frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_2} = 0, \quad \frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_3} = 0,$$

and hence we obtain three cyclic integrals:

$$\dot{\pi}_1 \equiv \omega_x = \text{const}, \quad \dot{\pi}_2 \equiv \omega_y = \text{const}, \quad \dot{\pi}_3 \equiv \omega_z = \text{const}.$$

Generalized energy integral. The equations of motion of a nonholonomic system in quasi-coordinates are

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} + \gamma_{kij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j - \frac{\partial T^*}{\partial \pi_k} = \Pi \tag{11.25}$$
$$(j, k = 1, 2, \ldots, m; \quad i = 1, 2, \ldots, n),$$

where the kinetic energy T^* is a function of the n coordinates $q_1, q_2, \cdots, q_n$ and, in general, all n quasi-coordinates $\dot{\pi}_1, \dot{\pi}_2, \cdots, \dot{\pi}_n$. We recall that the last $n - m$ quasi-velocities $\dot{\pi}_{m+1}, \dot{\pi}_{m+2}, \cdots, \dot{\pi}_n$ in the expressions for $\partial T^*/\partial \dot{\pi}_i$ can only be set equal to zero by virtue of the nonholonomic constraints after the differentiation has been completed.

We multiply each of the equations (11.25) by $\dot{\pi}_k$ and sum the resulting k expressions:

$$\dot{\pi}_k\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_k} + \gamma_{kij}\frac{\partial T^*}{\partial \dot{\pi}_i}\dot{\pi}_j\dot{\pi}_k - \frac{\partial T^*}{\partial \pi_k}\dot{\pi}_k = \Pi_k\dot{\pi}_k$$
$$(j, k = 1, 2, \ldots, m; \quad i = 1, 2, \ldots, n).$$

By virtue of the property (4.27) of the coefficients γ_{kij}

$$\gamma_{kij}\dot{\pi}_j\dot{\pi}_k = 0,$$

so that the corresponding sum vanishes and the remaining part can be transformed to

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\pi}_k}\dot{\pi}_k\right) - \left(\frac{\partial T^*}{\partial \pi_k}\dot{\pi}_k + \frac{\partial T^*}{\partial \dot{\pi}_k}\ddot{\pi}_k\right) = \Pi_k\dot{\pi}_k. \tag{11.26}$$

In this expression, the second sum is the derivative with respect to the time of the kinetic energy of the system

$$\frac{\partial T^*}{\partial \pi_k}\dot{\pi}_k + \frac{\partial T^*}{\partial \dot{\pi}_k}\ddot{\pi}_k = \frac{dT^*}{dt} \tag{11.27}$$

provided that T^* does not depend on the time explicitly.

The expression for the kinetic energy T^*, in which $\dot{\pi}_{m+1} = \cdots = \dot{\pi}_n = 0$, consists, in the general case, of three terms

$$T^* = T_2^* + T_1^* + T_0^*,$$

where T_2^* is a quadratic form in the quasi-velocities $\dot{\pi}_k$; T_1^* is a linear form in the quasi-velocities; and T_0^* is an expression that does not contain the quasi-velocities $\dot{\pi}_k$ $(k = 1, 2, \cdots, m)$. By Euler's well-known theorem on homogeneous functions, we have

$$\frac{\partial T^*}{\partial \dot{\pi}_k}\dot{\pi}_k = 2T_2^* + T_1^* \qquad (k = 1, 2, \ldots, m).$$

Substituting this into (11.26) and using (11.27), we obtain

$$\frac{d}{dt}(2T_2^{\bullet} + T_1^{\bullet} - T_2^{\bullet} - T_1^{\bullet} - T_0^{\bullet}) = \Pi_k \dot{\pi}_k$$

or

$$d\,(T_2^{\bullet} - T_0^{\bullet}) = \Pi_k\, d\pi_k \quad (k = 1, 2, \ldots, m). \tag{11.28}$$

We now assume that the given forces acting on the nonholonomic system have a potential and, consequently,

$$\Pi_k = -\frac{\partial V}{\partial \pi_k} = -\frac{\partial V}{\partial q_i} b_{ik} \quad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m).$$

where b_{ik} are the coefficients in the linear relations that express the generalized velocities $\dot{q}_i$ of the system in terms of the derivatives of the quasi-coordinates. Hence

$$\Pi_k d\pi_k = -\,dV,$$

so that the expression (11.28) takes the form

$$d(T_2^* - T_0^* + V) = 0.$$

By direct integration, we arrive at the first integral of the motion of the nonholonomic system:

$$T_2^* - T_0^* + V = \text{const}, \tag{11.29}$$

which is the generalized energy integral.

We should mention that this integral holds in the mechanics of nonholonomic systems under the same conditions as in the mechanics of holonomic systems.

This is explained by the fact that the reaction forces of the nonholonomic constraints do not do any work, as is reflected by the corresponding property (4.27) of the coefficients γ_{kij} and the property (5.17) of the nonholonomy terms, which, as we have seen earlier, are physically equivalent to gyroscopic forces. It is precisely for this reason that the nonholonomy is not manifested in the derivation of the generalized energy integral. In writing down the latter, we can, by virtue of this circumstance, take the function T^* to be the expression for the kinetic energy written down with allowance for the nonholonomic constraints, i.e., $T^* = T^*(q_1, q_2, \cdots, q_n, \dot{\pi}_1, \dot{\pi}_2, \cdots, \dot{\pi}_m)$. In this case, when the kinetic energy T^* is a homogeneous quadratic form in the quasi-velocities $\dot{\pi}_1, \dot{\pi}_2, \cdots, \dot{\pi}_m$, the generalized integral (11.29) is identical with the usual energy integral

$$T^* + V = \text{const}, \tag{11.30}$$

where V is the potential energy of the system.

We shall now discuss some examples of nonholonomic systems for which the mechanical energy is conserved during their motion.

EXAMPLE 3. *Disc on a rough plane.* In accordance with the expression (11.14) for the Lagrange function L^* written down with allowance for the nonholonomic constraints, the energy integral in this problem has the form

$$(A + ma^2)\,\dot{\theta}^2 + (C + ma^2)(\dot{\varphi} - \dot{\psi}\sin\theta)^2 + A\dot{\psi}^2\cos^2\theta + 2mga\cos\theta = h, \tag{11.31}$$

where h is an arbitrary constant determined by the initial conditions.

EXAMPLE 4. *Sphere on a rough plane.* Setting $\dot{\pi}_4 = 0$ and $\dot{\pi}_5 = 0$ in the expression for the kinetic energy (11.24), we obtain

$$T^* = \frac{1}{2}(a^2 + k^2)(\omega_x^2 + \omega_y^2) + \frac{1}{2}k^2\omega_z^2.$$

Since the potential energy does not change when a homogeneous sphere moves on a horizontal plane, the energy integral has the form

$$(a^2 + k^2)(\omega_x^2 + \omega_y^2) + k^2\omega_z^2 = h. \tag{11.32}$$

EXAMPLE 5. A homogeneous sphere of radius a and mass m rolls under the influence of the force of gravity on the surface of a fixed sphere of radius b (Figure 3.18).

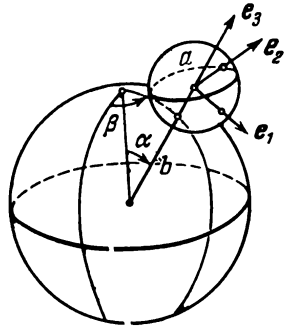

FIGURE 3.18

Let b, α, and β be the polar coordinates of the point of contact of the spheres (Figure 3.18). True coordinates determining the position of the rolling sphere are the Eulerian angles θ, φ, and ψ and the angles α and β. We introduce a moving system of coordinates with origin at the center of the moving sphere, as shown in Figure 3.18. The unit vector $\boldsymbol{e}_2$ is always horizontal and the vector $\boldsymbol{e}_3$ is directed along the line joining the centers of the spheres. Denoting by ω_1, ω_2, and ω_3 the projections of the angular velocity onto the axes of the moving system, we can write down an expression for the Lagrange function

$$L = \frac{m}{2}\left[(a+b)^2(\dot{\alpha}^2 + \dot{\beta}^2\sin^2\alpha) + \frac{2a^2}{5}(\omega_1^2 + \omega_2^2 + \omega_3^2\right] - mg(a+b)\cos\alpha. \tag{11.33}$$

We introduce the quasi-coordinates $\pi_1, \pi_2, \cdots, \pi_5$ by means of the relations

$$
\begin{aligned}
&\dot{\pi}_1 = \dot{\alpha}, \qquad \dot{\pi}_2 = \dot{\beta},\\
&\dot{\pi}_3 \equiv \omega_3 = \dot{\theta}\sin\alpha\cos(\psi-\beta) + \dot{\varphi}\sin\alpha\sin\theta\sin(\psi-\beta) +\\
&\qquad\qquad + (\dot{\varphi}\cos\theta + \dot{\psi})\cos\alpha,\\
&\dot{\pi}_4 = a\omega_2 + (a+b)\dot{\beta}\sin\alpha, \qquad \dot{\pi}_5 = a\omega_1 + (a+b)\dot{\alpha},
\end{aligned}
\tag{11.34}
$$

where π_1 and π_2 are true coordinates. The equations that express the condition of rolling without sliding of the point of contact of the moving sphere have the form

$$
\left.\begin{aligned} a\omega_2 + (a+b)\dot{\beta}\sin\alpha = 0,\\ a\omega_1 + (a+b)\alpha = 0. \end{aligned}\right\}
\tag{11.35}
$$

Using (11.34), we obtain from (11.33):

$$
\begin{aligned}
L^* = \frac{m}{2}\Big\{&(a+b)^2(\dot{\pi}_1^2 + \dot{\pi}_2^2\sin^2\alpha) + \frac{2}{5}[\dot{\pi}_4 - (a+b)\dot{\pi}_2\sin\alpha]^2\\
&+ [\dot{\pi}_5 - (a+b)\dot{\pi}_1]^2 + a^2\dot{\pi}_3^2\Big\} - mg(a+b)\cos\alpha.
\end{aligned}
\tag{11.36}
$$

It follows from (11.34) that the transformation matrix a_{ki} has the form

$\dot{\alpha}$	$\dot{\beta}$	$\dot{\theta}$	$\dot{\varphi}$	$\dot{\psi}$
1	0	0	0	0
0	1	0	0	0
0	0	$\sin\alpha\cos(\psi-\beta)$	$\sin\alpha\sin\theta\sin(\psi-\beta) + \cos\alpha\cos\theta$	$\cos\alpha$
0	$(a+b)\sin\alpha$	$\alpha\sin(\psi-\beta)$	$-a\sin\theta\cos(\psi-\beta)$	0
$a+b$	0	$a\cos(\psi-\beta)\cos\alpha$	$a[\cos\alpha\sin\theta\sin(\psi-\beta) - \cos\theta\sin\alpha]$	$-a\sin\alpha$

Hence, one can find the matrix b_{ik} of the inverse transformation, which expresses the generalized velocities $\dot{q}_i$ in terms of the quasi-velocities $\dot{\pi}_k$. Writing down the equations of motion in quasi-coordinates (11.17), we can readily prove the existence of the cyclic integral $\partial L^*/\partial\dot{\pi}_3 = \text{const}$; hence, $\omega_3 = \text{const}$. Thus, the energy integral in this example can be written in the form

$$
\dot{\alpha}^2 + \dot{\beta}^2\sin^2\alpha + \frac{10g\cos\alpha}{7(a+b)} = 2h = \text{const}.
\tag{11.37}
$$

§12. Theory of Čaplygin's reducing multiplier (last multiplier)

In certain cases, as was shown by Čaplygin, the equations of motion of nonholonomic systems can be reduced to the form of Lagrange equa-

tions of the second kind or to the form or ordinary canonical equations. This reduction is affected by the introduction of a new independent variable τ determined by the equation

$$d\tau = N\,dt, \tag{12.1}$$

where N is a function of the independent coordinates which must be determined. Čaplygin [85] gave the name *reducing multiplier* to the function N, which under certain conditions is a last multiplier.

In his paper [85], Čaplygin developed the theory of the reducing multiplier for the case of equations in true coordinates. However, even in the example of plane nonholonomic motion used to illustrate the theory of the reducing multiplier, he employed a variable which was, in fact, a quasi-coordinate. Although Čaplygin did not notice this circumstance and his derivation of the theory of the reducing multiplier was made only for true coordinates, his examples are correct. Moreover, it is not difficult to generalize the theory of Čaplygin's reducing multiplier to the case of quasi-coordinates. In the present section, we shall therefore develop the generalized theory.

We shall consider a nonholonomic Čaplygin system with two degrees of freedom. As free parameters, we choose the quasi-coordinates π_1 and π_2, in terms of which all the true coordinates are given by the expressions

$$\dot{q}_i = b_{i1}\dot{\pi}_1 + b_{i2}\dot{\pi}_2 \quad (i = 1, 2, \ldots, n). \tag{12.2}$$

Suppose that the Lagrange function L depends only on q_1 and q_2 and, in general, on all the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ and that the coefficients b_{i1} and b_{i2} are functions of q_1 and q_2. Čaplygin's equations (3.32) in quasi-coordinates, which describe the motion of the system, can be written in the form

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_1} - \frac{\partial L^*}{\partial \pi_1} &= \dot{\pi}_2 S, \\ \frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_2} - \frac{\partial L^*}{\partial \pi_2} &= -\dot{\pi}_1 S, \end{aligned}\right\} \tag{12.3}$$

where we have introduced the notation

$$S = \frac{\partial L}{\partial \dot{q}_i}\left(\frac{\partial b_{i2}}{\partial \pi_1} - \frac{\partial b_{i1}}{\partial \pi_2}\right) \quad (i = 1, 2, \ldots, n), \tag{12.4}$$

and the function $L^* = L^*(q_1, q_2, \dot{\pi}_1, \dot{\pi}_2)$ is obtained from L by replacing the generalized velocities $\dot{q}_i$ by their expressions (12.2). In order to

transform equations (12.3), we introduce a new independent variable τ by means of the relation (12.1).

Suppose that the expression for the kinetic energy T^* of the system is a quadratic form in the quasi-velocities $\dot{\pi}_1$ and $\dot{\pi}_2$. Using a prime to denote derivatives with respect to τ, we obtain

$$2T^* = L_1\dot{\pi}_1^2 + 2M\dot{\pi}_1\dot{\pi}_2 + L_2\dot{\pi}_2^2 = N^2(L_1\pi_1'^2 + 2M\pi_1'\pi_2' + L_2\pi_2'^2) = 2\widetilde{T},$$

$$\frac{\partial T^*}{\partial \dot{\pi}_\beta} = \frac{\partial \widetilde{T}}{\partial \pi_\beta'}\frac{\partial \pi_\beta'}{\partial \dot{\pi}_\beta} = \frac{1}{N}\frac{\partial \widetilde{T}}{\partial \pi_\beta'}, \quad \dot{\pi}_\beta = N\pi_\beta' \quad (\beta = 1, 2),$$

$$\frac{\partial T^*}{\partial q_\alpha} = \frac{\partial \widetilde{T}}{\partial q_\alpha} - \frac{1}{N}\frac{\partial N}{\partial q_\alpha}\frac{\partial \widetilde{T}}{\partial \pi_\sigma'}\pi_\sigma' \quad (\alpha, \sigma = 1, 2).$$

Hence, by virtue of (3.31), we have

$$\frac{\partial T^*}{\partial \pi_\beta} = \frac{\partial T^*}{\partial q_\alpha}b_{\alpha\beta} - \frac{1}{N}\frac{\partial N}{\partial q_\alpha}b_{\alpha\beta}\frac{\partial \widetilde{T}}{\partial \pi_\sigma'}\pi_\sigma',$$

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\beta} = \frac{d}{d\tau}\frac{\partial \widetilde{T}}{\partial \pi_\beta'} - \frac{1}{N}\frac{\partial \widetilde{T}}{\partial \pi_\beta'}\frac{\partial N}{\partial q_\alpha}b_{\alpha\sigma}\pi_\sigma' \quad (\alpha, \beta, \sigma = 1, 2).$$

Using these relations, we reduce the differential equations (12.3) to the form

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \pi_1'} - \frac{\partial \widetilde{L}}{\partial \pi_1} = \pi_2'R, \quad \frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \pi_2'} - \frac{\partial \widetilde{L}}{\partial \pi_2} = -\pi_1'R, \tag{12.5}$$

where $\widetilde{L} = \widetilde{T} - V$ and the function R is given by

$$R = NS + \frac{1}{N}\left(\frac{\partial \widetilde{T}}{\partial \pi_1'}\frac{\partial N}{\partial \pi_2} - \frac{\partial \widetilde{T}}{\partial \pi_2'}\frac{\partial N}{\partial \pi_1}\right). \tag{12.6}$$

Equation (12.5) takes the form of ordinary Lagrange equations of the second kind if the function N is defined in such a way that

$$R = 0. \tag{12.7}$$

We introduce the canonical variables

$$p_1 = \frac{\partial \widetilde{T}}{\partial \pi_1'} = N^2(L_1\pi_1' + M\pi_2'), \quad p_2 = \frac{\partial \widetilde{T}}{\partial \pi_2'} = N^2(M\pi_1' + L_2\pi_2'). \tag{12.8}$$

In the expressions for $\partial L/\partial \dot{q}_i$, we replace the quantities $\dot{q}_i$ by p_1 and p_2 by means of the relations (12.2) and (12.8), obtaining

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{1}{N}(A_ip_1 + B_ip_2), \tag{12.9}$$

where A_i and B_i are known functions of the generalized coordinates

q_1 and q_2. Substituting (12.9) into (12.6), we obtain a linear expression in p_1 and p_2. The condition (12.7) is certainly satisfied if the function N satisfies both the equations

$$\frac{1}{N}\frac{\partial N}{\partial \pi_1} = B_i\left(\frac{\partial b_{i2}}{\partial \pi_1} - \frac{\partial b_{i1}}{\partial \pi_2}\right), \quad \frac{1}{N}\frac{\partial N}{\partial \pi_2} = -A_i\left(\frac{\partial b_{i2}}{\partial \pi_1} - \frac{\partial b_{i1}}{\partial \pi_2}\right), \tag{12.10}$$

where

$$\frac{\partial b_{i\alpha}}{\partial \pi_\beta} = \frac{\partial b_{i\alpha}}{\partial q_\sigma} b_{\sigma\beta} \qquad (i = 1, 2, \ldots, n;\ \alpha, \beta, \sigma = 1, 2).$$

It is readily seen that equations (12.10) are identical with the well-known equations of Čaplygin for the reducing multiplier in the case when the quasi-coordinates π_1 and π_2 are true coordinates. For setting $\pi_1 = q_1$ and $\pi_2 = q_2$, we obtain $b_{11} = b_{22} = 1$, $b_{12} = b_{21} = 0$, and equations (12.8) take the form

$$\frac{1}{N}\frac{\partial N}{\partial q_1} = B_j\left(\frac{\partial b_{j2}}{\partial q_1} - \frac{\partial b_{j1}}{\partial q_2}\right), \quad \frac{1}{N}\frac{\partial N}{\partial q_2} = -A_j\left(\frac{\partial b_{j2}}{\partial q_1} - \frac{\partial b_{j1}}{\partial q_2}\right) \qquad (j = 3, 4, \ldots, n), \tag{12.11}$$

given by Čaplygin.

Comparison of the expressions (3.32) and (3.33) and also (12.10) and (12.11) shows that Čaplygin's equations and the equations for the reducing multiplier can be written in the same form for true coordinates as for quasi-coordinates. Hence, the extension of the theorem on the reducing multiplier to the case of quasi-coordinates is completely justified. However, in the case of quasi-coordinates, the expressions (3.31) must be used in the calculations of the partial derivatives with respect to the quasi-coordinates, in contrast to the case of true coordinates. Therefore, the final form of the equations given above for the case of quasi-coordinates may, in general, differ from the form of the equations in the case of true coordinates. At the same time, there is a class of Čaplygin systems for which the results are completely identical in the two cases. This class possesses two distinguishing features.

1. The number l of true coordinates on which the coefficients of the nonholonomic constraints and the Lagrange function depend is less than the number m of degrees of freedom of the system.

2. The number k of quasi-coordinates which, together with the l true coordinates, are chosen as free parameters does not exceed the number $m - l$.

We shall now show that if these conditions are satisfied equations

(3.32) and (12.10) are, respectively, identical with Čaplygin's equations (3.33) and (12.11). For suppose the first l $(l < m)$ quasi-coordinates in the relations (3.27) are true coordinates and the coefficients $b_{i\sigma}$ depend only on $q_1, q_2, \cdots, q_l$. It then follows that $b_{r\sigma} = \delta_{r\sigma}$ for $r, \sigma = 1, 2, \cdots, l$ ($\delta_{r\sigma}$ is the Kronecker delta) and $b_{r\sigma} = 0$ for $\sigma = l+1, l+2, \cdots, m$. Suppose that the Lagrange function contains, apart from the generalized velocities, only the coordinates $q_1, q_2, \cdots, q_l$. In the first l equations of (3.32) we have, in accordance with (3.31),

$$\frac{\partial L}{\partial \dot{q}_i}\left(\frac{\partial b_{i\sigma}}{\partial \pi_\alpha} - \frac{\partial b_{i\alpha}}{\partial \pi_\sigma}\right)\dot{\pi}_\sigma = \frac{\partial L}{\partial \dot{q}_j}\left(\frac{\partial b_{j\sigma}}{\partial q_r} b_{r\sigma} - \frac{\partial b_{j\alpha}}{\partial q_s} b_{s\sigma}\right)\pi_\sigma$$

$$= \frac{\partial L}{\partial q_j}\left(\frac{\partial b_{js}}{\partial q_r} - \frac{\partial b_{jr}}{\partial q_s}\right)\dot{q}_s$$

$$(j = l+1, l+2, \ldots, n;\ r, s = 1, 2, \ldots, l),$$

so that the l equations of (3.32) take the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_r} - \frac{\partial L^*}{\partial q_r} + \frac{\partial L}{\partial \dot{q}_j}\left(\frac{\partial b_{js}}{\partial q_r} - \frac{\partial b_{jr}}{\partial q_s}\right)\dot{q}_s = 0 \tag{12.12}$$

$$(j = l+1, \ldots, n; \qquad r, s = 1, 2, \ldots, l).$$

We see from equations (3.33) that, in the class of problems under consideration, the subscript j in equations (3.33) must take the values $j = l+1, l+2, \cdots, n$, since the $n - l$ generalized velocities $\dot{q}_{l+1}, \dot{q}_{l+2}, \cdots, \dot{q}_n$ can be expressed in terms of the independent parameters. Since the quantities $q_{l+1}, q_{l+2}, \cdots, q_m$ occur nowhere, the corresponding derivatives vanish and, hence, the first l equations of (3.33) are identical with (12.12).

It remains to prove that, when we write down the remaining $m - l$ equations, we arrive at identical expressions irrespective of whether we take equations (3.32) or (3.33) as the original equations. In accordance with (3.31), we make the following calculations:

$$\frac{\partial L^*}{\partial \pi_\rho} = \frac{\partial L^*}{\partial q_r} b_{r\rho} = 0, \qquad \frac{\partial b_{i\sigma}}{\partial \pi_\rho} = \frac{\partial b_{i\sigma}}{\partial q_r} b_{r\rho} = 0,$$

$$\frac{\partial b_{i\rho}}{\partial \pi_\sigma}\dot{\pi}_\sigma = \frac{\partial b_{i\rho}}{\partial q_r} b_{r\sigma}\dot{\pi}_\sigma = \frac{\partial b_{i\rho}}{\partial q_r}\dot{q}_r$$

$$\begin{pmatrix} i = 1, 2, \ldots, n;\ j = l+1, l+2, \ldots, n; \\ \rho = l+1, l+2, \ldots, m;\ r = 1, 2, \ldots, l. \end{pmatrix}$$

Substituting this into (3.32), we obtain the equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_\rho} - \frac{\partial L}{\partial \dot{q}_j}\frac{\partial b_{j\rho}}{\partial q_r}\dot{q}_r = 0 \tag{12.13}$$

$$\left(\begin{array}{c} j = l+1,\ l+2, \ldots, n; \\ \rho = l+1,\ l+2, \ldots, m;\ r = 1,\ 2, \ldots, l. \end{array}\right)$$

Now, taking (3.33) for the equations with subscript $l+1, l+2, \cdots, m$, we also arrive at (12.13) since the quantities $\dot{q}_\rho$ $(\rho = l+1, l+2, \cdots, m)$ have, in this case, exactly the same meaning as $\dot{\pi}_\rho$.

Turning to the equations for the reducing multiplier, we can show, in the same way, that the final equations are identical, irrespective of whether we take (12.10) or (12.11) as the original equations. For the case $m = 2$ with the presence of a single true coordinate q_1, we can only add a single quasi-coordinate π_2 if we do not wish to leave the class of Čaplygin systems. In accordance with (12.2), we have $b_{11} = 1$, $b_{12} = 0$, $\pi_1 = q_1$,

$$\frac{\partial b_{i2}}{\partial \pi_1} = \frac{\partial b_{i2}}{\partial q_1} = \frac{\partial b_{j2}}{\partial q_1},\quad \frac{\partial b_{i1}}{\partial \pi_2} = \frac{\partial b_{i1}}{\partial q_1} b_{12} = 0$$
$$(i = 1, 2, \ldots, n;\ j = 2, 3, \ldots, n),$$

and, hence, equations (12.10) can be transformed to the form

$$\frac{1}{N}\frac{\partial N}{\partial q_1} = B_j \frac{\partial b_{j2}}{\partial q_1},\quad \frac{1}{N}\frac{\partial N}{\partial \pi_2} = -A_j \frac{\partial b_{j2}}{\partial q_1} \tag{12.14}$$
$$(j = 2,\ 3, \ldots,\ n).$$

Now, taking equations (12.11) as the original equations, we must bear in mind that the subscript j takes the values $j = 2, 3, \cdots, n$ and that the coordinate q_2 does not occur explicitly in the coefficients b_{j2}. Hence, we obtain directly equations (12.14) for the reducing multiplier $N = N(q_1, \pi)$. Thus, we have found a class of problems for which the results are identical in the two cases. It so happens that Čaplygin's problem of plane-parallel nonholonomic motion belongs to this class of problem. We shall therefore take it as the first example to illustrate the use of the theorem on the reducing multiplier.

EXAMPLE 1. *Motion of a rigid body parallel to a plane.* Let us consider the free motion of a body on a horizontal plane, the body being supported on the plane by three points, of which two are freely sliding legs and the third is the point A of contact of a sharp semicircular knife edge. The position of the body can be specified by the horizontal coordinates x and y of the point of contact of the knife edge and the angle φ which an axis attached to the body and lying in the plane of the knife edge makes with the Ox axis. We shall denote by α and β the coordinates of

the center of mass of the body in a system of axes fixed to the body, the $A\xi$ axis being taken along the knife edge and the origin coinciding with the point A. We shall take the angle φ and the path length q of the trajectory of the point A as the free parameters. It is easily seen that the parameter q is a quasi-coordinate. The constraints of the system reduce to the two relations

$$\dot{x} = \dot{q}\cos\varphi, \quad \dot{y} = \dot{q}\sin\varphi. \tag{12.15}$$

The expression for the kinetic energy T written down without allowance for the nonholonomic constraints (12.15) is given by

$$T = \frac{1}{2} m \times \left\{[\dot{x} - \dot{\varphi}(\alpha\sin\varphi + \beta\cos\varphi)]^2 + [\dot{y} + \dot{\varphi}(\alpha\cos\varphi - \beta\sin\varphi)]^2 + k^2\dot{\varphi}^2\right\},$$

where m is the mass and k is the radius of gyration of the body about the vertical axis passing through the center of gravity. Hence, using the conditions (12.15), we find

$$\begin{aligned} T^* &= \frac{1}{2} m\,[\dot{q}^2 + (\alpha^2 + \beta^2 + k^2)\,\dot{\varphi}^2 - 2\beta\dot{\varphi}\dot{q}], \\ \frac{1}{m}\frac{\partial T}{\partial \dot{x}} &= \dot{q}\cos\varphi - \dot{\varphi}(\alpha\sin\varphi + \beta\cos\varphi), \\ \frac{1}{m}\frac{\partial T}{\partial \dot{y}} &= \dot{q}\sin\varphi + \dot{\varphi}(\alpha\cos\varphi - \beta\sin\varphi). \end{aligned} \tag{12.16}$$

We can obtain the function S from (12.4) by means of (12.15):

$$S = m\alpha\dot{\varphi} = mN\alpha\varphi',$$

where N is Čaplygin's reducing multiplier. We can find the functions p and p_1 on the basis of (12.8) from (12.16):

$$\begin{aligned} p &= N^2\,(mq' - m\beta\varphi'), \\ p_1 &= N^2\,[m\,(\alpha^2 + \beta^2 + k^2)\,\varphi' - m\beta q'], \end{aligned}$$

and hence

$$mN^2\varphi'(\alpha^2 + k^2) = p_1 + \beta p, \quad NS = mN^2\alpha\varphi' = \frac{\alpha\,(p_1 + \beta p)}{\alpha^2 + k^2}.$$

Substituting this expression into equation (12.7), we obtain the following partial differential equation for the reducing multiplier N:

$$p\,\frac{1}{N}\frac{\partial N}{\partial \varphi} - p_1\,\frac{1}{N}\frac{\partial N}{\partial q} = \alpha\,\frac{p_1 + \beta p}{\alpha^2 + k^2}.$$

We introduce the new variables ν and φ related to the old variables by the equations

$$N = e^{hu}, \quad u = \frac{\beta\varphi - q}{\sqrt{\alpha^2 + k^2}},$$

where

$$h = \frac{\alpha}{\sqrt{\alpha^2 + k^2}}.$$

In these variables, the expression (12.16) for the kinetic energy T^* takes the form

$$\widetilde{T}^* = \frac{m}{2} e^{2hu} (\alpha^2 + k^2) (u'^2 + \varphi'^2).$$

Solving this problem by the Ostrogradskiĭ-Jacobi method, we arrive at the following partial differential equation, which determines the characteristic function W:

$$\frac{e^{-2hu}}{m(\alpha^2 + k^2)} \left[\left(\frac{\partial W}{\partial u} \right)^2 + \left(\frac{\partial W}{\partial \varphi} \right)^2 \right] = \lambda.$$

The complete integral of this equation is

$$W = g\varphi + \int_{u_0}^{u} \sqrt{\mu e^{2hu} - g^2}\, du,$$

where we have introduced the notation $\mu = \lambda m(\alpha^2 + k^2)$, and λ and g are arbitrary constants. The integrals of the canonical system of equations of motion are

$$\frac{\partial W}{\partial g} = \varphi + \frac{1}{h} \sin^{-1} \frac{g e^{-hu}}{\sqrt{\mu}} = g_1,$$

$$\frac{\partial W}{\partial \lambda} = \tau - \lambda_1 = \int_{u_0}^{u} \frac{m(\alpha^2 + k^2) e^{2hu}\, du}{2\sqrt{\mu e^{2hu} - g^2}}.$$

Here, λ_1 and g_1 are arbitrary constants and the time is given by

$$dt = \frac{d\tau}{N} = \frac{m(\alpha^2 + k^2) e^{hu}\, du}{2\sqrt{\mu e^{2hu} - g^2}}.$$

Integrating this differential equation, we obtain

$$t = -\sqrt{\frac{m}{\lambda}} \frac{\alpha^2 + k^2}{2\alpha} \ln \left(\sqrt{\mu}\, e^{hu} - \sqrt{\mu e^{2hu} - g^2} \right) + C.$$

The solution of the problem is completed by the formulas which can be obtained from equations (12.15). In accordance with the parameters introduced, we have

$$\dot{x} = \dot{q} \cos\varphi = (\beta\dot{\varphi} - \dot{u}\sqrt{\alpha^2 + k^2}) \cos\varphi,$$
$$\dot{y} = \dot{q} \sin\varphi = (\beta\dot{\varphi} - \dot{u}\sqrt{\alpha^2 + k^2}) \sin\varphi,$$

from which we obtain the solution in quadratures

$$x = \beta \sin\varphi - \sqrt{\alpha^2 + k^2}\int \cos\varphi\, du,$$
$$y = -\beta \cos\varphi - \sqrt{\alpha^2 + k^2}\int \sin\varphi\, du.$$

In the special case when the axis of the knife edge passes through the projection of the center of gravity ($\alpha = 0$), the reducing multiplier is a constant; $N = \text{const}$. If the coordinate axes are chosen appropriately, we obtain $u = c_1 t$, $\varphi = c_2 t$, $q = (\beta c_2 - k c_1)t$, where c_1 and c_2 are arbitrary constants. Hence

$$x = \left(\beta - k\frac{c_1}{c_2}\right)\sin c_2 t, \quad y = -\left(\beta - k\frac{c_1}{c_2}\right)\cos c_2 t.$$

In this case, the motion of the body reduces to a rotation with a constant angular velocity c_2 about a fixed vertical axis which intersects the axis of the knife edge and whose position depends on the initial velocity of the latter.

EXAMPLE 2. *Rolling on a horizontal plane of an ellipsoid of revolution.* Let the center of gravity of the body coincide with the center O_1 of its ellipsoidal surface, whose symmetry axis is one of the principal axes of inertia.

We shall consider the general case when all three semiaxes of the central ellipsoid of inertia are different, i.e., the moments of inertia satisfy $A \neq B \neq C$. Apart from the constraint due to the rolling of the body on the plane without sliding, we shall assume that an additional constraint is imposed which ensures that the line of intersection of the equatorial plane of the ellipsoid of rotation with the horizontal has a fixed direction $\boldsymbol{n}^0$ in space.

We introduce a fixed system of coordinates $Oxyz$ with Oz axis upward and the xOy plane as the horizontal plane on which the body rolls, the Oy axis being parallel to $\boldsymbol{n}^0$. As moving coordinate axes $O_1\xi\eta\zeta$, we take the principal axes of the ellipsoid of inertia (Figure 3.19).

Let $O_1\zeta$ coincide with the symmetry axis of the bounding ellipsoid, whose equatorial and polar semiaxes we shall denote

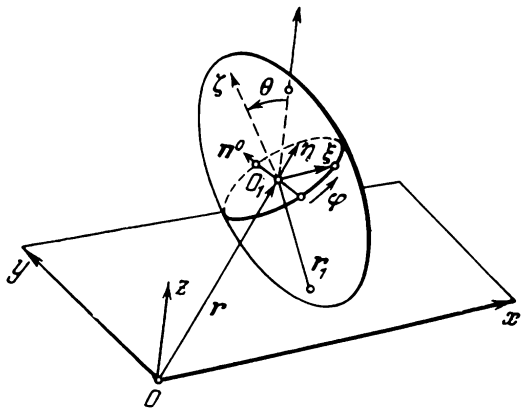

FIGURE 3.19

by c and c_1, respectively. We shall specify the position of the body by the coordinates x, y, and z of the center of gravity and the Eulerian angles θ and φ, where θ is the angle between Oz and $O_1\zeta$ and φ is the angle between Oy and $O_1\xi$. In accordance with the statement of the problem, the third Eulerian angle ψ remains constant and is a right angle. The condition of rolling without sliding leads to the following vector equation (Figure 3.19):

$$\dot{\boldsymbol{r}} + \omega \times \boldsymbol{r}_1 = 0. \tag{12.17}$$

We introduce a moving system of coordinates $O_1x_1y_1z_1$ such that the O_1y_1 axis is always horizontal and the O_1z_1 axis coincides with the $O_1\zeta$ axis. The point of contact of the body and the plane then lies on an ellipse whose equation is in parametric form is

$$z_1 = c_1\cos\alpha, \quad x_1 = c\sin\alpha, \tag{12.18}$$

where α is a running parameter whose meaning is clear from Figure 3.20. It is readily seen that α and θ are related by the equation

$$c\cot\alpha = c_1\cot\theta,$$

which can be used to transform (12.18) to the form

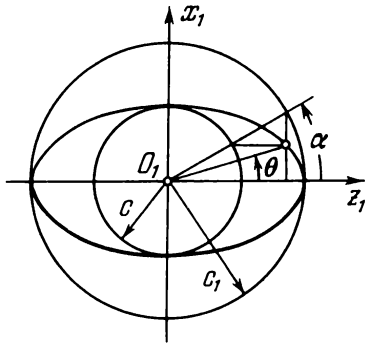

FIGURE 3.20

$$z_1 = \frac{c_1^2}{h}\cos\theta, \qquad x_1 = \frac{c^2}{h}\sin\theta,$$

where $h^2 = c^2\sin^2\theta + c_1^2\cos^2\theta$. The components of the vectors that occur in equation (12.17) are

$$\left.\begin{aligned} &\boldsymbol{r}\{\dot{x},\ \dot{y},\ \dot{z}\}, \quad \boldsymbol{\omega}\{\dot{\varphi}\sin\theta,\ \dot{\theta},\ \dot{\varphi}\cos\theta\}, \\ &\boldsymbol{r}_1\left\{\frac{c^2 - c_1^2}{h}\sin\theta\cos\theta,\ 0,\ -h\right\}, \end{aligned}\right\} \tag{12.19}$$

where the components of $\boldsymbol{r}_1$ are found from the relations

$$x = -z_1\sin\theta + x_1\cos\theta, \quad z = -x_1\sin\theta - z_1\cos\theta, \quad y = 0.$$

In accordance with (12.19), equation (12.17) is equivalent to the three scalar equations

$$\left.\begin{aligned} &\dot{x} - h\dot{\theta} = 0, \\ &\dot{y} = \dot{\varphi}\left(h\sin\theta + \frac{c^2 - c_1^2}{h}\sin\theta\cos^2\theta\right) = \frac{c^2}{h}\dot{\varphi}\sin\theta, \end{aligned}\right\} \tag{12.20}$$

(12.21) $$\dot{z} - \frac{c^2}{h}\dot{\theta}\sin\theta\cos\theta = 0,$$

of which only equation (12.20) is nonintegrable. Equation (12.21) reduces to $z = h$ and, hence, the coordinate z, as also x, can be expressed in terms of the parameter θ. Thus, one can take θ, φ, and y as generalized coordinates of the system and θ and φ as independent parameters. The kinetic energy of the body, written down without allowance for the nonholonomic constraint (12.20), is

$$2T = m\,(h^2\dot{\theta}^2 + \dot{h}^2 + \dot{y}^2) + (A\cos^2\varphi + B\sin^2\varphi)\,\dot{\theta}^2 + C\dot{\varphi}^2.$$

With allowance for the nonholonomic constraint (12.20), this becomes

$$2T^* = \left(C + m\frac{c^4}{h^2}\sin^2\theta\right)\dot{\varphi}^2 + \left(A\cos^2\varphi + B\sin^2\varphi + mc_1^2 + mc^2\frac{c^2 - c_1^2}{h^2}\sin^2\theta\right)\dot{\theta}^2.$$

Introducing the notation $q_1 = \theta$, $q_2 = \varphi$, and $q_3 = y$, we find, in accordance with (12.2) and (12.20), that only the following coefficients are nonvanishing:

$$b_{11} = b_{22} = 1, \quad b_{32} = \frac{c^2}{h}\sin\theta.$$

Hence,

$$\frac{\partial T}{\partial \dot{y}} = m\dot{y} = \frac{1}{h}mc^2\dot{\varphi}\sin\theta,$$

and, in accordance with (12.4), we obtain

$$S = -\frac{mc^4c_1^2\sin\theta\cos\theta}{h^4}\dot{\varphi} = -\frac{N}{2}\varphi'\frac{d}{d\theta}\left(\frac{mc^4}{h^2}\sin^2\theta\right).$$

We introduce the canonical variables p and p_1:

$$p = \left(A\cos^2\varphi + B\sin^2\varphi + mc_1^2 + mc\frac{c^2 - c_1^2}{h^2}\sin\theta\right)N^2\varphi',$$

$$p_1 = \left(C + \frac{mc^4}{h^2}\sin^2\theta\right)N^2\varphi',$$

and, hence,

$$NS = -\frac{p_1}{2}\frac{d}{d\theta}\ln\left(C + \frac{mc^4}{h^2}\sin^2\theta\right).$$

Thus, the reducing multiplier must be chosen such that the following equation holds:

(12.22) $$p\frac{1}{N}\frac{\partial N}{\partial\varphi} - p_1\left[\frac{1}{2}\frac{d}{d\theta}\ln\left(C + \frac{mc^4}{h^2}\sin^2\theta\right) + \frac{1}{N}\frac{\partial N}{\partial\theta}\right] = 0.$$

Equation (12.22) will be satisfied if N does not depend on φ and, in addition, it is a solution of the equation

$$\frac{1}{2}\frac{d}{d\theta}\ln\left(C+\frac{mc^4}{h^2}\sin^2\theta\right)+\frac{1}{N}\frac{\partial N}{\partial\theta}=0.$$

Hence,

$$N=\left(C+\frac{mc^4}{h^2}\sin^2\theta\right)^{-1/2},$$

and the kinetic energy $\widetilde{T}$ of the system can be written in the form

$$\widetilde{T}=\frac{1}{2}(\alpha\dot{\theta}'^2+\varphi'^2),$$

where we have introduced the notation

$$\alpha=\frac{(A\cos^2\varphi+B\sin^2\varphi+mc_1^2)h^2+mc^2(c^2-c_1^2)\sin^2\theta}{Ch^2+mc^4\sin^2\theta}. \tag{12.23}$$

The solution of the problem reduces to finding the complete integral of the following partial differential equation:

$$\frac{1}{2}\left[\alpha\left(\frac{\partial W}{\partial\theta}\right)^2+\left(\frac{\partial W}{\partial\varphi}\right)^2\right]+mgh=\lambda, \tag{12.24}$$

where $\alpha=\alpha(\theta,\varphi)$, $h=h(\theta)$, and λ is an arbitrary constant. Hence, and by (12.23), it follows that the problem reduces very simply to quadratures in two cases.

1) The case when the principal moments of inertia A and B are equal, $A=B$, i.e., when the ellipsoid of inertia is an ellipsoid of revolution.

2) The case when the bounding surface of the figure of revolution is spherical, i.e., $c=p_1$.

In the first case, the quantity α is a function of the single variable θ. The complete integral of equation (12.24) can be written in the form

$$W=\int_{\theta_0}^{\theta}\sqrt{\frac{1}{\alpha(\theta)}\{2[\lambda-mgh(\theta)]-\mu^2\}}\,d\theta+\mu\varphi+\text{const},$$

where μ is an arbitrary constant. In the second case

$$\alpha=\frac{A\cos^2\varphi+B\sin^2\varphi+mc^2}{C+mc^2\sin^2\theta}=\frac{\alpha_1(\varphi)}{\alpha_2(\theta)},$$

and equation (2.24) can be reduced to the form

$$\frac{1}{\alpha_2(\theta)}\left(\frac{\partial W}{\partial\theta}\right)^2+\frac{1}{\alpha_1(\theta)}\left[\left(\frac{\partial W}{\partial\varphi}\right)^2-\lambda_2\right]=0.$$

Seeking the complete integral of this equation in the form

$$W = W_1(\theta) + W_2(\varphi),$$

we obtain

$$W = \mu_1 \int_{\theta_0}^{\theta} \sqrt{\alpha_2(\theta)}\, d\theta + \int_{\varphi_0}^{\varphi} \sqrt{\lambda_1 - \mu_1^2 \alpha_1(\varphi)}\, d\varphi + \text{const},$$

where μ_1 and λ_1 are arbitrary constants.

CHAPTER IV

VALIDITY OF THE MATHEMATICAL MODELS IN THE MECHANICS OF NONHOLONOMIC SYSTEMS

In this chapter, we shall, taking a number of concrete examples, study the influence of small parameters on the dynamics of nonholonomic systems. This study will cast light on the problem of the error introduced by neglecting small physical parameters in the construction of a mathematical model of a nonholonomic system. The validity of some given idealization can be verified not only by comparing the results of the theoretical investigation with experimental data but also by comparing the results of two different theories, one of which is based on the chosen idealization and another which dispenses with the idealization. In the present chapter we shall consider several rather different examples; however, they are all united by a common approach based on a division of the phase space into subspaces of slow and rapid motions [10]. If the slow motions are stable with respect to the rapid motions, the neglect of the small physical parameter is completely justified. In the opposite case, the small parameter plays an important role.

As the first example chosen to illustrate the role of the parameters which are always neglected in an idealization of a real physical system, we shall consider the influence of the usually ignored dimension of the area of contact of a body which rolls on another body. In order to simplify the mathematical calculations, we shall study the simplest example of the rolling of a homogeneous sphere on a horizontal rough plane.

The problem of rolling of a homogeneous sphere on a horizontal plane has been considered by many authors, for example, Coriolis [116], Hemming [195], Routh [392], Synge [451], and others. The assumption that the sphere and plane make contact at a point leads to the conclusion that the vertical component of the angular velocity remains constant in an inertial motion of the sphere and that the rotation of the sphere about the vertical axis does not influence the trajectory of the motion of the center of the sphere. In reality, however, the sphere touches the plane in a circular area of small diameter. As a result,

friction leads not only to a resultant force applied to the center of the area but also to a nonvanishing moment of forces about the vertical axis through the center of the area. The friction of sliding (the resultant force) and the friction of pivoting (the resultant torque) are related to each other and, hence, a variation of the vertical component of the angular velocity of the sphere has a general influence on the form of the trajectory of the motion of the center of the sphere. At the same time, the limiting motions of the sphere obtained when the area of contact tends to zero go over into the motion of a sphere with a point contact, i.e., the idealization of the point contact of the surfaces is justified.

As a second example illustrating the role of small parameters in the dynamics of nonholonomic systems, we shall consider the Appell-Hamel example of a mechanical system with a nonlinear nonholonomic constraint.

A great many investigations have been devoted to the derivation of the equations of motion of mechanical systems with nonlinear nonholonomic constraints. A systematic development of the methods of analytical mechanics for nonholonomic systems with nonlinear constraints can be found in the works of Johnsen [**220**], Hamel [**192**], Četaev [**111**], and Novoselov [**345, 347, 348, 352**]. Despite the fact that there was a lively discussion at that time concerning the possibility of the realization of nonlinear nonholonomic constraints, in which Appell, Delassus, Beghin, and others participated, the works of these authors do not, in fact, contain examples of systems with nonlinear ideal nonholonomic constraints differing essentially from the example given by Appell in 1911 [**19**].

Appell's example attracted the interest of a number of authors. It was, in particular, considered in detail by Hamel, who derived the equations of motion for it on the basis of the accepted definition of virtual displacements for systems with nonlinear nonholonomic constraints.

A more soundly based approach to the investigation of the system in the Appell-Hamel example leads to motions that cannot be described by Hamel's equations. This is because the nonholonomic system with nonlinear constraints found by Appell is a limiting case of a nonholonomic system with linear constraints. It turns out that the passage to the limit is accompanied by a reduction in the order of the system of differential equations of motion and that the limiting motions do not coincide with the motions of the system obtained by the limiting idealization.

In the last example considered in this chapter, we shall show that

Carathéodory's conclusion that nonholonomic constraints cannot be realized by arbitrarily large forces of viscous friction is erroneous.

§1. Errors arising from neglect of the finite size of the region of contact of rolling bodies

Let us consider the inertial motion of a homogeneous heavy sphere on a horizontal rough plane. Let the radius a of the area of contact of the sphere and the plane be much less than the radius R of the sphere. In order to simplify the problem, we shall assume that the curvature of the surface of contact can be neglected and assume that the pressure on the area of contact is distributed uniformly. We introduce a fixed system of coordinates $Oxyz$ (Figure 4.1) and the following notation: $V_c(u, v, 0)$ is the velocity of the center of the sphere; $\omega(\omega_x, \omega_y, \omega_z)$ is the angular velocity of the sphere; m is the mass and ρ the radius of gyration of the sphere; $F(F_x, F_y, 0)$ is the resultant of the forces of friction; M is the moment of the forces of friction about the vertical axis passing through the center of the area of contact; finally, $V(V\cos\theta, V\sin\theta, 0)$ is the velocity of the point of the sphere at the center of the area of contact. The equations of motion of the center of mass are

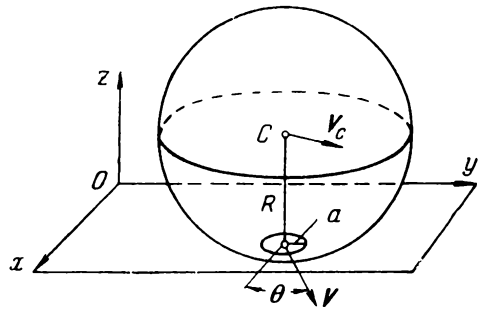

FIGURE 4.1

$$m\frac{du}{dt} = F_x, \qquad m\frac{dv}{dt} = F_y \tag{1.1}$$

and the equations that determine the variation of the angular velocity of the sphere are

$$m\rho^2\frac{d\omega_x}{dt} = RF_y, \quad m\rho^2\frac{d\omega_y}{dt} = -RF_x, \quad m\rho^2\frac{d\omega_z}{dt} = M. \tag{1.2}$$

These equations must be complemented by the kinematic relations

$$\boldsymbol{V} = \boldsymbol{V}_c + \omega \times \boldsymbol{R}$$

or, projected onto the coordinate axes,

$$V\cos\theta = u - R\omega_y, \quad V\sin\theta = v + R\omega_x. \tag{1.3}$$

In order to determine F_x, F_y, and M, we introduce the force of friction f per unit area. We shall assume that the forces of friction are forces

of Coulomb (dry) friction. Let S be the area of the surface of contact; then the force of friction acting on the infinitesimal area dS is

$$d\boldsymbol{F} = -f\frac{\boldsymbol{V}_A}{V_A}dS,$$

where V_A is the velocity of the infinitesimal area. Hence

$$\boldsymbol{F} = -f\int_S \frac{\boldsymbol{V}_A}{V}dS, \quad \boldsymbol{M} = \int_S (\boldsymbol{r}_A \times d\boldsymbol{F}), \tag{1.4}$$

where $\boldsymbol{r}_A$ is the radius vector from the center of the area of contact to the infinitesimal area dS. It follows from the expressions (1.4) that the velocity field on the area of contact must be known in order to calculate the resultant force F and the moment M. Kinematically, this field is equivalent to the velocity field of a circular plate of radius r that is displaced along a plane. Therefore, performing calculations similar to those given in Lur'e's book [**282**], we obtain the following expressions

$$\begin{aligned} &F = -\frac{4}{3\pi}mg\varkappa k f_1(k), \quad M = -\frac{4}{9\pi}mg\varkappa a f_3(k) \quad \text{if } V \ll a\omega_z, \\ &F = -\frac{4}{3\pi}mg\varkappa f_1(k), \quad M = -\frac{4}{9\pi}mg\varkappa a f_2(k), \quad \text{if } V \geqslant a\omega_z, \end{aligned} \tag{1.5}$$

where $k = 1/k_1 = V/a\omega_z$ and the functions

$$\begin{aligned} f_1(k) &= (k^{-2}+1)E(k) - (k^{-2}-1)K(k), \\ f_2(k) &= k^{-1}[(4-2k^{-2})E(k) - (k^{-2}-1)(3k^2-2)K(k)], \\ f_3(k) &= (4-2k^2)E(k) - (1-k^2)K(k), \end{aligned} \tag{1.6}$$

can be expressed in terms of the complete elliptic integrals $K(k)$ and $E(k)$ of modulus k (Figure 4.2). In equations (1.5), the quantity f is expressed in terms of the usual coefficient $\varkappa$ of dry friction, which is equal to the ratio of the force of friction to the force of pressure. In our case,

$$F_{\text{fr}} = \varkappa \cdot mg.$$

In this equation, the field of the relative velocities of the friction surfaces is assumed to be homogeneous; therefore, $F_{\text{fr}} = \pi a^2 f$ and, hence, $f = mg\varkappa/\pi a^2$. The projections of the resultant $\boldsymbol{F}$ of the forces of friction onto the coordinate axes are $F_x = F\cos\theta$, $F_y = F\sin\theta$.

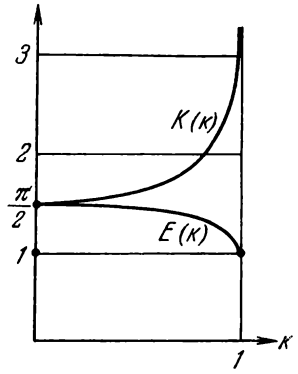

FIGURE 4.2

We now substitute this into (1.2) and then differentiate the relations (1.3) with respect to the time, replacing their right-hand sides by the expression (1.2). Solving the resulting equations for the derivatives, we find

$$\frac{dV}{dt} = \frac{1}{m}\left(1 + \frac{R^2}{\rho^2}\right)F, \quad V\frac{d\theta}{dt} = 0. \tag{1.7}$$

It follows from the second equation of (1.7) that $\theta = \theta_0 = \text{const}$, i.e., the direction of the velocity of sliding of the center of the area of contact remains fixed. The first equation of (1.7) and the third equation of (1.2) form a closed system of equations for V and ω_z. Thus, the study of the motion of the sphere reduces to an investigation of the system of equations

$$\frac{dV}{dt} = \frac{1}{m}\left(1 + \frac{R^2}{\rho^2}\right)F, \quad m\rho^2\frac{d\omega_z}{dt} = M, \tag{1.8}$$

where F and M are determined by the expressions (1.5).

Once the time dependence of the quantities V and ω_z and, consequently, F and M have been found, the final solution of the problem reduces to quadratures.

In order to simplify the notation, we introduce the quantities

$$\Omega = a\omega_z, \quad \mu = \frac{a^2}{3(\rho^2 + R^2)} \tag{1.9}$$

and a new time $\tau = (4\kappa g/3\pi)(1 + R^2/\rho^2)t$. The system of equations (1.8) and (1.5) can then be written in the form

$$\dot{V} = -kf_1(k), \quad \dot{\Omega} = -\mu f_3(k), \text{ if } V \leqslant \Omega, \tag{1.10}$$

$$\dot{V} = -f_1(k_1), \quad \dot{\Omega} = -\mu f_2(k_1), \text{ if } V \geqslant \Omega, \tag{1.11}$$

where $k = 1/k_1 = V/\Omega$ and the dot denotes differentiation with respect to the time τ; the functions f_1, f_2, and f_3 are given by the expressions (1.6). It follows from (1.10) and (1.11) that the qualitative behavior of the system is determined by the single physical parameter μ ($\mu > 0$), which can be expressed in terms of the parameters of the system by the relation (1.9). The motion of the sphere with sliding can be represented by the displacement of the representative point in the first quadrant of the (V, Ω) plane. This quadrant is divided into two regions S and S_1 by the bisector $\Omega = V$. In the region S, the motion of the representative point is described by equations (1.10) and, in the region S_1, by equations (1.11). At the boundary $\Omega = V$, the solutions of equations (1.10) and (1.11) must be "fitted" by the continuity condition.

Investigation of the functions $f_1(k)$, $f_2(k)$, and $f_3(k)$ shows (see the Appendix at the end of this section) that the corresponding curves have the form shown in Figure 4.3. Hence, and by equations (1.10) and (1.11), it follows that the values of V and Ω decrease monotonically with the course of time.

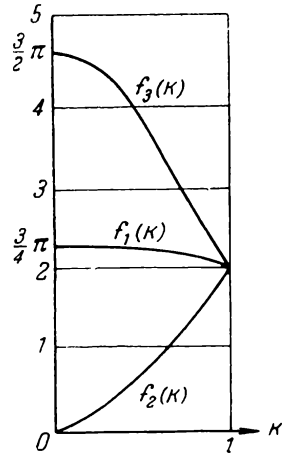

FIGURE 4.3

Let us consider the qualitative behavior of the integral curves in the (V, Ω) plane. The differential equation of the integral curves can be obtained by dividing the second equation of (1.10) [or, respectively, (1.11)] by the first equation:

$$\frac{d\Omega}{dV} = \mu \frac{1}{k} \frac{f_3(k)}{f_1(k)} \text{ in the region } S\left(k = \frac{V}{\Omega}\right), \tag{1.12}$$

$$\frac{d\Omega}{dV} = \mu \frac{f_2(k_1)}{f_1(k_1)} \text{ in the region } S_1\left(k_1 = \frac{\Omega}{V}\right). \tag{1.13}$$

It follows from (1.12) and (1.13) that the family of isoclines is a bundle of straight lines passing through the origin. In accordance with Figure 4.3, the functions on the right-hand side of equations (1.12) and (1.13) are represented by the curves shown in Figure 4.4. These curves enable one to associate the angle of the slope of the tangent to the integral curves with the angle of the slope of the corresponding isocline and, by the same token, construct the vector field of the tangents in the

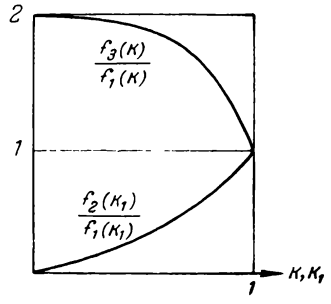

FIGURE 4.4

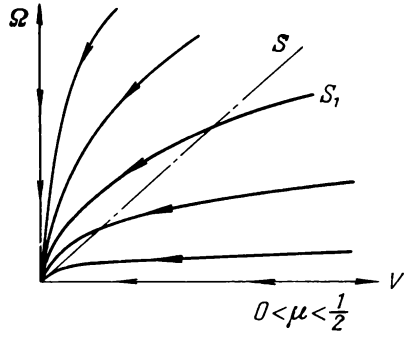

FIGURE 4.5

(V, Ω) plane for different values of the parameter $\mu > 0$. It is then found that for values of μ in the interval $(1/2 < \mu < 8/3)$ one of the integral curves in the (V, Ω) plane is a straight line passing through the origin.

The decomposition of the (V, Ω) plane into the trajectories for small values of μ is shown in Figure 4.5 from which it can be seen that all trajectories reach the origin. The coordinate axes $V = 0$ and $\Omega = 0$ are also trajectories of the motion of the representative point.

The following conclusion can be drawn from our investigation: For all initial conditions for which the velocity V of sliding and the angular velocity ω_z of pivoting of the sphere $(a \neq 0)$ do not vanish, the sphere moves in such a way that the pivoting and sliding, which decrease with the time, always cease simultaneously. This result can be explained physically by the fact that when there is pivoting, the dry friction for the translational motion is, effectively, annihilated and transformed into viscous friction. Conversely, the presence of sliding gives rise to the same effect with regard to the pivoting motion. The mutual influence means that the sliding and pivoting can only cease at the same time.

Once the representative point has reached the origin, which happens after a finite time for $a \neq 0$, we have $V = \Omega = 0$ and the sphere rolls without sliding. At the same time, as follows from equations (1.8), $F = M = 0$; from (1.1), we obtain $u = u_0 = \text{const}$, $v = v_0 = \text{const}$, and from (1.3) we find the corresponding values of the components ω_x and ω_y of the angular velocity of rotation of the sphere.

Thus, the center of the sphere moves along a straight line if it rolls without sliding.

Let us now consider the limiting motion of a sphere which slides when the area of the surface of contact tends to zero. We shall first study the motion of the sphere for $a = 0$, which corresponds to the case of point contact. It follows from the expressions (1.9) that, in this case, $\Omega = \mu = 0$ and the motion of the sphere is described by equations (1.11) in which one must make the passage to the limit $a \to 0$. From the second equation of (1.11), we then obtain $\omega_z = \omega_{z_0} = \text{const}$ and integration of the first equation of (1.11) yields $V = V_0 - 3\pi\tau/4$. On the (V, Ω) plane, the representative point moves along the axis $\Omega = 0$, reaching the origin after the finite interval of time $T = 4V_0/3\pi$. At the same time, the center of the sphere describes a parabola and then, after the time T has elapsed, it moves along a straight line.

Let us now consider the motion of the sphere in the case of a finite area of the surface of contact, letting the area of contact tend to zero. Let us assume that the parameter a is very small but nonvanishing. In this case, the motion of the sphere is described by the complete system of equations (1.10) and (1.11) in which $\mu \ll 1$. The system of

equations (1.12) and (1.13) can be written in the form of a single equation

$$\frac{d\Omega}{dV} = \mu F(k) \quad \left(k = \frac{V}{\Omega}\right), \tag{1.14}$$

where the function $F(k)$, as follows from Figure 4.4 and the right-hand sides of equations (1.12) and (1.13), is contained between the functions $3k/8$ and $2/k$. Consequently, the solution of equation (1.14) is contained between the solutions [**165**]

$$\Omega_1 = \frac{\Omega_0}{V_0^{3\mu/8}} V^{3\mu/8}, \quad \Omega_2 = \frac{\Omega_0}{V_0^{2\mu}} V^{2\mu}, \tag{1.15}$$

where Ω_0 and V_0 are the initial values. For all initial conditions for which $\Omega_0 \neq 0$, the representative point always lies in the region S. Therefore, for the sake of generality, we shall assume that at the initial instant the representative point is situated in the region S_1 (i.e., $V_0 > \Omega_0$). We substitute the limiting solutions (1.15) into the first equations of (1.10) and (1.11) and estimate the time during which the sphere slides during its motion. In accordance with Figure 4.4, the solution of the first equation of (1.11) is enclosed between the solutions $V = V_0 - 3\pi\tau/4$ and $V = V_0 - 2\tau$. Hence, the time T_1 during which the representative point moves in the region S_1 is contained in the interval

$$\frac{4}{3\pi}(V_0 - V_1) < T_1 < \frac{1}{2}(V_0 - V_1),$$

where $V_1 = \Omega_1$ is the value on the boundary of the regions S and S_1. The first equation of (1.10), which describes the motion of the representative point in the region S, is majorized by the equations $\dot{V} = -3\pi k/4$ and $\dot{V} = -2k$. Substituting the solutions of (1.15) into these expressions, we obtain

$$\dot{V} = -ACV^{(1-B)}, \tag{1.16}$$

where the constants A, B, and C are, respectively, equal to one of the two numbers

$$A = 2, \ \frac{3\pi}{4}; \quad B = \frac{3\mu}{8}, \ 2\mu; \quad C = \Omega_1^{\left(\frac{3\mu}{8}-1\right)}, \ \Omega_1^{(2\mu-1)}. \tag{1.17}$$

The solution of equation (1.16) has the form

$$V^B = \Omega_1^B - ABC\tau.$$

Substituting $V = 0$ into this equation, we find the time T taken by the representative point in passing from the boundary $V_1 = \Omega_1$ to the

origin. In accordance with (1.17), the value of T is contained in the interval

$$\frac{2\Omega_1\left(1-\frac{13\mu}{8}\right)}{3\pi\mu} < T < \frac{4\Omega_1\left(1+\frac{13\mu}{8}\right)}{3\mu},$$

from which it follows that $\lim_{a\to 0} T = \infty$. Thus, for an arbitrarily small but nonvanishing area of contact, the time the sphere slides during its motion can be made arbitrarily large. In this we see the qualitative difference between the motion of a sphere with surface contact of vanishingly small area and the motion of a sphere with point contact.

Finally, let us consider the motion of a homogeneous sphere with point contact when it rolls on a horizontal plane of which part is rough and part is smooth.

We note first that during the motion of the sphere on the smooth part of the plane its center moves along a straight line with the constant velocity V_{c0} imparted to the center of the sphere at the initial instant of time. Therefore, we shall begin our consideration at the instant at which the sphere arrives at the boundary of the rough part of the plane. In the case of a point contact of the sphere with the rough plane, the force of dry friction is given by the expression $\boldsymbol{F}_{\text{fr}} = \kappa mg\boldsymbol{V}/V$. Substituting this expression into (1.8), we find

$$\frac{dV}{dt} = -\varkappa g\left(1+\frac{R^2}{\rho^2}\right). \tag{1.18}$$

Thus, the direction of the velocity $\boldsymbol{V}$ of sliding of the sphere remains fixed ($\theta = \text{const}$) and the magnitude of the velocity V of sliding decreases in accordance with the linear law (1.18). The motion of the center of the sphere is determined by the equation

$$\frac{d\boldsymbol{V}_c}{dt} = -\varkappa g\frac{\boldsymbol{V}}{V}. \tag{1.19}$$

We take the Oy axis along the fixed direction of V and project equation (1.19) onto the Ox and Oy axes:

$$\frac{du_c}{dt} = 0, \quad \frac{dv_c}{dt} = -\varkappa g. \tag{1.20}$$

Hence, it follows that $u_c = u_{c0} = \text{const}$ and that the value of v_c decreases with the time in accordance with the linear law (1.20). Equations (1.18) and (1.19) enable one to construct the trajectory of the motion of the center of the sphere for all initial values V_{c0} and V_0. Several cases are possible; these are shown in Figure 4.6, in which the thick straight lines indicates the change in the value of V and the thin straight lines

the change in the value of v_c. At the instant of time $t = t_0$, the sliding

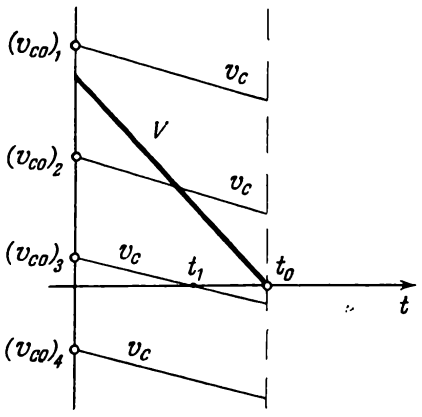

FIGURE 4.6

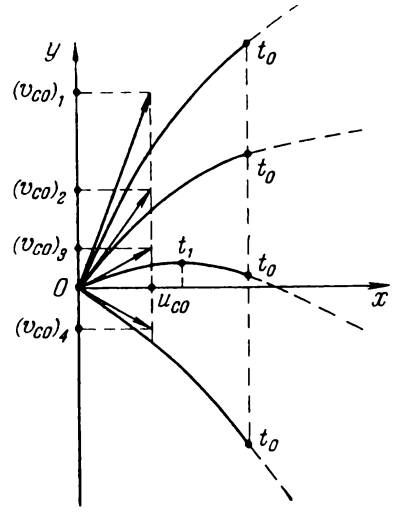

FIGURE 4.7

ceases and the center of the sphere then moves along a straight line. During the interval of time $0 \leqq t \leqq t_0$, the center of the sphere describes a section of a parabola whose parameters are determined by the relations between the initial values of V_0, u_{c0}, and v_{c0}. In Figure 4.7, we show the trajectories of the center of the sphere corresponding to the different cases of Figure 4.6, constructed for the same initial value u_{c0}. The axes of all the parabolas are parallel to Oy and the apex of the parabola is reached at the time $t = t_1$.

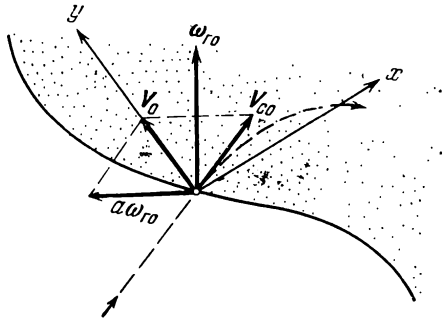

FIGURE 4.8

Suppose that at the initial instant of time $t = 0$ we have a given velocity V_{c0} of the center of the sphere and angular velocity ω_0 of rotation of the sphere. Then, decomposing the vector ω_0 into vertical ω_v and horizontal ω_h components, we obtain $\omega_v = \text{const}$ and the value of the initial velocity of sliding V_0 is determined by the vectors ω_{h0} and V_{c0}. The construction of the vector V_0 by geometrical composition of the corresponding vectors is shown in Figure 4.8, in which the rough part of the plane is shaded.

Appendix. Here, we shall make an investigation of the qualitative behavior of the functions $f_1(k)$, $f_2(k)$, and $f_3(k)$ in the interval $(0 \leqq k \leqq 1)$. The curves of the complete elliptic integrals $K(k) = \int_0^{\pi/2}(1 - k^2\sin^2\varphi)^{-1/2}d\varphi$ and $E(k) = \int_0^{\pi/2}(1 - k^2\sin^2\varphi)^{1/2}d\varphi$ are shown in Figure 4.2. In addition, we shall require the following expressions

$$\frac{dK}{dk} = \frac{1}{k}\left[\frac{1}{1-k^2}E(k) - K(k)\right] > 0, \tag{1.21}$$

$$\frac{dE}{dk} = \frac{1}{k}[E(k) - K(k)] < 0. \tag{1.22}$$

Let us consider the behavior of the following functions.

1)

$$f_1(k) = \left(\frac{1}{k^2} + 1\right)E(k) - \left(\frac{1}{k^2} - 1\right)K(k). \tag{1.23}$$

It follows from (1.23) that $f_1(0) = 3\pi/4$, $f_1(1) = 2$. Also, $f_1'(k) = -2k^{-3}[(1 - \frac{1}{2}k^2)E - (1 - k^2)K] < 0$; therefore, we have $\varphi(k) \equiv (1 - \frac{1}{2}k^2)E - (1 - k^2)K > 0$. For $\varphi(0) = 0$, $\varphi'(k) = (3k/2)(K - E) > 0$. Calculating the second derivative, we find $f_1''(k) = -6k^{-4}[(1 - k^2/2)K - E] < 0$. Indeed $\varphi_1(k) \equiv (1 - \frac{1}{2}k^2)K - E > 0$, since $\varphi_1(0) = 0$ and $\varphi_1'(k) = \frac{1}{2}k \times [E/(1 - k^2) - K] > 0$. Thus, the function $f_1(k)$ decreases monotonically and its first derivative decreases monotonically from the value $f_1'(0) = 0$ to the value $f_1'(1) = -1$.

2)

$$f_2(k) \equiv \frac{1}{k}\left[\left(4 - \frac{2}{k^2}\right)E(k) - \left(\frac{1}{k^2} - 1\right)(3k^2 - 2)K(k)\right]. \tag{1.24}$$

From the expression (1.24), we find $f_2(0) = 0$, $f_2(1) = 2$, and $f_2'(k) = 6k^{-4}[(1 - k^2/2)E - (1 - k^2)K] > 0$; therefore, we have $\varphi_2(k) \equiv (1 - k^2/2)E - (1 - k^2)K > 0$. For $\varphi_2(0) = 0$, we have $\varphi_2'(k) = (3k/2)(K - E) > 0$. Further, $f_2''(k) = 3k^{-5}[(8 - 3k^2)K - (8 - k^2)E] > 0$, since $\varphi_3(k) \equiv (8 - 3k^2)K - (8 - k^2)E > 0$. For $\varphi_3(0) = 0$,

$$\varphi_3'(k) = k\left(\frac{8 - 3k^2}{1 - k^2}E - 4K\right) > 0,$$

since the expression in the brackets is a sum of two positive terms:

$$\frac{8 - 3k^2}{1 - k^2}E - 4K = 4\left(\frac{1}{1 - k^2}E - K\right) + \frac{4 - 3k^2}{1 - k^2}E.$$

Thus, the function $f_2(k)$ increases monotonically. Its first derivative also increases monotonically from the value $f_2'(0) = 9\pi/32$ to the value $f_2'(1) = 3$.

3)

$$f_3(k) = (4 - 2k^2)E(k) - (1 - k^2)K(k). \tag{1.25}$$

We find $f_3(0) = 3\pi/2$, $f_3(1) = 2$, $f_3'(k) = -3k^{-1}[(1 - k^2)K - (1 - 2k^2)E] < 0$, since

$$(1 - k^2)K - (1 - 2k^2)E = (1 - k^2)(K - E) + k^2E > 0.$$

The second derivative $f_3''(k) = 3k^{-2}[(1 + 2k^2)K - (1 + 4k^2)E]$ vanishes only once because

$$f_3''(0) = -\frac{9\pi}{4} < 0, \quad f_3''(1) = +\infty$$

and

$$f_3'''(k) = \frac{3}{k^4}\left(\frac{1}{1 - k^2}E - K\right) > 0.$$

Thus, the curve of $f_3 = f_3(k)$ has a tangent with zero slope at the point $(0, 3\pi/2)$. The angle of slope decreases in the interval $(0 \leqq k < k^*)$, where k^* is a root of the equation $(1 + 2k^2)K - (1 + 4k^2)E = 0$. At the value $k = k^*$, the curve of $f_3 = f_3(k)$ has a point of inflection. In the interval $(k^* < k \leqq 1)$ the angle of slope of the tangent increases to a value for which the derivative is $f_3'(1) = -3$. The qualitative behavior of the functions $f_1(k)$, $f_2(k)$, and $f_3(k)$ is shown in Figure 4.3.

§2. The Appell-Hamel example of a system with a nonlinear nonholonomic constraint

In the investigations of Appell [19] and Hamel [193], a study is made of the nonholonomic system shown in Figure 4.9. A weight of mass m hangs on a thread which passes around the pulleys and is wound round the drum of radius b. The drum is fixed to a wheel of radius a which rolls without sliding on a horizontal plane, touching it at the point B. The legs of the frame that support the pulleys and keep the plane of the wheel vertical slide on the horizontal plane without friction. Let x_B and y_B be the coordinates of the point B; θ, the angle between the plane of the wheel and the Ox axis (Figure 4.9); φ, the angle of the rotation of the wheel in its own plane;

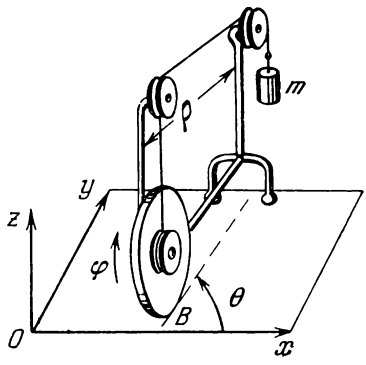

FIGURE 4.9

and x, y, and z, the coordinates of the mass m. It follows from Figure 4.9 that

$$dz = b\,d\varphi \quad (b > 0). \tag{2.1}$$

The coordinates x_B and y_B and x and y are related, in accordance with Figure 4.9, as follows:

$$x = x_B + \rho\cos\theta, \qquad y = y_B + \rho\sin\theta. \tag{2.2}$$

The condition of rolling without sliding leads to the equations of the nonholonomic constraints:

$$dx_B = a\cos\theta\,d\varphi, \qquad dy_B = a\sin\theta\,d\varphi. \tag{2.3}$$

We shall denote by m_1, A, and C the mass and the central moments of inertia of the wheel and, neglecting the mass of the frame, write the Lagrange function of the system in the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}m_1(\dot{x}_B^2 + \dot{y}_B^2) + \frac{1}{2}(A\dot{\theta}^2 + C\dot{\varphi}^2) - mgz.$$

In the variables φ and θ, we obtain the following equations of motion of the system:

$$\begin{gathered}(A + m\rho^2)\,\ddot{\theta} + m\,a\rho\dot{\theta}\dot{\varphi} = 0,\\ [(m + m_1)\,a^2 + mb^2 + C]\ddot{\varphi} - ma\,\rho\dot{\theta}^2 + mgb = 0.\end{gathered} \tag{2.4}$$

Following Hamel, we shall neglect the mass of the wheel ($m_1 = A = C = 0$); then, instead of (2.4), we obtain the equations

$$\rho\ddot{\theta} + a\,\dot{\theta}\dot{\varphi} = 0, \quad (a^2 + b^2)\,\ddot{\varphi} - a\rho\dot{\theta}^2 = -gb, \tag{2.5}$$

which, together with the linear nonholonomic constraints (2.3), describe the inertial motion of the system.

Equations (2.1) – (2.5) are identical with those given in Hamel's book [193] for the case of inertial motion. The diagrams in Hamel's book correspond to the case $\rho < 0$; in what follows, we shall therefore consider separately the case $\rho > 0$ and the case $\rho < 0$. We shall say that the system whose motion is described by equations (2.3) and (2.5) is nondegenerate.

Hamel obtained the equations of motion of a nonholonomic system with nonlinear constraints from equations (2.5) by making the passage to the limit $\rho \to 0$. For $\rho \to 0$, we obtain from (2.5)

$$\dot{\theta} = 0, \quad (a^2 + b^2)\,\ddot{\varphi} = -gb. \tag{2.6}$$

We now consider the variables x, y, and z. From (2.3) and (2.2) for $\rho = 0$, we obtain

$$\dot{x} = a\dot{\varphi}\cos\theta, \quad \dot{y} = a\dot{\varphi}\sin\theta. \tag{2.7}$$

Eliminating the variables φ and θ by means of (2.1) from these equations and using the relation

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = \frac{a^2}{b^2}\dot{z}^2\dot{\theta},$$

which can be obtained from (2.7), we arrive at a nonlinear equation of a nonholonomic constraint

$$\ddot{x}^2 + \dot{y}^2 = \frac{a^2}{b^2}\dot{z}^2 \tag{2.8}$$

and equations of motion (1.6) in the form

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = 0, \quad (a^2 + b^2)\ddot{z} = -gb^2. \tag{2.9}$$

Equations (2.9) (with allowance for the generalized forces corresponding to the coordinates x and y) were obtained by Hamel. In addition, he deduced the same equations by Gauss's principle and this convinced him of their validity. Thus, the system with nonlinear nonholonomic constraints considered by Appell and Hamel is obtained from a nonholonomic system with linear constraints by means of the passage to the limit $\rho \to 0$. However, as a result of this limiting process, the order of the system of differential equations is reduced, i.e., they become degenerate, and, therefore, it is not a priori clear whether the motions of the limiting ($\rho = 0$) system coincide with the limiting motions of the nondegenerate system for $\rho \to 0$. In this connection, we do not know to what extent the equations of motion of the degenerate system ($\rho = 0$) correctly describe the motion of the original system with vanishingly small ρ.

The answer to this question is given by the following investigation, which is based on a study of the motions of the nondegenerate system for $\rho > 0$ and $\rho < 0$, the limiting motions of the nondegenerate system as $|\rho| \to 0$, and also the motions of the degenerate system. It will be seen that the motions of the degenerate system differ essentially from the limiting motions and, hence, the example of a nonholonomic system with nonlinear nonholonomic constraints is incorrect.

1. Dynamics of the nondegenerate system. We introduce new variables ξ and η by means of the relations $\dot{\theta} = \alpha\xi$ and $\dot{\varphi} = \beta\eta$, where

$$\alpha = \left(1 + \frac{b^2}{a^2}\right)^{1/2}, \quad \beta = \frac{gb}{a^2 + b^2}.$$

Then, denoting the unique parameter of the system by μ, where

$$\mu = \frac{|\rho|}{a\beta} \geqslant 0,$$

we can write the equations of motion (2.5) in the form

$$\mu\dot{\xi} = \mp\,\xi\eta, \quad \dot{\eta} = -1 \pm \mu\xi^2, \tag{2.10}$$

where the upper sign refers to the case $\rho > 0$ and the lower sign to the case $\rho < 0$. The motion of the system can be represented by the motion of a point on the (ξ, η) plane, which we shall call the phase plane. Let us consider the decomposition of the (ξ, η) plane into trajectories. Dividing the second equation of (2.10) by the first, we obtain the equation

$$\frac{d\eta}{d\xi} = \mu \frac{-1 \pm \mu\xi^2}{\mp\,\xi\eta} \tag{2.11}$$

with separable variables. Hence, we find the family of integral curves

$$\eta^2 = \mu\,(\pm \ln \xi^2 - \mu\xi^2) + \text{const}. \tag{2.12}$$

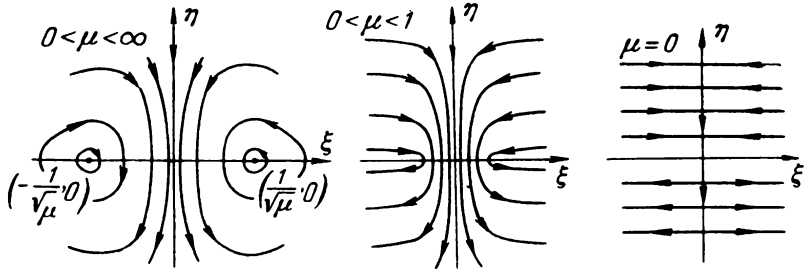

FIGURE 4.10

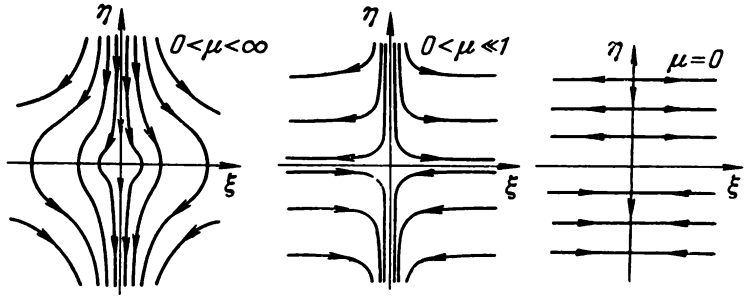

FIGURE 4.11

These curves divide the (ξ, η) plane into the trajectories shown in Figures 4.10 and 4.11, in which the arrows indicate the direction of motion of the representative point.

1) *The case* $\rho > 0$. The motion of the representative point on the (ξ, η) plane is described by the equations

$$\mu\dot{\xi} = -\xi\eta, \quad \dot{\eta} = -1 + \mu\xi^2. \tag{2.13}$$

Expressing the arbitrary constant in equation (2.12) in terms of the initial values $\xi = \xi_0$ and $\eta = \eta_0$, we obtain

$$\eta^2 = \mu\left[\ln\frac{\xi^2}{\xi_0^2} - \mu(\xi^2 - \xi_0^2)\right] + \eta_0^2. \tag{2.14}$$

The phase trajectories on the (ξ, η) plane are closed curves contained in each other and surrounding the singular points $(\mu^{-1/2}, 0)$ and $(-\mu^{-1/2}, 0)$, which are also phase trajectories (Figure 4.10a). Substituting the relations (2.14) into the first equation of (2.13), we obtain

$$\mu\frac{d\xi}{dt} = \mp\xi\left\{\mu\left[\ln\frac{\xi^2}{\xi_0^2} - \mu(\xi^2 - \xi_0^2)\right] + \eta_0^2\right\}^{1/2}$$

or

$$\mu\frac{d\xi}{dt} = \mp\xi\Phi\left(\Phi = \left\{\mu\left[\ln\frac{\xi^2}{\xi_0^2} - \mu(\xi^2 - \xi_0^2)\right] + \eta_0^2\right\}^{1/2}\right).$$

In these expressions, the upper sign corresponds to motion of the representative point along the upper half of the closed integral curve (Figure 4.10a) passing through the point (ξ_0, η_0), and the lower sign to the lower half of the curve. Hence, we obtain the dependence $\xi = \xi(t)$:

$$\mp\mu\int_{\xi_0}^{\xi}(\xi\Phi)^{-1}\,d\xi = t - t_0. \tag{2.15}$$

In order to find the time dependence of the other variables, it is now sufficient to express them in terms of ξ. Thus, for the angle θ of rotation of the plane of the wheel, we have $\dot{\theta} = \alpha\xi$. Hence, we obtain

$$\mp\alpha\mu\int_{\xi_0}^{\xi}\Phi^{-1}d\xi = \theta - \theta_0. \tag{2.16}$$

For the angle φ of the rotation of the wheel in its own plane, we have the relation

$$\dot{\varphi} = \beta\eta = \pm\beta\Phi,$$

and, hence,

$$\varphi - \varphi_0 = \mp \mu\beta \ln \frac{\xi}{\xi_0}. \tag{2.17}$$

The corresponding expressions for the coordinates x_B and y_B of the point B of contact of the wheel and the plane can be obtained from equations (2.3). The modulus of the velocity v_B of the point B is

$$v_B = \sqrt{\dot{x}_B^2 + \dot{y}_B^2} = a\,|\dot{\varphi}| = a\beta\Phi, \tag{2.18}$$

from which it follows that, at the instant of time at which ξ takes the values $\xi = \xi_1$ and $\xi = \xi_2$, where ξ_1 and ξ_2 are the roots of the equation

$$\mu\left[\ln \frac{\xi^2}{\xi_0^2} - \mu(\xi^2 - \xi_0^2)\right] + \eta_0^2 = 0, \tag{2.19}$$

which expresses the condition that the integral curve intersects the axis $\eta = 0$, the point B comes to rest instantaneously. Since the angular velocity does not vanish, $\dot{\theta} \neq 0$, at these instants of time and $\dot{\varphi}$ changes sign, the trajectory of the point B is beak-shaped with a cusp-like singular point. The trajectory between these singular points is a section of a spiral.

Our results enable us to give a qualitative description of the behavior of the system and to determine the nature of the trajectory of the point B for the different initial conditions.

The simplest type of motion of the system arises in the case when the representative point is situated at the singular point $(\pm \mu^{-1/2}, 0)$ on the (ξ, η) plane. In this case, $\dot{\varphi} = 0$; the point B does not move; and the mass m, which remains at a constant height $z = z_0$, rotates with a constant angular velocity $\dot{\theta} = \alpha/\sqrt{\mu}$ about an axis passing through the point B. This type of motion is realized physically when the moment of the force of friction $F_{fr} = m\rho\dot{\theta}^2$ about the center of the wheel compensates the moment of the force of gravity mg, i.e., $a\rho\theta^2 = gb$, or, equivalently, $\xi = \mu^{-1/2}$.

Another very simple type of motion of the system is obtained when the representative point moves along the axis $\xi = 0$ on the (ξ, η) plane. In this case, $\theta = \theta_0 = \text{const}$, i.e., the trajectory of the point B is a straight line. It follows from the equation of motion $\dot{\eta} = -1$ that, at the same time, the mass m descends with the constant acceleration

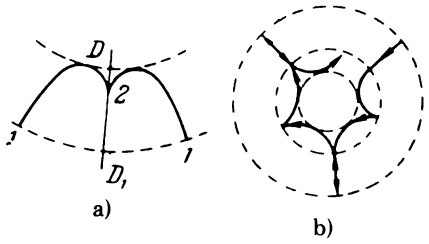

FIGURE 4.12

(2.20) $$\ddot{z} = -\frac{gb^2}{a^2+b^2}.$$

All the remaining motions of the system correspond to a displacement of the representative point on the (ξ, η) plane around a closed curve surrounding one of the points $(\pm \mu^{-1/2}, 0)$. At the same time, the angular velocity $\dot{\theta}$ of rotation of the plane of the wheel fluctuates about a certain mean value $\dot{\theta}_{av} = \text{const}$ and the angular velocity $\dot{\varphi}$ about the value 0. The foregoing consideration shows that when the representative point passes once round the closed curve in Figure 4.12a, the point B of contact of the wheel describes a curve which we shall call a cell. One ofi the possible cells is depicted in Figure 4.12a, in which the point 1 corresponds to the value $\xi = \xi_1$ and the point 2 to the value $\xi = \xi_2$. The complete trajectory consists of identical cells $(1-2-1)$, which are arranged along the strip indicated in Figure 4.12a by the dashed lines. Since the cell is symmetric with respect to the straight line DD_1, the dashed lines can only be circles. Thus, the trajectory of the point B is always restricted to an annular region. Depending on the initial conditions and the relations between the parameters of the system, the trajectory of the point B is either a closed curve or a quasi-periodic curve which fills the annular region densely at all points.

2) *The case* $\rho < 0$. The equations of motion (2.1) in this case have the form

(2.21) $$\mu\dot{\xi} = \xi\eta, \quad \dot{\eta} = -1 - \mu\xi^2,$$

where the parameter $\mu = |\rho|/a\beta$ takes only positive values.

The phase trajectories on the (ξ, η) plane are open curves arranged symmetrically with respect to the coordinate axes (Figure 4.11a). The equation (2.12) of the integral curve passing through the point $\xi = \xi_0$, $\eta = \eta_0$ is

(2.22) $$\eta^2 = -\mu\left[\ln\frac{\xi^2}{\xi_0^2} + \mu(\xi^2 - \xi_0^2)\right] + \eta_0^2.$$

In order to obtain the dependence of ξ and the other relevant quantities on the time, one can proceed as for the case $\rho > 0$. However, it is readily seen that all these relations can be obtained from (2.15) – (2.18) by replacing μ by $-\mu$. As a result, equation (2.19) is replaced by

(2.23) $$\mu\left[\ln\frac{\xi^2}{\xi_0^2} + \mu(\xi^2 - \xi_0^2)\right] = \eta_0^2,$$

which has only the single root $\xi^2 = (\xi^2)^*$. It follows from the expression (2.18) and

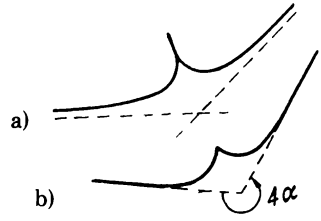

FIGURE 4.13

Figure 4.11a that, depending on the initial conditions, the trajectory of the point B has only one beak-shaped singular point if $\eta_0 > 0$ or has no singular points at all if $\eta_0 < 0$. Since the infinite branches of all the integral curves in Figure 4.11 tend to the axis $\xi = 0$, the trajectory of the point B also has infinite branches tending to asymptotes as $t \to -\infty$ and $t \to +\infty$. Hence, the complete $(-\infty < t < +\infty)$ trajectory of the point B has the form shown in Figure 4.13a. Thus, the behavior of the system in the case $\rho < 0$ differs significantly from the behavior in the case $\rho > 0$. Only for the special initial conditions when $\xi_0 = 0$, when the trajectory of the point B is a straight line in both cases, does the system behave in the same way.

2. Dynamics of the degenerate ($\rho = 0$) system. Limiting motion of the nondegenerate systems as $|\rho| \to 0$. As we have shown, the equations of motion (2.6) or, equivalently, (2.9) of the system with the nonlinear nonholonomic constraint (2.8) can be obtained from equations (2.5) or (2.10) of the nondegenerate system for $\rho \to 0$. In the limiting case, equations (2.10) take the form

$$\xi = 0, \quad \dot{\eta} = -1. \tag{2.24}$$

It follows that the phase space of the degenerate system is the straight line $\xi = 0$. For all initial conditions, the inertial motion of the system corresponds to a displacement of the representative point along the phase straight line $\xi = 0$ with the constant velocity $\dot{\eta} = -1$. This means that the trajectory of the point B of contact of the wheel and the plane is always a straight line, along which the wheel rolls with the constant acceleration $\dot{\varphi} = -\beta$.

We shall now show that the motion described by equations (2.24) differs from the limiting motion which the system executes as the quantity $|\rho|$ tends to zero. For this purpose, it is sufficient to investigate the dynamics of the nondegenerate system as $|\rho| \to 0$. As before, we shall consider the cases $\rho > 0$ and $\rho < 0$ separately.

1) *The case* $\rho > 0$. It follows from the first equation of (2.13) that

$$\lim_{\mu \to 0} \dot{\xi} = -\lim_{\mu \to 0} \frac{\xi\eta}{\mu} = \begin{cases} -\infty, & \text{if} \quad \xi\eta > 0 \ (\xi \neq 0), \ (\eta \neq 0), \\ +\infty, & \text{if} \quad \xi\eta < 0, \end{cases}$$

i.e., the rate of change of the coordinate ξ on the (ξ, η) plane as $\mu \to 0$ increases to infinity for all values $\xi \neq 0$, $\eta \neq 0$. It follows from this result and from equations (2.11) and (2.14) that the way in which the phase plane is divided into trajectories changes as $\mu \to 0$ in accordance with Figures 4.10b and 4.10c. In the limiting case ($\mu = 0$), the complete

phase plane (ξ, η) is a region of rapid motions with the exception of the straight line $\xi = 0$, which is an axis of slow motions. At the same time, the axis $\xi = 0$ of slow motions is stable with respect to the rapid motions in the region $\eta > 0$ but is unstable in the region $\eta < 0$.

We shall now find the limiting motion to which the motion of the nondegenerate system tends as $\mu \to 0$. For this purpose, we shall consider the motion of the representative point on the (ξ, η) plane along one of the closed integral curves. The limiting position of the integral curve is shown in Figure 4.14. For it follows from equation (1.14) that the curve degenerates into the pair of straight lines $\eta = \pm \eta_0$ as $\mu \to 0$. At the same time, the roots ξ_1 and ξ_2 of equation (2.19) tend to zero and infinity, respectively, in accordance with the law

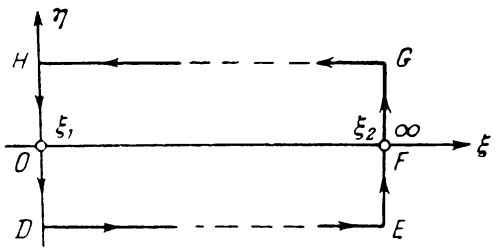

FIGURE 4.14

$$|\xi_1| \approx \exp\left(-\frac{\eta_0^2}{2\mu}\right), \quad |\xi_2| \approx \frac{\eta_0}{\mu}. \tag{2.25}$$

We now find the intervals in which the time t and also the angles θ and φ vary as the representative point passes along the sections OD, DE, and EF in Figure 4.14. Denoting by 1, 2, and 3 the variations of the quantities on the respective sections, we obtain the following results.

On the section OD we have $\xi = 0$; it therefore follows from the second equation of (2.13) that $\int_0^{-\eta_0} d\eta = -\int_0^{t_1} dt$ or $t_1 = \eta_0$. Since $\dot{\theta} = \alpha\xi$, we have $\theta_1 = 0$. In order to calculate the value of φ_1 we shall use the integral relation $\varphi = -\beta(\eta^2 + \mu^2\xi^2)/2 + \text{const}$; hence $\varphi_1 = -\beta\eta^2/2$.

On the section DE the relations (2.15) yield

$$t_2 = \lim_{\mu \to 0} \frac{\mu}{\eta_0} \int_{\xi_1}^{\xi_2} \frac{d\xi}{\xi} = \lim_{\mu \to 0} \frac{\mu}{\eta_0} \left[\ln \frac{\eta_0}{\mu} + \frac{\eta_0^2}{2\mu}\right] = \frac{1}{2}\eta_0.$$

Using the relation $\dot{\theta} = \alpha\xi$ and the first equation of (2.13), we obtain

$$\theta_2 = \alpha \int_0^{t_2} \xi\, dt = \lim_{\mu \to 0} \frac{\alpha\mu}{\eta_0} \int_{\xi_1}^{\xi_2} d\xi = \alpha.$$

Finally,

$$\varphi_2 = -\beta\eta_0 \int_0^{t_2} dt = -\frac{1}{2}\beta\eta_0^2;$$

On the section EF the second equation of (2.13) yields

$$\int_{-\eta_0}^{0} d\eta = \frac{\eta_0^2}{\mu}\int_0^{t_3} dt \quad \text{or} \quad t_3 = \lim_{\mu\to 0}\frac{\mu}{\eta_0} = 0.$$

Further,

$$\theta_3 = \lim_{\mu\to 0}\alpha\int_0^{t_3}\xi\, dt = \lim_{\mu\to 0}\alpha\cdot\frac{\eta_0}{\mu}\cdot\frac{\mu}{\eta_0} = \alpha,$$

$$\varphi_3 = \lim_{\mu\to 0}\beta\cdot(\eta)_{\text{av}}\cdot\frac{\mu}{\eta_0} = 0.$$

The motion is symmetric on the sections *EG, GH* and *HO*. These results enable us to represent the form of the trajectory of the point B of contact of the wheel and the plane in the limiting case $\mu = 0$ as well as the limiting motion of the system. As the representative point moves along the contour *ODEFGHO* (Figure 4.14) the system moves as follows: During the time $t_1 = \eta_0$, the wheel turns through the angle $\varphi_1 = -\beta\eta_0^2/2$, moving along a straight line ($\theta_1 = 0$); then, during the time $t_2 = \eta_0/2$, it turns through the same angle $\varphi_2 = \varphi_1 = -\beta\eta_0^2/2$, its plane turning through the angle $\theta_2 = \alpha$. The trajectory of the point B on this section has the form of an arc of length equal to the foregoing rectilinear section. The plane of the wheel then turns instantaneously through the angle $\theta_3 + \theta_4 = 2\alpha$ and the direction of the rotation of the wheel in its own plane is reversed. At the same time, the trajectory of the point B is a salient point with the finite angle $\theta = 2\alpha$. Then, the wheel again describes an arc-shaped trajectory, turning through the angle $\varphi_5 = \beta\eta_0^2/2$ during the time $t_5 = \eta_0/2$ while its plane turns through an angle $\theta_5 = \alpha$. Finally, during the time $t_6 = \eta_0$, the wheel moves along a straight line, turning through an angle $\varphi_6 = \beta\eta_0^2/2$. At the end of this section of the trajectory the direction of the rotation of the wheel in its own plane is reversed and the motion is repeated. The complete trajectory has the form of a rosette, which may be either a closed curve or an open curve covering a certain annular region densely at every point. The dimensions of this region are proportional to the initial value $|\eta_0|$ of the angular velocity of the rotation of the wheel in its own plane. The form of the trajectory of the limiting motion is shown in Figure 4.12b.

2) *The case* $\rho < 0$. In this case, the first equation of (2.21) yields

$$\lim_{\mu\to 0}\dot{\xi} = \lim_{\mu\to 0}\frac{\xi\eta}{\mu} = \begin{cases} +\infty, & \text{if} \quad \xi\eta > 0, \\ -\infty, & \text{if} \quad \xi\eta < 0 \end{cases} \qquad (\xi \neq 0,\ \eta \neq 0).$$

As $\mu \to 0$, the way in which the phase plane is divided into trajectories changes in the way shown in Figures 4.11b and 4.11c.

In the limiting case ($\mu = 0$), the complete phase plane is a region of rapid motions and the axis $\xi = 0$ a region of slow motions. In contrast to the case $\rho > 0$, the half-axis $\xi = 0$, $\eta > 0$ is unstable with respect to the rapid motions and the half-axis $\xi = 0$, $\eta < 0$ is stable. In order to find the limiting motion of the nondegenerate system as $\mu \to 0$, we consider the displacement of the representative point along the integral curve whose limiting position is shown in Figure 4.15, in which the values $\xi = \xi_1$ and $\xi = \xi_2$ are determined, as before, by the expressions (2.25). The intervals of variation of the time t and also the angles θ and φ corresponding to the passage of the representative point along the sections *DE*, *EF*, *FG*, and *GH* (Figure 4.15) can be found in the same way as in the case $\rho > 0$. The nature of the limiting motion of the system and the form of the trajectory of the point B are as follows. As the representative point moves down the axis $\xi = 0$ to the point D, the wheel moves along a straight line with constant acceleration. Then, during a time $t_1 = \eta_0/2$ the wheel turns through an angle $\varphi_1 = \beta\eta_0^2/2$ and its plane through an angle $\theta_1 = \alpha$. On this section, the trajectory of the point B has the form of an arc. Thereafter, the plane of the wheel turns instantaneously through an angle $\theta_2 + \theta_3 = 2\alpha$ and the direction of the rotation in the plane of the wheel is reversed. The trajectory of the point B describes a salient point. Then, the wheel again describes an arc, turning through an angle $\varphi_4 = -\beta\eta_0^2/2$ during the time $t_4 = \eta_0/2$ while its plane turns through an angle $\theta_4 = \alpha$. After this, the wheel moves along a straight line that forms an angle of $\theta = 4\pi$ with the original straight line. The form of the limiting trajectory in the case $\rho < 0$ is shown in Figure 4.13b.

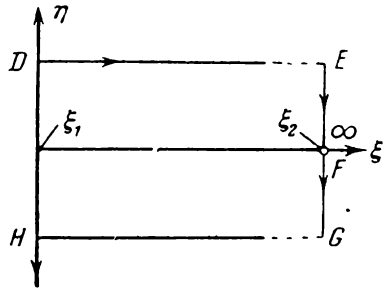

FIGURE 4.15

§3. On the realizability of a nonholonomic constraint by forces of anisotropic viscous friction

In 1933, Carathéodory [88] considered the inertial motion of Čaplygin's sleigh [85] in the special case when the center of mass of the system lies on the straight line passing through the plane of the knife edge. In this case, the motion of the sleigh is described by the differential equations

$$\dot{u} = a\omega^2, \quad ak^2 \neq \omega = -u\omega. \tag{3.1}$$

Here, u is the magnitude of the velocity of the point of contact of the knife edge and the plane; ω is the angular velocity of rotation of the sleigh; and $k^2 = 1 + J/ma^2$, where J is the central moment of inertia, m is the mass, and a is the distance between the point (x, y) of contact of the knife edge and the plane and the center of mass C of the sleigh (Figure 4.16). The force of reaction R prevents the sleigh from sliding in the direction perpendicular to the plane of the knife edge, i.e., for the usual motion of the sleigh described by equations (3.1), the velocity component v vanishes. Assuming that R has the nature of a force of viscous friction, Carathéodory represented it in the form

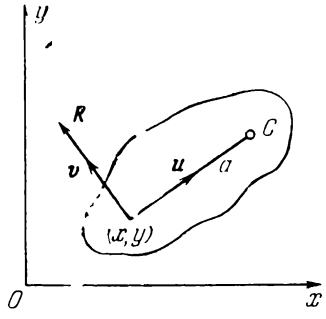

FIGURE 4.16

$$R = -Nv, \tag{3.2}$$

where N is the coefficient of viscous friction, a very large number. Introducing the small parameter $\epsilon = JN^{-1}a^{-2}$ and assuming $v \neq 0$, we can write the equations of motion of the sleigh in the form

$$v = \varepsilon a\dot{\omega}, \quad \dot{u} = a\omega^2 + \varepsilon a\omega\dot{\omega}, \quad ak^2\dot{\omega} + u\omega = -\varepsilon a\ddot{\omega}. \tag{3.3}$$

It is immediately clear from equations (3.3) that they go over into equations (3.1) for $\epsilon = 0$. Carathéodory also put forward the following argument. Consider the motion of the sleigh for the initial conditions $u_0 = 0$, $v_0 = 0$, and $\omega_0 = kc$, where $c \neq 0$ is some constant. From equations (3.3) we then obtain the initial values $\dot{\omega}_0 = 0$ and $\ddot{\omega}_0 = 0$ for $\dot{\omega}$ and $\ddot{\omega}$, but, from equations (3.1), the values $\dot{\omega}_0 = 0$ and $\ddot{\omega}_0 = -kc^3$. Carathéodory therefore concluded that for any arbitrary small $\epsilon \neq 0$ the trajectory of the sleigh differs from the trajectory of the sleigh for $\epsilon = 0$ and, hence, the nonholonomic constraint cannot be realized by the forces of viscous friction.

Carathéodory's argument cannot be regarded as convincing, since it must be borne in mind when one is comparing the motions described by equations (3.1) and (3.3) that the latter describes motion of the representative point in three-dimensional space whereas the former describes motion in a two-dimensional space. Hence, the system becomes degenerate on the transition from (3.3) to (3.1). Let us reconsider the system of equations (3.3). We introduce the small parameter $\mu = \epsilon a$ and

the variable $\sigma = \dot{\omega}$; equations (3.3) can then be written in the form of three first-order equations

$$\dot{u} = a\omega^2 + \mu\omega\sigma, \quad \dot{\omega} = \sigma, \quad \mu\dot{\sigma} = -ak^2\sigma - u\omega, \tag{3.4}$$

wfiich determine the motion of the representative point in the space (u, ω, σ), and the first equation of (3.3) is transformed into the simple relation $v = \mu\sigma$ between the variables v and σ. It follows from the last equation of the system (3.4) that the space (u, ω, σ) is transformed into the region of rapid motions (with respect to the coordinate σ) as $\mu \to 0$, with the exception of the surface

$$ak^2\sigma + u\omega = 0, \tag{3.5}$$

which becomes a region of slow motions and, for the rapid motions, a region of stable equilibrium states (see Figure 4.17). As $\mu \to 0$, the

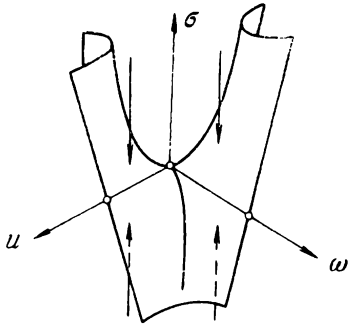

FIGURE 4.17

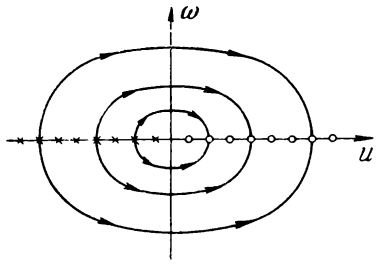

FIGURE 4.18

representative point arrives from any point of the space (u, ω, σ) at the surface (3.5) with the velocity

$$\dot{\sigma} = -\lim_{\mu \to 0} \frac{ak^2\sigma + u\omega}{\mu} = \begin{cases} -\infty, & \text{if} \quad ak^2\sigma + u\omega > 0, \\ +\infty, & \text{if} \quad ak^2\sigma + u\omega < 0, \end{cases}$$

and then moves on this surface in accordance with equations (3.3), which can be obtained from (3.4) for $\dot{\sigma} = 0$. Equations (3.1) describe the slow motion of the representative point on the surface (3.5) projected onto the plane (u, ω). Dividing the first equation of (3.1) by the second, we obtain the differential equation

$$\frac{du}{d\omega} = -a^2k^2\frac{\omega}{u},$$

from which we find the family of integral curves $u^2 + a^2k^2\omega^2 = \text{const}$, which divide the (u, ω) plane into trajectories (Figure 4.18). The straight

line $\omega = 0$ is occupied by equilibrium states of the system (3.1), unstable equilibrium states occurring on the ray ($\omega = 0$, $u < 0$) and stable states on the ray ($\omega = 0$, $u > 0$). The points on the axis $\omega = 0$ correspond to rectilinear motion of the sleigh with the constant velocity $u = u_0 = \text{const}$. This motion of the sleigh is unstable if the knife edge is situated in front of the center of mass and stable if it is behind the center of mass. As the representative point moves along the arc of the ellipse, the knife edge of the sleigh describes one of the beak-shaped curves for which an analytic expression was found by Carathéodory.

In the case of small values of the parameter μ ($\mu \neq 0$), the representative point in the space (u, ω, σ) arrives in a μ-neighborhood of the surface (3.5) from points outside a finite neighborhood of the latter after a time of order $\mu \ln(1/\mu)$ and remains in it thereafter [10].

We now consider Carathéodory's argument. We note first of all that for Carathéodory's initial conditions ($u_0 = 0$, $\sigma_0 = 0$, $\omega_0 = kc$) the representative point in the (u, ω, σ) space is situated on the surface (3.5) and, hence, in the region of slow motions, which, for $\mu \neq 0$ ($\mu \ll 1$), is the μ-neighborhood of the surface (3.5). In accordance with what we have said, the representative point subsequently remains in this region. Let us also study the behavior of the function $\dot{\sigma} = \dot{\sigma}(t)$. For this purpose, we consider the motion of the representative point in the $(u, \omega, \dot{\sigma})$ space. Differentiating the last equation of (3.4) and eliminating the variable σ, we obtain the following system of differential equations from (3.4):

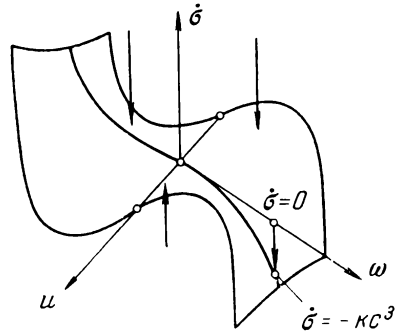

FIGURE 4.19

$$
\begin{aligned}
ak^2 \frac{du}{dt} &= ak^2\omega^2 - \mu\omega(u\omega + \mu\dot{\sigma}),\\
ak^2 \frac{d\omega}{dt} &= -u\omega - \mu\dot{\sigma}, \qquad (3.6)\\
\mu ak^2 \frac{d\dot{\sigma}}{dt} &= [\mu(u + \mu\omega^2) - a^2k^4]\,\dot{\sigma} - a^2k^2\omega^3 + \omega u^2 + \mu u\omega^3,
\end{aligned}
$$

which describe the motion of the representative point in the $(u, \omega, \dot{\sigma})$ space. It follows from the last equation of (3.6) that, as $\mu \to 0$, the $(u, \omega, \dot{\sigma})$ space is transformed into the space of rapid motions (with respect to the coordinate $\dot{\sigma}$), with the exception of the surface

$$a^2k^4\dot{\sigma} = \omega u^2 - a^2k^2\omega^2, \tag{3.7}$$

which becomes the region of slow motions (Figure 4.19). These slow motions are stable with respect to the rapid motions. In the case of small values of the parameter μ ($\mu \neq 0$), the region of slow motions is the μ-neighborhood of the surface

$$[\mu\,(u + \mu\omega^2) - a^2k^4]\dot{\sigma} - a^2k^2\omega^3 + \omega u^2 + \mu u\omega^3 = 0. \tag{3.8}$$

In a bounded region of the $(u, \omega, \dot{\sigma})$ space, the surface (3.8) differs from (3.7) by a small quantity of the order μ and tends to the surface (3.7) as $\mu \to 0$. For Carathéodory's initial conditions ($u_0 = 0$, $\omega_0 = kc$, $\dot{\sigma}_0 = 0$), the representative point in the $(u, \omega, \dot{\sigma})$ space is situated outside a finite neighborhood of the surface (3.8) and, in accordance with the foregoing, it arrives in the μ-neighborhood of this surface after a time of order $\mu \ln(1/\mu)$. For the values $u = 0$ and $\omega = kc$ on the surface (3.7), we have $\dot{\sigma} = -kc^3$. Consequently, as $\mu \to 0$, the variation of $\dot{\sigma}$ from $\dot{\sigma} = 0$ to $\dot{\sigma} = -kc^3$ is transformed into an instantaneous jump (Figure 4.19).

Thus, Carathéodory's contradiction can be resolved as follows. Although the values of $\ddot{\omega} \equiv \dot{\sigma}$ at the initial instant of time $t = 0$ are indeed different in the cases $\mu = 0$ and $\mu \neq 0$ ($\mu \ll 1$), their difference disappears during an interval of time of order $\mu \ln(1/\mu)$. As $\mu \to 0$, the region of slow motions contracts to a surface and the limiting (slow) motion of the nondegenerate system (3.4) coincides with the motion of the limiting (degenerate) system (3.1). At the same time, the slow motion is stable with respect to the rapid motions.

Our consideration shows that the nonholonomic constraint arising during the motion of a knife edge on a surface can be realized by the forces of viscous friction and corresponds to the case when the coefficient of viscous friction has an infinitely large value.

Thus, a more thorough analysis of the question of the realization of the nonholonomic constraint by means of anisotropic forces of viscosity reveals that Carathéodory's assertion is unfounded.

CHAPTER V

SMALL OSCILLATIONS AND THE STABILITY OF NONHOLONOMIC SYSTEMS

In this chapter, we shall consider the stability and small oscillations of nonholonomic systems about equilibrium positions and also about steady motion. In view of the existence of the general theory of stability and the theory of small oscillations of dynamical systems with a finite number of degrees of freedom, it may be asked whether it is necessary to undertake an additional investigation. However, it is made necessary by the special features of nonholonomic systems and also by a number of important practical applications, which we shall consider in the next chapter. Particular attention is devoted to the investigation of the stability of equilibrium states of nonholonomic systems. This is made necessary by the fact that no unified approach has been developed in the literature regarding this question and that contradictions have arisen in the investigation of stability and in the interpretation of the nature of the vanishing roots of the characteristic equation, etc. In order to be able to give a unified exposition of this problem, we shall quote certain general results from the theory of small oscillations and the theory of stability in the first section of this chapter.

§1. General results from the theory of small oscillations and the theory of stability

1. Equations of motion and equilibrium states of a holonomic system. Let $q_1, q_2, \cdots, q_n$ be the generalized coordinates of a holonomic system with scleronomous constraints. The equations of motion of such a system, written down in the form of the Lagrange equations of the second kind, are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, \ldots, n), \tag{1.1}$$

where

$$T = \frac{1}{2} A_{ik}(q_1, \ldots, q_n)\, \dot{q}_i \dot{q}_k \quad (A_{ik} = A_{ki})$$

is the kinetic energy of the system, and

$$Q_i = Q_i(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$

are the generalized forces.

It follows from equations (1.1) that

$$(1.2) \quad Q_i(q_1, q_2, \ldots, q_n; 0, 0, \ldots, 0) = 0 \qquad (i = 1, 2, \ldots, n)$$

if the system is in equilibrium. In the cases usually encountered, this system of n equations has one or several isolated solutions. A sufficient condition for a solution $q_1^0, q_2^0, \cdots, q_n^0$ to be isolated is, of course, that the Jacobian of the system $|\partial Q_i/\partial q_k|$ does not vanish at this point. Each such isolated solution corresponds to an isolated equilibrium state, i.e., a motion of the system for which all the coordinates $q_1, q_2, \cdots, q_n$ remain constant. Without loss of generality, we shall henceforth assume that the values of $q_1, q_2, \cdots, q_n$ are equal to zero in the equilibrium position under consideration.

2. Linearization of the equations of motion. Equations of small oscillations. The characteristic equation and the form of the general solution. Stability. Limits of applicability of the linearized equations. Let us consider motions for which the system is near an equilibrium position and all its points have small velocities. These motions, which are called small oscillations of the system about the equilibrium position, are described by equations (1.1). However, since the quantities q_i and $\dot{q}_i$ $(i = 1, 2, \cdots, n)$ are now small, equations (1.1) can be simplified and terms of second and higher orders in q_i and $\dot{q}_i$ neglected. The equations obtained in this manner are called the linearized equations of motion. In order to obtain the linearized equations, one can, prior to writing down equations (1.1), expand the function

$$T = \tfrac{1}{2} A_{ik} \dot{q}_i \dot{q}_k$$

in a Maclaurin series, retaining only the terms of lowest order. Then,

$$T = \tfrac{1}{2} a_{ik} \dot{q}_i \dot{q}_k,$$

where $a_{ik} = A_{ik}(0, \cdots, 0; 0, \cdots, 0)$, i.e., the kinetic energy is transformed into a quadratic form in the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ with constant coefficients.

In the case of a conservative system

$$Q_i = -\frac{\partial V}{\partial q_i},$$

where $V = V(q_1, q_2, \cdots, q_n)$ is the potential energy. In an equilibrium position

$$\frac{\partial V}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, n). \tag{1.3}$$

In this case, one can, instead of directly linearizing the right-hand sides of equations (1.1), expand the expression for the potential energy $V(q_1, q_2, \cdots, q_n)$ in a Maclaurin series, as for the kinetic energy. In this expansion, the constant term can be omitted, since it does not affect the equations of motion. In accordance with (1.3), there are no linear terms in the expansion. Hence, the expansion of V begins with terms of second order in the generalized coordinates $q_1, q_2, \cdots, q_n$. Neglecting small terms of higher order, we obtain

$$V = \frac{1}{2} c_{ij} q_i q_j,$$

where

$$c_{ij} = c_{ji} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_{q_s=0}.$$

Thus, the potential energy is transformed into a quadratic form in the generalized coordinates $q_1, q_2, \cdots, q_n$ with constant coefficients.

In accordance with the approximate expressions obtained for T and V, the linearized equations of motion (1.1) of a conservative system near an equilibrium position have the form

$$a_{ij} \ddot{q}_j + c_{ij} q_j = 0 \qquad (i = 1, 2, \cdots, n). \tag{1.4}$$

If the system is also subjected to forces of viscous friction, they may be taken into account by the introduction of a dissipation function (Rayleigh function)

$$F = \tfrac{1}{2} B_{ij}(q_1, q_2, \cdots, q_n)\, \dot{q}_i \dot{q}_j.$$

The generalized forces Q_i in equations (1.1) are then

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial F}{\partial \dot{q}_i}.$$

In order to obtain the linearized equations of motion, we expand the function F in a series in $q_1, q_2, \cdots, q_n$, $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ and retain only the terms of lowest order:

$$F = \tfrac{1}{2} b_{ij} \dot{q}_i \dot{q}_j,$$

where $b_{ij} = B_{ij}(0, 0, \cdots, 0)$. Thus, the dissipation function F is transformed into a quadratic form in the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$

with constant coefficients. If forces of viscous friction act in the system, then, in accordance with this result, the linearized equations of motion (1.1) take the form

$$a_{ij}\ddot{q}_j + b_{ij}\dot{q}_j + c_{ij}q_j = 0 \quad (i = 1, 2, \ldots, n). \tag{1.5}$$

Equations (1.5) are called the equations of small oscillations of the system about the equilibrium position. In the case of arbitrary generalized forces, the linearized equations of motion also have the form (1.5), but the relations $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, and $c_{ij} = c_{ji}$ are not, in general, satisfied.

The study of small oscillations of a holonomic system about an equilibrium position reduces to the study of the system of n linear equations (1.5) with constant coefficients. Of course, a particular solution of equations (1.5) can be written in the form $q_i = A_i e^{pt}$. Substituting this into (1.5), we obtain a system of n linear homogeneous equations in the constants A_i and the quantity p:

$$(a_{ij}p^2 + b_{ij}p + c_{ij})A_i = 0.$$

A necessary condition for this system to have a nontrivial solution is the vanishing of its determinant. Equating the determinant to zero, we obtain the characteristic equation of the system

$$\mathrm{Det}\|a_{ij}p^2 + b_{ij}p + c_{ij}\| = 0. \tag{1.6}$$

Let us suppose that the roots of this equation $p_1, p_2, \cdots, p_{2n}$ are all different. This means that not all the minors of the determinant (1.6) vanish. Assuming that the minor $\Delta_n(p_k)$ obtained by removing the last row and the last column does not vanish, we find from the first $n-1$ equations

$$\frac{A_i^{(k)}}{A_n^{(k)}} = \frac{\Delta_i(p_k)}{\Delta_n(p_k)} \quad (i = 1, 2, \ldots, n),$$

where $\Delta_i(p_k)$ is the minor obtained from the determinant (1.6) by removing the ith column and the last row.

Hence,

$$\frac{A_1^{(k)}}{\Delta_1(p_k)} = \frac{A_2^{(k)}}{\Delta_2(p_k)} = \ldots = \frac{A_n^{(k)}}{\Delta_n(p_k)}.$$

Denoting these ratios by c_k, we obtain the kth particular solution in the form

$$q_i^{(k)} = c_k\Delta_i(p_k)e^{p_k t} \quad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, 2n)$$

and, accordingly, the general solution of the system of equations (1.5) in the form

$$q_i = \sum_{k=1}^{2n} c_k \Delta_i(p_k)\, e^{p_k t}, \tag{1.7}$$

where c_k are arbitrary constants determined by the initial conditions.

The study of linearized equations of motion is called the theory of small oscillations. It should be borne in mind that, by the nature of these equations, the results of the theory of small oscillations remain valid only as long as the quantities q_i and $\dot{q}_i$ are small. In particular, if the general solution (1.7) contains terms that increase unboundedly with the time, the quantities q_i and $\dot{q}_i$ can only be small for a finite (restricted) interval of time and the results of the theory of small oscillations will only be valid for this finite interval of time. In the case when there are no increasing solutions, it is natural to assume that the solutions of the linearized equations of motion are valid for an unrestricted interval of time.

It should also be noted that the theory of small oscillations is inapplicable in the case when the system contains so-called nonlinearizable nonlinearities associated with the presence of dry friction, free play, hysteresis, etc.

3. Classification of reactions and geometrical representation of couplings. Theorems on stability, instability, and stabilization. By writing the equations of motion in the form (1.5) for arbitrary forces, we obtain an oscillator representation of the system under consideration. If all the coordinates in the system apart from the ith are kept fixed, the system degenerates into a partial oscillator

$$a_{ii}\ddot{q}_i + b_{ii}\dot{q}_i + c_{ii}q_i = 0.$$

We shall say that a_{ii} is the characteristic mass, b_{ii} the coefficient of the characteristic friction, and c_{ii} the coefficient of the characteristic stiffness of the ith partial oscillator. In accordance with (1.5), the ith oscillator acts on the jth oscillator with the generalized force

$$Q_{ji} = -a_{ji}\ddot{q}_i - b_{ji}\dot{q}_i - c_{ji}q_i$$

and the jth oscillator acts on the ith oscillator with the generalized force

$$Q_{ij} = -a_{ij}\ddot{q}_j - b_{ij}\dot{q}_j - c_{ij}q_j.$$

We shall classify all the reactions between the oscillators as coordinate, velocity, or acceleration reactions. Accordingly, we obtain the classification of the coefficients a_{ij}, b_{ij}, and c_{ij} given in Table 1.

TABLE 1.

Force of reaction (or coupling)	Coefficients of coordinate coupling	Coefficients of velocity coupling	Coefficients of acceleration coupling
Mutual	Of elasticity $c_{ij} = c_{ji}$	$b_{ij} = b_{ji}$ of viscous friction $b_{ij} = -b_{ji}$ gyroscopic	$a_{ij} = a_{ji}$ inertial
Directed from ith to jth oscillator	$c_{ij} = 0$ $c_{ji} \neq 0$	$b_{ij} = 0$ $b_{ji} \neq 0$	$a_{ij} = 0$ $a_{ji} \neq 0$

We shall distinguish directed and mutual reactions: for the coefficients of the mutual couplings we retain the notation a_{ij}, b_{ij}, and c_{ij} and we shall denote the coefficients of the directed couplings by the letters α_{ij}, β_{ij}, and γ_{ij}, respectively. It is readily seen that the division of the reactions into mutual and directed couplings is not by itself unique. However, in the consideration of specific systems, it is often predetermined by the construction of the system.

Let us assume that we have made such a division. Then, multiplying equations (1.5) successively by $\dot{q}_i$ and then summing, we obtain an equation of the form

$$\frac{d}{dt}(T + V) = -2F - A - B - C, \tag{1.8}$$

where

$$2T = a_{ij}\dot{q}_i\dot{q}_j, \quad 2V = c_{ij}q_iq_j, \quad 2F = b_{ij}\dot{q}_i\dot{q}_j,$$

$$A = \alpha_{ij}\dot{q}_i\ddot{q}_j, \quad B = \beta_{ij}\dot{q}_i\dot{q}_j, \quad C = \gamma_{ij}\dot{q}_iq_j.$$

We shall call the quantity $H = T + V$ the energy of the system. We shall classify forces that do not change the value of H as conservative; forces (couplings) that always decrease H as dissipative; and forces (couplings) which may increase H, as active. It follows immediately from equations (1.8) that characteristic and mutual forces of elasticity (stiffness), inertial forces, and gyroscopic couplings are conservative; characteristic and mutual forces of friction for which the corresponding function F is positive are dissipative and for which it is negative, active; all directed couplings are active, since the forms A, B, and C may take negative values. Since directed couplings and negative friction

are active, they can only be realized if external energy sources are present. On the basis of the classification of the reactions, one can represent them as follows. In accordance with the oscillator representation of the system, we represent each partial oscillator by a point. If there is a mutual coupling between the ith and jth oscillators, we join the corresponding points by a section of a curve; if there is a directed coupling, we join the points by a section of a curve, indicating the direction of the coupling by an arrow. We shall call the resulting representation a geometrical diagram. We shall assume that the nondirected sections of the diagram can be traversed in either direction and the directed sections only in the direction of the arrow. A closed contour without intersections composed of nondirected sections and, at the most, a single directed section will be called a cycle.

Using the geometrical diagram and the concepts introduced, we can formulate the following theorems [**323**], which we quote here without proof.

THEOREM 1. *If $H = T + V$ is a positive definite function, a necessary and sufficient condition for instability is the presence of negative friction or a cycle. Here, sufficiency means that the system becomes unstable if the coefficients of the friction or the directed couplings of the cycle are appropriately chosen.*

THEOREM 2. *A necessary condition for a system whose potential energy can take negative values to be stable is the presence of either negative friction or a cycle.*

Essentially, these theorems state the simple fact that the supply of energy to the system necessary to send it out of equilibrium in the first case or to retain it in the equilibrium position in the second is impossible without negative friction or a cycle. Similarly, energy can only be extracted from a system if there is positive friction or a cycle is present.

EXAMPLE 1. *Shimmy of an automobile.* By the shimmy of an automobile, we mean the self-excitation of vibrations of the front suspension which can arise during fast driving and lead to directional instability. Let θ be the angle of rotation of the wheels about the vertical and φ the angle of rotation of the suspension (which is not assumed to be independent) about the longitudinal axis of the automobile (Figure 5.1a). The study of small vibrations of the suspension about its center of

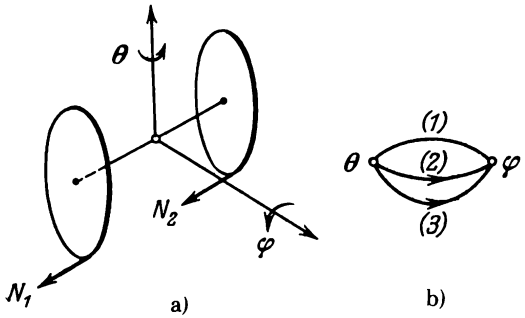

FIGURE 5.1

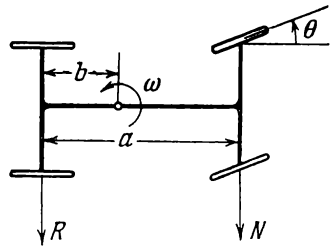

FIGURE 5.2

mass reduces to the consideration of a system of two coupled oscillators for θ and φ possessing inertia, positive friction, and stiffness (the elasticity of the tires, the springs, and the steering gear). Because of the rotation of the wheels, gyroscopic coupling is realized between the oscillators. In addition, there is a directed coupling from θ to φ, since a change in the angle θ causes a turning of the automobile and, consequently, the appearance of forces of reaction of the road, which create a moment about the φ axis. The gyroscopic coupling and the directed coupling form a cycle, which, in accordance with Theorem 1, may lead to instability of the suspension. The corresponding geometrical diagram is shown in Figure 5.1b, in which (1) is the directed coupling, (2) is the directed coordinate coupling, and (3) is the directed velocity coupling.

Let us consider the case when the chassis is rigidly fixed to the rear axle. Assuming that the motion of the wheels satisfies the classical equations of the nonholonomic constraints of the rolling of a disc on a rough plane, we obtain the following equations of small vibrations of the suspension [**323**]:

$$\begin{aligned} &A_1\ddot{\theta} + h_1\dot{\theta} + k_1\theta = -A\frac{v}{r}\dot{\varphi}, \\ &A_2\ddot{\varphi} + h_2\dot{\varphi} + k_2\varphi = A\frac{v}{r}\dot{\theta} + \frac{rv}{a^2}\left[mbv\theta + (J_c + mb^2)\dot{\theta}\right], \end{aligned} \tag{1.9}$$

where we have introduced the notation of Figure 5.2. In order to study the conditions of self-excitation of autovibrations of the suspension, we write down the characteristic equation of the system (1.9)

$$A_1A_2p^4 + (A_1h_2 + A_2h_1)\,p^3 + \left[A_1k_2 + A_2k_1 + h_1h_2 + A^2\frac{v^2}{r^2} + A\frac{v^2}{a^2}(J_c + mb^2)\right]p^2 + \left(h_1k_2 + h_2k_1 + A\frac{mb^2}{a^2}v^3\right)p + k_1k_2 = 0$$

and construct the Vyšnegradskiĭ diagram for the parameters

$$\tau = A\left(\frac{A}{r^2} + \frac{J_c + mb^2}{a^2}\right)v^2, \quad \nu = A\frac{mb^2}{a^2}v^3.$$

As the velocity of motion v of the automobile increases from zero, the representative point on the plane of the parameters (τ, ν) moves along the curve Γ (Figure 5.3). It follows from Figure 5.3 that the onset of shimmy occurs for $v > v_{cr}$. We now take into account the degrees of freedom of the chassis of the automobile (Figure 5.4a). If, in accordance with our assumption, the wheels do not drift or slide, there will be directed coupling from θ to y, α, and γ. Apart from these directed couplings, there may be different (elastic, inertial, frictional) couplings between x and y, z, α, β, and γ and couplings between α and φ, γ and φ, y and φ, and also a gyroscopic coupling between φ and θ. It follows from the geometrical diagram (Figure 5.4b) that there are no cycles if the gyroscopic coupling of φ and θ is eliminated.

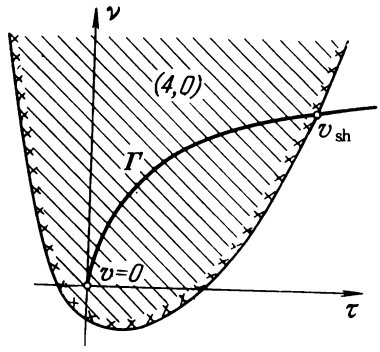

FIGURE 5.3

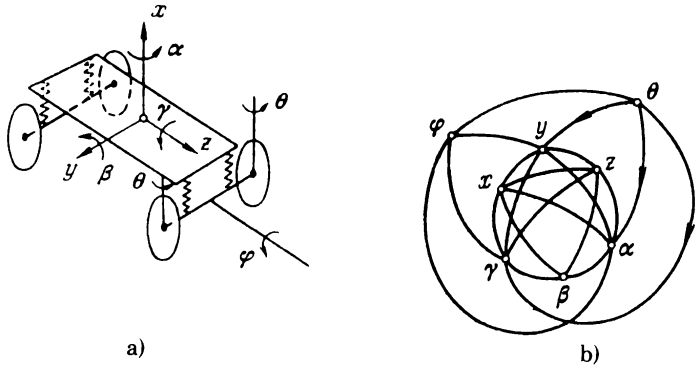

FIGURE 5.4

As an application of Theorem 2, let us consider the phenomenon of stabilization of a bicycle. The stability of a bicycle is connected with the presence of the very same cycle which is responsible for the shimmy of the front suspension of an automobile. Let θ be the angle through

which handlebars are turned and φ be the angle of deflection of the bicycle from the vertical (Figure 5.5). Turning of the handlebars (a variation of θ) causes the appearance of directed forces of reaction which tend to turn the bicycle and act on φ. In addition,

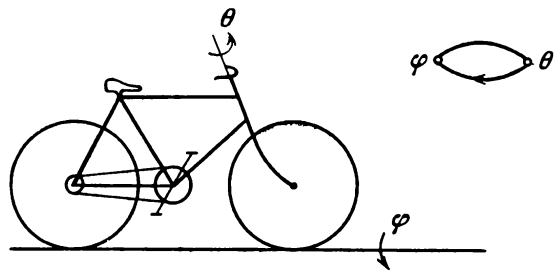

FIGURE 5.5

the rotation of the front wheel gives rise to a gyroscopic coupling between φ and θ. The gyroscopic and directed couplings between φ and θ form a cycle. If the values of the parameters are appropriately chosen, the cycle can stabilize the bicycle; for the equations of motion of the bicycle can, with appropriate simplifications, be written in the form [323]

$$J\ddot{\varphi} - mgh\varphi = -\frac{mv^2}{a}\theta - \frac{mbv}{a}\dot{\theta} + G\frac{v}{a}\dot{\theta},$$

$$A\ddot{\theta} + B\dot{\theta} - (C - ev^2)\theta = -G\frac{v}{a}\dot{\varphi}.$$

An investigation of the characteristic equation of this system shows that a necessary condition for the stability of the bicycle is the fulfilment of the inequalities

$$B > 0, \quad C = ev^2 > 0, \quad v_{cr} < v < v_{sh}.$$

4. Linear conservative systems. Natural frequencies and normal oscillations. Dependence of the natural frequencies on the parameters of the system. According to the results of subsection 2 of this section, the problem of small oscillations of a conservative system about an equilibrium position reduces to the integration of Lagrange equations in which the kinetic energy T is a homogeneous quadratic form in the generalized velocities and the potential energy V is a homogeneous quadratic form in the generalized coordinates with constant coefficients. At the same time, the kinetic energy is a positive definite form. The equations of small oscillations of a conservative system can be written in the form

$$\sum_{j=1}^{n} (a_{ij}\ddot{q}_j + c_{ij}q_j) = 0 \quad (i = 1, 2, \ldots, n). \tag{1.10}$$

We multiply equations (1.10) by undetermined numbers m_i and add the resulting expressions

$$\sum_{i,j=1}^{n} (m_i a_{ij} \ddot{q}_j + m_i c_{ij} q_j) = 0.$$

We now choose the multipliers m_i in such a way that the relation

$$\sum_{i,j=1}^{n} m_i a_{ij} q_j = \frac{1}{\lambda} \sum_{i,j=1}^{n} m_i c_{ij} q_j$$

holds for all values of the variables $q_1, q_2, \cdots, q_n$. For the determination of the unknown multipliers m_i, we obtain the system of equations

$$\sum_{i=1}^{n} m_i (\lambda a_{ij} - c_{ij}) = 0 \quad (j = 1, 2, \ldots, n), \tag{1.11}$$

which has a nontrivial solution if λ is a root of the equation

$$\Delta(\lambda) = \mathrm{Det} \| \lambda a_{ij} - c_{ij} \| = 0.$$

Since the quadratic form $T = a_{ij} \dot{q}_i \dot{q}_j / 2$ is positive definite, all the elementary divisors of the matrix $\Delta(\lambda)$ are linear [**502**] and, hence, equations (1.11) have n nontrivial solutions $m_1^{(k)}, m_2^{(k)}, \cdots, m_n^{(k)}$ $(k = 1, 2, \cdots, n)$, a new variable x_k corresponding to each of the solutions. In the new variables $x_1, x_2, \cdots, x_n$, the expressions for the kinetic and potential energy are

$$T = \sum \frac{1}{2} \dot{x}_k^2, \quad V = \frac{1}{2} \sum \lambda_k x_k^2 \quad (k = 1, 2, \ldots, n),$$

and equations (1.10) take the form

$$\ddot{x}_k + \lambda_k x_k = 0 \qquad (k = 1, 2, \cdots, n). \tag{1.12}$$

The variables $x_1, x_2, \cdots, x_n$ are called normal coordinates. According to Sylvester's theorem, all the n roots of the equation $\Delta(\lambda) = 0$ are real, so that the transformation to the normal coordinates is also real. In the normal coordinates, the variables occur independently in the equations of small oscillations [the kth normal coordinate occurs only in the kth differential equations of (1.12)]. This means that, in the linear approximation, the variations of the normal coordinates are independent of each other and that an arbitrary small oscillation of the system is a superposition of normal oscillations. We recall that the kth normal oscillation of a system is a particular solution of the equations (1.12) of the form

$$x_1 = x_2 = \ldots = x_{k-1} = x_{k+1} = \ldots = x_n = 0,$$

$$x_k = A_k e^{\sqrt{-\lambda_k}t} + B_k e^{-\sqrt{-\lambda_k}t}.$$

In the case when $\lambda_k > 0$, the corresponding normal oscillation is an harmonic oscillation with frequency $\sqrt{\lambda_k} = \omega_k$ and amplitude $\sqrt{A_k^2 + B_k^2}$. If all the normal oscillations are harmonic, i.e., $\lambda_k > 0$ for all k, the equilibrium state is said to possess conservative stability. If $\lambda_k < 0$ for certain k, the corresponding x_k will in general (i.e., for $A_k \neq 0$) increase unboundedly in accordance with the linearized equations.

Let us now consider how the spectrum of natural frequencies ω_k is altered by the introduction of an additional constraint, i.e., when the number of degrees of freedom of the holonomic system is reduced. Let us suppose that the new constraint is compatible with the equilibrium position which we are considering. For small deviations, the new constraint can be expressed in the normal coordinates by the equation

$$(*) \qquad \sum_{k=1}^{n} A_k x_k = 0,$$

where the constants A_k do not all vanish. Writing down the equations of motion with undetermined multipliers, we now have, in place of (1.12),

$$(1.13) \qquad \ddot{x}_k + \omega_k^2 x_k + \mu A_k = 0 \quad (k = 1, 2, \ldots, n).$$

In order to find the characteristic exponents, we set

$$x_k = C_k e^{i\omega t}, \qquad \mu = M e^{i\omega t}$$

and substitute these values into (1.13) and into the equation of the new constraint (*). Eliminating the quantities $C_1, \cdots, C_n$, and M from the resulting equations, we obtain

$$\sum_{k=1}^{n} \frac{A_k^2}{\omega^2 - \omega_k^2} = 0.$$

These equations for ω have $n - 1$ positive roots, which, as follows from the equations, lie in the intervals between the numbers $\omega_1, \omega_2, \cdots, \omega_n$, arranged in increasing order. Consequently, the $n - 1$ frequencies of the normal oscillations of the system with an additional constraint lie between the n frequencies of the original system. This result is a special case of Courant's theorem.

Suppose that a linear conservative system S possesses the natural frequencies $\omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_n$. If a new system S' is obtained by imposing r holonomic constraints

$$\sum_{k=1}^{n} A_{ik}x_k = 0 \quad (i = 1, 2, \ldots, r)$$

on the original system S, then the natural frequencies ω_k' $(k = 1, 2, \cdots, n - r)$ of the system S' satisfy the relations $\omega_k \leqq \omega_k' \leqq \omega_{k+r}$.

Two other theorems of Courant, which we shall quote here without proof, concern the influence on the natural frequencies of a conservative system of a change in the inertia and stiffness.

Suppose that a system S' has a greater inertia than the system S, i.e., $T' \geqq T$, $V' = V$; then the natural frequencies are not increased (they decrease or remain the same).

Suppose that the coefficients of stiffness in a system S are increased, i.e., $V' \geqq V$, $T' = T$; then the natural frequencies of the system S' are not decreased.

Courant's theorems enable one to draw conclusions about the qualitative influence of the parameters of a system on its natural frequencies.

5. Influence of gyroscopic and dissipative forces on the nature of small oscillations. Splitting of frequencies. Let us consider the influence of gyroscopic forces. By definition, the forces $Q_i(q, \dot{q})$ $(i = 1, 2, \cdots, n)$ are said to be gyroscopic if they satisfy the condition

$$\sum_{i=1}^{n} Q_i\dot{q}_i = 0. \tag{1.14}$$

This means that the work done by these forces in a real displacement of the system is equal to zero. For small deviations of the system from an equilibrium position, the gyroscopic forces may be represented in the form

$$Q_i = g_{ij}\dot{q}_j,$$

where the coefficients g_{ij} satisfy, in accordance with equation (1.14), the relation $g_{ij} = -g_{ji}$ $(g_{ii} = 0)$ and may therefore be expressed in the form $g_{ij} = x_{ij} - x_{ji}$, where $x_{ij} > 0$ are certain positive quantities. We now write down equations (1.10) with allowance for the gyroscopic forces:

$$a_{ij}\ddot{q}_j + c_{ij}q_j + (x_{ij} - x_{ji})\dot{q}_j = 0. \tag{1.15}$$

Because of the presence of the gyroscopic terms $(x_{ij} - x_{ji})\dot{q}_j$, the system

of equations (1.15) cannot be transformed to the form (1.12) by the introduction of new coordinates. However, Whittaker showed that this can be achieved by means of a contact transformation. It follows that the basic property of oscillations about an equilibrium position is also preserved in a system with gyroscopic forces, i.e., every oscillation may be regarded as a superposition of harmonic normal oscillations.

Let us consider the question of the influence of gyroscopic forces on the spectrum of the natural frequencies, taking as an example a system with two degrees of freedom. Suppose the system consists of two harmonic oscillators which are coupled to each other by only gyroscopic reaction forces. The equations of motion of such a system are

$$a_{11}\ddot{q}_1 + c_{11}q_1 + \gamma\dot{q}_2 = 0,\ a_{22}\ddot{q}_2 + c_{22}q_2 - \gamma\dot{q}_1 = 0, \tag{1.16}$$

where $a_{ii} > 0$, $c_{ii} > 0$. In the absence of the gyroscopic coupling ($\gamma = 0$), the normal frequencies ω_1 and ω_2 are determined by the expressions $\omega_1^2 = c_{11}/a_{11} = \lambda_1^0$, $\omega_2^2 = c_{22}/a_{22} = \lambda_2^0$. In order to find the characteristic exponents of the system (1.16), we assume $q_i = c_i\exp(\sqrt{-\lambda}t)$ $(i = 1, 2)$. Substituting these values into (1.16), we obtain an equation for which a nontrivial solution exists if

$$\lambda^2 - (\lambda_1^0 + \lambda_2^0 + g)\,\lambda + \lambda_1^0\,\lambda_2^0 = 0, \tag{1.17}$$

where we have introduced the notation $g = \gamma^2/a_{11}a_{22} > 0$. Assuming $g \ll 1$, we find the following approximate values of the roots of equation (1.17):

$$\lambda_1 = \lambda_1^0\left(1 + \frac{g}{\lambda_1^0 - \lambda_2^0}\right), \quad \lambda_2 = \lambda_2^0\left(1 - \frac{g}{\lambda_1^0 - \lambda_2^0}\right).$$

Thus, the introduction of a gyroscopic coupling between the oscillations with different ($\lambda_1^0 \neq \lambda_2^0$) normal frequencies gives rise to a shift of these frequencies in opposite directions. In the case of a gyroscopic coupling of oscillators with equal natural frequencies, we obtain a characteristic equation of the form

$$\lambda^2 - (2\lambda_0 + g)\lambda + \lambda_0^2 = 0,$$

where

$$\lambda_0 = \lambda_1^0 = \lambda_2^0.$$

Hence,

$$\lambda_{12} = \lambda_0 + \frac{g}{2} \pm \sqrt{g\lambda_0 + \left(\frac{g}{2}\right)^2},$$

i.e., the otherwise equal frequencies are split as a result of the introduction of a gyroscopic coupling.

Let us now consider the influence of dissipative forces on the small oscillations of a holonomic system about an equilibrium position. Let us assume that, apart from potential forces, a material system is subjected to dissipative forces represented by the Rayleigh function

$$2F = b_{ij}\dot{q}_i\dot{q}_j.$$

On the transition to the variables x_k, the Rayleigh function takes the form $2F = h_{ij}\dot{x}_i\dot{x}_j$, where $h_{ii} > 0$ and the equations of small oscillations of the system about the equilibrium position take the form

$$\ddot{x}_i + h_{ij}\dot{x}_j + \lambda_i x_i = 0 \quad (i, j = 1, 2, \ldots, n). \tag{1.18}$$

Seeking a solution of the system (1.18) in the form $x_i = Ce^{pt}$ and writing down the appropriate determinant, we obtain the characteristic equation

$$\begin{vmatrix} p^2 + h_{11}p + \lambda_1, & h_{12}p, & \ldots, & h_{1n}p \\ h_{21}p, & p^2 + h_{22}p + \lambda_2, & \ldots, & h_{2n}p \\ \ldots & \ldots & \ldots & \ldots \\ h_{n1}p, & h_{n2}p, & \ldots, & p^2 + h_{nn}p + \lambda_n \end{vmatrix} = 0. \tag{1.19}$$

If $h_{ij} = 0$ $(i, j = 1, 2, \cdots, n)$, the characteristic equation (1.19) can be simplified, taking the form

$$\prod_{i=1}^{n} (p^2 + \lambda_i) = 0. \tag{1.20}$$

Let us assume that $\lambda_i > 0$ for all i, i.e., the system possesses conservative stability. In this case, the solution of equation (1.20) can be written in the form

$$p_k^0 = \pm i\omega_k \quad (k = 1, 2, \cdots, n),$$

where $\omega_k = \sqrt{\lambda_k}$ are the frequencies of the normal oscillations of the system.

Now suppose that there is slight damping in the system, i.e., $h_{ij} \ll 1$. The characteristic equation (1.19) can then be represented in the form

$$\prod_{i=1}^{n} (p^2 + h_{ii}p + \lambda_i) = 0, \tag{1.21}$$

from which it immediately follows that, to within small terms of first order inclusively, the characteristic exponents of equation (1.19) are given by the expressions

$$p_k = \pm i\omega_k - \tfrac{1}{2}h_{kk}. \tag{1.22}$$

Thus, allowance for damping gives a zeroth-order correction to the frequency in the linear approximation and the general solution is a superposition of n damped oscillations.

6. Linear systems which are almost conservative. Effect of proximity of natural frequencies. Let us consider the small oscillations of a system with two degrees of freedom. According to subsection 3, such a system can be represented as two coupled oscillators. Assuming that the system is nearly a linear conservative system, we shall find the stability conditions and show that the occurrence of instability of such systems is governed not so much by the magnitude of the couplings but rather by the magnitude of the connections (the concept of connection was introduced by L. I. Mandel'štam).

The equations of motion of a system of two coupled oscillators can be written in the form

$$\begin{aligned} &a_{11}\ddot{q}_1 + b_{11}\dot{q}_1 + c_{11}q_1 + a_{12}\ddot{q}_2 + b_{12}\dot{q}_2 + c_{12}q_2 = 0, \\ &a_{21}\ddot{q}_1 + b_{21}\dot{q}_1 + c_{21}q_1 + a_{22}\ddot{q}_2 + b_{22}\dot{q}_2 + c_{22}q_2 = 0. \end{aligned} \tag{1.23}$$

Let us consider the case when, in accordance with the classification adopted in subsection 3, there is only a derivative coupling and a directed coordinate coupling. In such a system, instability is obviously possible either in the presence of negative friction or because of the directed coupling. We shall also assume that the system of two coupled oscillators is almost conservative. In this case, the system of differential equations (1.23) may be reduced to the form

$$\begin{aligned} &a_1\ddot{q}_1 + \delta_1\dot{q}_1 + k_1q_1 + (h+g)\dot{q}_2 + \lambda q_2 = 0, \\ &a_2\ddot{q}_2 + \delta_2\dot{q}_2 + k_2q_2 + (h-g)\dot{q}_1 - \lambda q_1 = 0, \end{aligned} \tag{1.24}$$

where the parameters δ_1, δ_2, h, and λ are small. If $\delta_1 = \delta_2 = h = \lambda = 0$, we have a conservative system. Suppose that the directed coupling is absent ($\lambda = 0$). Then the characteristic equation of the system (1.24) takes the form

$$\begin{aligned} p^4 + (r_1 + r_2)p^3 + \left(\omega_1^2 + \omega_2^2 + \frac{g^2}{a_1a_2} + r_1r_2 - r^2\right)p^2 \\ + (r_1\omega_1^2 + r_2\omega_2^2)p + \omega_1^2\omega_2^2 = 0, \end{aligned} \tag{1.25}$$

where we have introduced the notation

$$r_1 = \frac{\delta_1}{a_1}, \quad r_2 = \frac{\delta_2}{a_2}, \quad r = \frac{h}{\sqrt{a_1a_2}}, \quad \omega_1^2 = \frac{k_1}{a_1}, \quad \omega_2^2 = \frac{k_2}{a_2},$$

and ω_1 and ω_2 are the natural frequencies of the corresponding conservative systems in the absence of the gyroscopic coupling ($g = 0$). In accordance with (1.25), the stability conditions are given by the inequalities

$$\begin{gathered} r_1 + r_2 > 0,\ r_1\omega_2^2 + r_2\omega_1^2 > 0,\ \omega_1^2 + \omega_2^2 - \frac{g^2}{a_1a_2} + r_1r_2 - r^2 > 0, \\ (r_1 + r_2)\left(\omega_1^2 + \omega_2^2 + \frac{g^2}{a_1a_2} + r_1r_2 - r^2\right) - (r_1\omega_2^2 + r_2\omega_1^2) > 0, \\ (r_1 + r_2)(r_1\omega_2^2 + r_2\omega_1^2)\left(\frac{g^2}{a_1a_2} + r_1r_2 - r^2\right) + r_1r_2(\omega_1^2 - \omega_2^2)^2 > 0. \end{gathered} \tag{1.26}$$

Considering these stability conditions, one can draw the following conclusions.

1) If $r_1 > 0$ and $r_2 > 0$, an increase of g^2, i.e., an increase in the strength of the gyroscopic coupling, extends the region of stability.

2) If $\omega_1^2 = \omega_2^2$ the stability conditions are

$$r_1 + r_2 > 0, \quad \frac{g^2}{a_1a_2} + r_1r_2 - r^2 > 0.$$

A violation of the equation $\omega_1^2 = \omega_2^2$ with $r_1r_2 > 0$ extends the region of stability relative to the remaining parameters.

We now turn to the study of the influence of a directed coupling. In this case, when $\lambda \neq 0$, the characteristic equation of the system (1.24) can be written in the form

$$\begin{aligned} -\frac{1}{a_1a_2}(gp + \lambda)^2 = {} & p^4 + (r_1 + r_2)p^3 + (\omega_1^2 + \omega_2^2 + r_1r_2 - r^2)p^2 \\ & + (r_1\omega_2^2 + r_2\omega_1^2)p + \omega_1^2\omega_2^2. \end{aligned} \tag{1.27}$$

The stability condition corresponding to positiveness of the last Routh-Hurwitz determinant can, on the assumption that r_1, r_2, r, and λ are small, be written in the form

$$\begin{aligned} & r_1r_2(\Omega_1^2 - \Omega_2^2) > \Omega_1^2\Omega_2^2(r_1 - r_2) \\ & + \left[\frac{g^2}{a_1a_2}(gr_1 - 2\lambda) + \omega_1^2(r_1 - r_2)\right]\left[\frac{g^2}{a_1a_2}(gr_2 - 2\lambda) + \omega_2^2(r_2 - r_1)\right] \end{aligned} \tag{1.28}$$

in which terms of higher order have been neglected. In the special case $r_1 = r_2 > 0$, this expression can be greatly simplified and reduces to the condition

$$|\Omega_1^2 - \Omega_2^2| > \left|\frac{g^2}{a_1a_2}\left(g - \frac{2\lambda}{r_1}\right)\right|. \tag{1.29}$$

From an analysis of the D-decomposition[1] for the parameters g and λ shown in Figure 5.6 for the case of positive coefficients of the left-hand side of equation (1.27) and the stability conditions (1.28) and (1.29), one can draw the following conclusions.

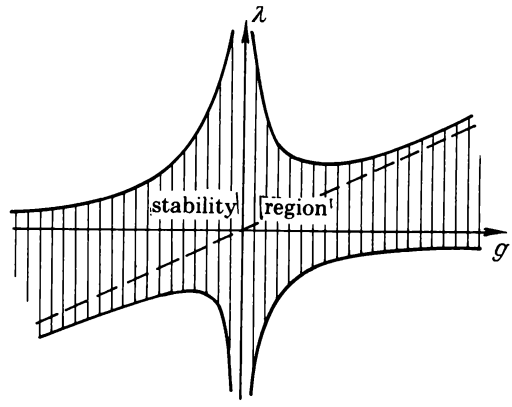

FIGURE 5.6

1) If there is no directed coordinate coupling, the system is stable provided $r_1 > 0$, $r_2 > 0$, and $r_1 r_2 - r^2 > 0$.

2) Stability is preserved if the signs of the quantities g and λ are simultaneously reversed.

3) For any magnitude of the gyroscopic coupling g, instability can be achieved by varying the value of λ.

4) For any value of λ, the system may be made stable by changing the magnitude of the gyroscopic coupling.

5) If the system is nearly conservative (r_1, r_2, and r are sufficiently small) the width of the stability region relative to the parameter λ for $g = 0$ is given by

$$\frac{2\sqrt{r_1 r_2 a_1 a_2}}{r_1 + r_2} |\omega_1^2 - \omega_2^2| .$$

6) In the case of a system which is nearly conservative and in which there is a gyroscopic coupling ($g \neq 0$) and a weak directed coupling ($\lambda \ll 1$), a sufficiently close approach of the natural frequencies ω_1 and ω_2 of the corresponding conservative systems results in instability. One can show that the region of stability on the (r_1, r_2) plane is only a narrow wedge, whose faces converge as $\Omega_1^2 \to \Omega_2^2$. In connection with

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[1] *Translator's Note.* The D-decomposition method, developed by Ju. I. Neĭmark, is assumed to be familiar to the reader of the present book, although there appears to be no detailed exposition in English. The reader may consult the summaries, with definitions and descriptions, given in Mathematical Reviews for Neĭmark's two articles: "The structure of the D-decomposition of a space of polynomials and the diagrams of Vyšnegradskiĭ and Nyquist." Dokl. Akad. Nauk SSSR (N.S.) **59** (1948), 853-856. MR **9**, 428; "D-decomposition of the space of quasi-polynomials (On the stability of linearized distributive systems)." Prikl. Mat. Meh. **13** (1949), 349-380. MR **11**, 355.

this result, we must point out, as was first done by L. I. Mandel'štam, that two different concepts must be distinguished: a large coupling and a large connection (the concept of connection was introduced for conservative systems; as the frequencies approach, the connection increases unboundedly).

The investigation of systems that are nearly conservative shows that it is the connection and not the coupling (which may be very weak) that plays the main role in the occurrence of instability as a result of a directed coupling.

7. General theorems on the stability of an isolated equilibrium state. Before considering the stability theorems, we shall give a definition of the concept of stability itself.

An equilibrium state of a system specified by null values of the coordinates ξ_k $(k = 1, 2, \cdots, 2n)$ in the phase space $\xi_1 = q_1, \cdots, \xi_n = q_n$, $\xi_{n+1} = \dot{q}_1, \cdots, \xi_{2n} = \dot{q}_n$ (where q_i are generalized coordinates) is said to be *stable* (according to Ljapunov) if for any arbitrarily small positive number $\epsilon > 0$ one can find a positive number $\eta > 0$, which depends on ϵ, such that at all times the coordinates ξ_k of the perturbed motion satisfy the inequalities $|\xi_k| < \epsilon$ if their initial values ξ_k^0 are chosen in accordance with the conditions $|\xi_k^0| < \eta$.

If, in addition, the perturbed motion is, for all initial conditions satisfying the conditions $|\xi_k^0| < \eta$, such that $\xi_k \to 0$ as $t \to +\infty$, the equilibrium state $\xi_1 = \cdots = \xi_{2n} = 0$ is said to be *asymptotically stable.*

In his classical work "The general theory of the stability of motion". Ljapunov developed two methods of solving stability problems of motion: the so-called "direct method" and the "first approximation" method of investigation.

a) *General theorems of Ljapunov's direct method on the stability of an equilibrium state.* Let $V(\xi_1, \xi_2, \cdots, \xi_{2n})$ be a continuous function of the variables ξ_k that vanishes at the point $\xi_1 = \cdots = \xi_{2n} = 0$, i.e., at the origin of the coordinates of the space $(\xi_1, \xi_2, \cdots, \xi_{2n})$. A function V is said to be of *fixed sign* if at all points of a certain neighborhood G of the origin it takes values of the same sign, vanishing only at the origin. If these values are positive, the function V is said to be *positive definite*; if negative, then *negative definite.* If the function V vanishes at certain other points of the region G apart from the origin, keeping the same sign at the remaining points of G, it is said to be of *constant sign.*

LJAPUNOV'S STABILITY THEOREM. *If the differential equations of motion of a system*

$$\dot{\xi}_k = f_k(\xi_1, \xi_2, \ldots, \xi_{2n}) \qquad (k = 1, 2, \ldots, 2n), \tag{1.30}$$

where $f_k(0, 0, \cdots, 0) = 0$, *are such that one can find a function* $V(\xi_1, \cdots, \xi_{2n})$ *of fixed sign whose derivative* dV/dt *is, by virtue of the equations of motion, a function of constant sign opposite to that of* V *or which vanishes identically, the equilibrium state is stable.*

A function V satisfying the conditions of the theorem is called a Ljapunov function. In this case when the derivative dV/dt is a function of fixed sign opposite to the fixed sign V, the equilibrium state is asymptotically stable.

b) *Ljapunov's theorem on linear approximation stability.* Suppose the differential equations (1.30) of the perturbed motion can be represented in the form

$$\frac{d\xi_i}{dt} = a_{ij}\xi_j + \Sigma_i, \tag{1.31}$$

where a_{ij} are certain constants and Σ_i are real functions of the variables $\xi_1, \cdots, \xi_{2n}$ whose expansions in powers of ξ_i begin with terms of not lower than second order. If we neglect the terms Σ_i in equations (1.31), we obtain a linearized system of equations of the perturbed motion or the so-called first approximation system of equations

$$\frac{d\xi_k}{dt} = a_{kj}\xi_j$$

with the characteristic equation

$$\mathrm{Det}\|a_{ij} - \delta_{ij}p\| = 0, \tag{1.32}$$

where δ_{ij} is the Kronecker delta.

THEOREM 1. *If the real parts* α_k *of all the roots* $p_k = \alpha_k + i\beta_k$ *of the characteristic equation* (1.32) *are negative, then the equilibrium position of the original system is asymptotically stable, irrespective of the terms* Σ_i *of higher than first order.*

THEOREM 2. *If at least one of the roots of the characteristic equation has a positive real part, the equilibrium position is unstable, irrespective of the terms of higher order* Σ_k.

If even one root of the characteristic equation (1.32) has a real part that vanishes, the remaining roots having negative real parts, we have a so-called critical case. In this case, the stability cannot be established from the first approximation. In a critical case, one must take into account the nonlinear terms Σ_i in the system of equations (1.31).

We shall now give various results of the investigation of the problem of stability in critical cases. Suppose that the differential equations of the perturbed motion can be transformed to the form

$$\begin{aligned} \dot{x} &= X(x;\ x_1,\ldots,\ x_n), \\ \dot{x}_s &= \sum_{k=1}^{n} a_{sk}x_k + X_s(x; x_1, \ldots, x_n) \quad (s = 1, 2, \ldots, n), \end{aligned} \tag{1.33}$$

where X and X_s are analytic functions of the variables $x, x_1, \cdots, x_n$ whose expansions begin with terms of order not lower than the second. Let us denote by $u_k(x)$ $(k = 1, 2, \cdots, n)$ the solution of the system of equations

$$a_{sk}x_k + X_s = 0.$$

Substituting this solution into the first equation of (1.33), we obtain

$$\dot{x} = X(x; u_1(x), \cdots, u_n(x)). \tag{1.34}$$

a) Let us assume that $X(x; u_1(x), \cdots, u_n(x)) \not\equiv 0$. Suppose that the lowest term in the expansion of the right-hand side of equation (1.34) is gx^m. If m is odd and $g < 0$, the equilibrium state of the system is asymptotically stable if the roots of the equation

$$\mathrm{Det}\,\|a_{sk} - \delta_{sk}p\| = 0 \tag{1.35}$$

have negative real parts. In other cases, the equilibrium is unstable.

b) If $X(x; u_1(x), \cdots, u_n(x)) \equiv 0$, we have a so-called singular case. If the roots of equation (1.35) have negative real parts, the equilibrium state is always stable in a singular case, but not asymptotically stable. This result was subsequently generalized by Malkin [**286**] to the case of several vanishing roots: Suppose the differential equations of the perturbed motion have the form

$$\begin{aligned} \frac{dy_i}{dt} &= Y_i(x_1, \ldots, x_n, y_1, \ldots, y_k) && (i = 1, 2, \ldots, k), \\ \frac{dx_s}{dt} &= \sum a_{sk}x_k + X_s(x_1, \ldots, x_n, y_1, \ldots, y_k) && (s = 1, 2, \ldots, n), \end{aligned}$$

where X_s and Y_i are analytic and bounded functions in the neighborhood of the origin whose expansions with respect to x_s and y_i are of order not lower than the second.

Suppose $Y_i(0, \cdots, 0, y_1, \cdots, y_k) \equiv X_s(0, \cdots, 0, y_1, \cdots, y_k) = 0$, and the coefficients a_{sk} are such that the characteristic equation has roots with only negative real parts. The equilibrium state is then stable but not asymptotically stable.[2)]

8. Stability of an equilibrium state in the case of potential forces. Influence of dissipative and gyroscopic forces on the stability of an equilibrium state. Let us apply Ljapunov's theorems to the investigation of the stability of an equilibrium state of a conservative system and to the study of the influence of dissipative and gyroscopic forces on the stability of equilibrium states. Suppose that the kinetic energy T of the system described by equations (1.1) is a homogeneous positive definite form in the generalized velocities and that the generalized forces Q_i can be represented in the form

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad (i = 1, 2, \ldots, n), \tag{1.36}$$

where the function V, which depends on the generalized coordinates $q_1, q_2, \cdots, q_n$, is called the potential energy of the system. In this case, the energy $T + V$ is conserved. In the equilibrium position $q_1 = \cdots = q_n = \dot{q}_1 = \cdots = \dot{q}_n = 0$, the generalized forces vanish, $Q_i = 0$, and, hence, the potential energy has a stationary value. Regarding the function $T + V$ as a Ljapunov function, we arrive immediately at the following theorem, which is known as the Lejeune-Dirichlet theorem.

LEJEUNE-DIRICHLET THEOREM. *If the potential energy of a conservative system has an isolated minimum at an equilibrium position, the equilibrium is stable in this position.*

It should be noted that the equilibrium position is stable whatever the order of the minimum, i.e., whatever the order of the first even derivative that does not vanish.

Ljapunov proved the opposite theorem. *The equilibrium of a conservative system is unstable if the potential energy has a maximum in the equilibrium position.*

[2)] In [286], the Ljapunov-Malkin theorem is stated under the more general assumptions when X_s and Y_i depend on the time, and the stability of steady motion is considered.

Let us assume that, apart from potential forces, the system is also subject to dissipative forces characterized by the Rayleigh function

$$F = \frac{1}{2} \sum_{i,j=1}^{n} h_{ij} \dot{q}_i \dot{q}_j. \tag{1.37}$$

Using the equations of motion (1.1), in which

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial F}{\partial \dot{q}_i}, \tag{1.38}$$

we find that

$$\frac{d}{dt}(T + V) = -2F. \tag{1.39}$$

Hence, in the case when the function V has a minimum at the equilibrium state, the function $H = T + V$ is positive definite in the neighborhood of the equilibrium state and its derivative dH/dt is negative. Therefore, the dissipative forces do not disturb the stability. Moreover, in the case when dissipation is complete, i.e., Rayleigh's function is a positive definite function of the generalized velocities, an equilibrium state that is stable under the potential forces becomes asymptotically stable when the dissipative forces are added. Similarly, it can be shown that an equilibrium state that is unstable under potential forces cannot be stabilized by means of dissipative forces.

Let us assume, finally, that gyroscopic forces are added to the potential and dissipative forces, i.e.,

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial F}{\partial \dot{q}_i} - G_i.$$

By virtue of the fact that the gyroscopic forces satisfy the condition

$$\sum G_i \dot{q}_i = 0,$$

equation (1.39) holds, as before, showing that the addition of gyroscopic forces does not disturb the stability of an equilibrium state. In contrast to dissipative forces, the addition of gyroscopic forces may make an unstable state stable.

In this connection, the following assertions are true.

1. *If the instability of an isolated equilibrium position under the influence of potential forces alone has an odd degree (i.e., the number of negative* λ_i *is odd), the equilibrium cannot be stabilized gyroscopically.*

2. *Under certain equilibrium conditions, instability under the influence of potential forces alone can be changed into stability by the addition of*

suitable gyroscopic forces if the degree of instability is not odd and if dissipative forces with complete dissipation are not added at the same time.

As a simple example of gyroscopic stabilization, let us consider the system described by the equations [112]

$$\ddot{x}_1 - \alpha x_1 - g\dot{x}_2 = 0, \quad \ddot{x}_2 - \beta x_2 + g\dot{x}_1 = 0 \quad (\alpha > 0, \beta > 0),$$

whose characteristic equation is

$$p^4 + (g^2 - \alpha - \beta)p^2 + \alpha\beta = 0.$$

For $g = 0$, the system is unstable. However, if the constant g does not vanish, i.e., $g \neq 0$, and satisfies the inequality

$$(g^2 - \alpha - \beta)^2 > 4\alpha\beta,$$

the system becomes conservatively stable.[3)]

§2. Stability and small oscillations of nonholonomic systems near equilibrium states

Despite the fact that there is an extensive literature on nonholonomic systems, there have been relatively few investigations devoted to the stability of such systems. Stability questions and small oscillations of nonholonomic systems have been studied by Whittaker [502], Bottema [71,72], Aĭzerman and Gantmaher [7], Obmoršev [357,358], Knjazev [238], and others. However, the methods of investigation proposed in these studies are mutually contradictory. It is well known that Whittaker assumed that one could integrate the linearized equations of nonholonomic constraints, the difference between holonomic and nonholonomic systems then vanishing. In this way, Whittaker essentially reduced the problem of the stability of equilibrium states of a nonholonomic system to the corresponding problem for a holonomic system with the number of generalized coordinates reduced by the number of equations of nonholonomic constraints. Bottema questioned the validity of Whittaker's conclusions and, noting an inaccuracy in his argument, showed that, in contrast to a holonomic system, the characteristic determinant of a nonholonomic system is asymmetric and that the characteristic equation of a nonholonomic system possesses vanishing roots, their number being equal to the number of equations of non-

[3)] Here, and in what follows, we shall say that a conservative system that is stable in the linear approximation is conservatively stable.

holonomic constraints. In view of this result, Bottema assumed that one is dealing with a critical case of stability theory, and he concluded that the question of the stability of equilibrium of a nonholonomic system remains open in the first approximation, since conditions for the stability of a system whose characteristic equation has an arbitrary number of vanishing roots have not yet been established for the general case. Aĭzerman and Gantmaher drew attention to the fact that, in this case, the problem of stability can be completely solved. They showed that the problem can be reduced to the singular case, which was completely investigated by Ljapunov and Malkin. Aĭzerman and Gantmaher established that an equilibrium state of a nonholonomic system is stable (but not asymptotically stable) if all the roots of the characteristic equation, apart from the vanishing roots, whose number is equal to the number of equations of nonholonomic constraints, have negative real parts. On the basis of this result, Knjazev proposed that cases should only be regarded as critical when the number of vanishing roots of the characteristic equations is greater than the number of equations of the nonholonomic constraints. In his investigation, Knjazev considered the case when the number of roots is greater than the number of equations of the nonholonomic constraints by one. Finally, we mention the paper of Obmoršev, in which he derived linearized equations for the small oscillations of a nonholonomic system about an equilibrium state in the general case and also the equations of motion about a steady motion for Čaplygin systems. In connection with the vanishing roots, Obmoršev noted that Bottema did not integrate the linearized equations of the nonholonomic constraints and incorrectly obtained vanishing roots of the characteristic equation as a result.

This review of the literature indicates not only the absence of a unified approach to the question of the stability of equilibrium states of nonholonomic systems but also a number of contradictions in the method of investigation of stability. For if Whittaker was correct in integrating the linearized equations of the nonholonomic constraints, Bottema erred in not doing so, obtaining as a result vanishing roots. However, if Bottema was correct, then Whittaker made a fundamental error in the investigation of the stability of the equilibrium of a nonholonomic system. In this second case, the interpretation of the vanishing roots remains unclear. Bottema and Aĭzerman and Gantmaher attribute the appearance of the vanishing roots to a critical case in the sense of Ljapunov. Knjazev does not regard this case as critical and Obmoršev

assumes that the vanishing roots only occur because of a confusion associated with the fact that the linearized equations of the nonholonomic constraints are not integrated.

In this section, we shall show that in the study of the stability and small oscillations of nonholonomic systems about equilibrium states we encounter a circumstance which forces us to state the problem in a manner that differs somewhat from the usual statement of the problem of stability. We shall see that equilibrium states of a nonholonomic system cannot be isolated but, rather, they form a manifold whose dimension is equal to the number of nonholonomic constraints. It is this circumstance that is responsible for the occurrence of vanishing roots of the characteristic equation.[4]

1. Derivation of the equations of small oscillations. Singularity of the coefficient matrix. Possibility of application of the general theory. The meaning of the results obtained. In the study of small oscillations of a nonholonomic system about an equilibrium position, one can use the usual approach of the theory of small oscillations. This approach is general enough to reveal various special features inherent in nonholonomic systems.

Suppose that the motion of the system whose configuration is determined by n generalized coordinates $q_1, q_2, \cdots, q_n$ is restricted by m $(m < n)$ nonholonomic constraints expressed by the equations

$$\omega_{ik}(q_1, \ldots, q_n)\,\dot{q}_k = 0 \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n). \tag{2.1}$$

Assuming that the forces are potential forces and that dissipation of energy in the system can be represented by a dissipation function F, we obtain the following equations of motion of the nonholonomic system:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\lambda_i \omega_{ij} - \frac{\partial F}{\partial \dot{q}_j} \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n). \tag{2.2}$$

Equations (2.1) and (2.2) enable one to determine the $n + m$ unknowns $q_1, \cdots, q_n$, $\lambda_1, \cdots, \lambda_m$ as functions of the time. We now assume that the point $q_1 = q_2 = \cdots = q_n = 0$ is an equilibrium position of the system. It follows from equations (2.1) and (2.2) that the relations

$$\frac{\partial V(0, \ldots, 0)}{\partial q_j} + \lambda_i \omega_{ij}\,(0, \ldots, 0) = 0, \tag{2.3}$$

[4] A similar situation is found to obtain in the investigation of the stability of steady motions of both nonholonomic and holonomic systems.

where $V = V(q_1, \cdots, q_n)$ is the potential energy of the system, hold in the equilibrium position. It is immediately clear from (2.3) that, in contrast to the equilibrium of a nonholonomic system, the quantities $\partial V/\partial q_k$ do not, in general, vanish in an equilibrium position of a nonholonomic system. Consequently, the series expansion of the function $V(q_1, \cdots, q_n)$ in the neighborhood of the equilibrium position contains linear terms such as

$$V = V(0, \ldots, 0) + \Sigma c_k q_k + \Sigma c_{ij} q_i q_j + \ldots,$$

where

$$c_k = \left(\frac{\partial V}{\partial q_k}\right)_0, \quad c_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0.$$

As in the case of a holonomic system, for small values of $q_1, \cdots, q_n$, $\dot{q}_1, \cdots, \dot{q}_n$, the functions T and F are quadratic forms in the generalized velocities with constant coefficients:

$$2T = a_{ik}\dot{q}_i\dot{q}_k, \quad 2F = b_{ik}\dot{q}_i\dot{q}_k, \tag{2.4}$$

where

$$a_{ik} = a_{ki} = \left(\frac{\partial^2 T}{\partial q_i \partial q_k}\right)_0, \quad b_{ik} = b_{ki} = \left(\frac{\partial^2 F}{\partial q_i \partial q_k}\right)_0.$$

Linearization of the terms $\lambda_i\omega_{ij}$ yields

$$\lambda_i\omega_{ij} = (\lambda_i^0 + \lambda_i')(\omega_{ij}^0 + \omega_{ij}^k q_k + \ldots) = \lambda_i^0\omega_{ij}^0 + \omega_{ij}^k\lambda_i^0 q_k + \lambda_i'\omega_{ij}^0 + \ldots,$$

where

$$\omega_{ij}^0 = \omega_{ij}(0, \ldots, 0), \quad \omega_{ij}^k = \frac{\partial \omega_{ij}}{\partial q_k}, \quad \lambda_i^0$$

are the values of the undetermined multipliers λ_i in the equilibrium position of the system. Substituting these into (2.2) and using the expressions (2.4), we obtain

$$a_{jk}\ddot{q}_k + b_{jk}\dot{q}_k + c_{jk}q_k + c_j = -\lambda_i^0\omega_{ij}^0 - \lambda_i^0\omega_{ij}^k q_k - \omega_{ij}^0\lambda_i'.$$

In these equations, $c_j = -\lambda_i^0\omega_{ij}^0$ by virtue of (2.3). Therefore, we obtain

$$a_{jk}\ddot{q}_k + b_{jk}\dot{q}_k + (c_{jk} + \lambda_i^0\omega_{ij}^k)q_k + \omega_{ij}^0\lambda_i' = 0. \tag{2.5}$$

To these equations, we must add the linearized equations of the nonholonomic constraints

$$\omega_{ik}^0\dot{q}_k = 0. \tag{2.6}$$

Equations (2.5) and (2.6) may be called the equations of small oscillations of a nonholonomic system near an equilibrium position

$$q_1 = q_2 = \cdots = q_n = 0.$$

They form a system of $n + m$ linear differential equations with constant coefficients for the unknowns $q_1, \cdots, q_n, \lambda_1', \cdots, \lambda_m'$. Substituting a solution of the form

$$q_k = C_k e^{pt}, \qquad \lambda_i' = d_i e^{pt}$$

into these equations, we arrive at the characteristic equation

$$p^m \begin{vmatrix} P_{11}, P_{12}, \ldots, P_{1n}, \omega_{11}^0, \omega_{21}^0, \ldots, \omega_{m1}^0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{n1}, P_{n2}, \ldots, P_{nn}, \omega_{1n}^0, \omega_{2n}^0, \ldots, \omega_{mn}^0 \\ \omega_{11}^0, \omega_{12}^0, \ldots, \omega_{1n}^0, 0, 0, \ldots, 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \omega_{m1}^0, \omega_{m2}^0, \ldots, \omega_{mn}^0, 0, 0, \ldots, 0 \end{vmatrix} = 0, \tag{2.7}$$

where

$$P_{jk} = a_{jk}p^2 + b_{jk}p + c_{jk} + \lambda_i^0 \omega_{ij}^k.$$

From the form of the characteristic equation (2.7) we can immediately deduce that: a) The characteristic equation has m vanishing roots; b) the coefficient matrix of equation (2.7) is not symmetric, as is the case for the corresponding holonomic systems.

These properties of the characteristic equation, which were first noted by Bottema, express one of the features peculiar to a nonholonomic system.

The study of the small oscillations of a nonholonomic system on the basis of the linear differential equations (2.5) and (2.6) does not differ essentially from the similar investigation of the linearized equations of motion of a holonomic system. As in the case of a holonomic system, thc results of such an investigation are valid only for a finite interval of time etc if there are solutions that increase with the time. In this sense, all the results of the ordinary theory of small oscillations can be extended to nonholonomic systems. As regards the connection between the linearized equations (2.5) and (2.6) and the motion of the original nonholonomic system, we do encounter a feature that is inherent only in nonholonomic systems. This is the occurrence of vanishing roots and the asymmetry of the coefficient matrix of the characteristic equation

in the case of a conservative system. The usual approach from the stand point of the theory of small oscillations fails to give an adequate answer to the question of stability and does not enable one to establish the nature of the vanishing roots. As we shall see, these problems are intimately related. A more detailed consideration of the question of stability and the small oscillations of nonholonomic systems enables one not only to explain the nature of the vanishing roots of the characteristic equation, but also to discover another particular feature of a nonholonomic system, namely, the existence of a manifold of equilibrium states and also a manifold of steady motions.[5)]

2. Manifold of equilibrium states of a nonholonomic system. Equations of small oscillations near a manifold of equilibrium states. The concept of the stability of a manifold of equilibrium states. Suppose that the motion of a system with the Lagrange function $L = L(q_1, \cdots, q_n; \dot{q}_1, \cdots, \dot{q}_n)$ and generalized forces $Q_\beta(q_1, \cdots, q_n; \dot{q}_1, \cdots, \dot{q}_n)$ $(\beta = 1, 2, \cdots, n)$ is subject to nonholonomic constraints which can be expressed by the equations

$$\omega_{\alpha\beta}(q_1, \ldots, q_n)\,\dot{q}_\beta = 0 \quad (\alpha = 1, 2, \ldots, m;\ \beta = 1, 2, \ldots, n). \tag{2.8}$$

The equations of motion with undetermined multipliers are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\beta} - \frac{\partial L}{\partial q_\beta} = Q_\beta + \lambda_\alpha \omega_{\alpha\beta}. \tag{2.9}$$

The system of equations (2.8) and (2.9) enables one to determine the quantities $q_1, \cdots, q_n$, $\lambda_1, \cdots, \lambda_m$ as functions of the time and the initial values. It follows from equations (2.8) and (2.9) that the equilibrium states of the nonholonomic system are determined by the n equations

$$\frac{\partial L}{\partial q_\beta} + Q_\beta + \lambda_\alpha \omega_{\alpha\beta} = 0 \tag{2.10}$$

for the $n + m$ unknowns $q_1, \cdots, q_n$, $\lambda_1, \cdots, \lambda_m$, i.e., m quantities can take arbitrary values. By virtue of this result, we have, in the general case, a *manifold* of equilibrium states of a nonholonomic system. It forms a surface O_m of dimension m in the n-dimensional configuration space. For, expressing the generalized coordinates $q_1, \cdots, q_n$ in terms of $\lambda_1, \cdots, \lambda_m$ by means of equations (2.10), we obtain the surface O_m in the parametric representation

$$q_\beta^0 = q_\beta^0(\lambda_1, \cdots, \lambda_m) \quad (\beta = 1, 2, \cdots, n).$$

[5)] It will be shown in subsection 5 of this section that holonomic systems also possess a manifold of steady motions.

We should mention that in specific problems (see the examples given below) it may be found that not all the equations (2.10) are independent. In such a case, the number of dimensions of the manifold of equilibrium states is greater than m.

Because of the presence of a manifold of equilibrium states, it is meaningless to speak of the stability of an isolated equilibrium state of a nonholonomic system, since a nonholonomic system does not possess isolated equilibrium states. In this connection, the very formulation of the problem of the small oscillations of a nonholonomic system must be changed. The properly posed problem is the study of the small oscillations of a nonholonomic system about the manifold of equilibrium states and not in the neighborhood of an individual (isolated) equilibrium state. It follows that the question of the stability of a nonholonomic system can only be posed with respect to the manifold of equilibrium states and not for individual equilibrium states.

In accordance with what we have said, we shall write down the equations of small oscillations of a nonholonomic system about the surface O_m of equilibrium states.

From equations (2.9) we eliminate the undetermined multipliers $\lambda_1, \cdots, \lambda_m$ and write down the equations of motion of the nonholonomic system in the normal form

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_{2n-m}) \qquad (i = 1, 2, \ldots, 2n - m), \tag{2.11}$$

where we have denoted by x_i the phase variables $q_1, q_2, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_{n-m}$. In the phase space $\Phi(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_{n-m})$, let the surface O_m be defined by the equations

$$x_i^0 = x_i^0(u_1, u_2, \cdots, u_m),$$

where the variables $u_1, u_2, \cdots, u_m$ are running parameters of the surface O_m. Together with the variables $u_1, \cdots, u_m$, we introduce the new variables $v_1, v_2, \cdots, v_{2(n-m)}$ by means of the relations

$$x_i = x_i^0(u_1, \ldots, u_m) + \gamma_{ij}(u_1, \ldots, u_m)\, v_j,$$

where $\gamma_{ij}(u_1, \cdots, u_m)$ are certain functions of the variables $u_1, \cdots, u_m$. In the new variables u and v, equations (2.11) can be written in the form

$$\frac{du_i}{dt} = G_i(u, v), \qquad \frac{dv_j}{dt} = H_j(u, v). \tag{2.12}$$

We now linearize the equations of motion (2.12) in the neighborhood

of the surface O_m of equilibrium states. Expanding the right-hand sides of equations (2.12) in a series in the small quantities $v_1, v_2, \cdots, v_{2(n-m)}$, we obtain

$$\begin{aligned} \frac{du_i}{dt} &= a_i(u_1, \ldots, u_m) + a_{ij}(u_1, \ldots, u_m)v_j + O(\|v\|^2), \\ \frac{dv_j}{dt} &= b_j(u_1, \ldots, u_m) + b_{jk}(u_1, \ldots, u_m)v_k + O(\|v\|^2), \end{aligned} \tag{2.13}$$

where

$$\|v\| = (v_1^2 + \cdots + v_{2n-2m}^2)^{1/2},$$

and the expression $O(\|v\|^2)$ stands for terms of not lower than second order in $\|v\|$. It is readily seen that the expansion coefficients a_i and b_j in equations (2.13) vanish by virtue of the fact that the quantities $v_1, \cdots, v_{2(n-m)}$, $\dot{v}_1, \cdots, \dot{v}_{2(n-m)}$, $\dot{u}_1, \cdots, \dot{u}_m$ vanish on the surface O_m.

Thus, the equations of motion (2.13), linearized in a small neighborhood of the surface O_m (with respect to the variables $v_1, \cdots, v_{2(n-m)}$), have the form

$$\begin{aligned} \frac{du_i}{dt} &= a_{ij}(u_1, \ldots, u_m)v_j \\ \frac{dv_j}{dt} &= b_{jk}(u_1, \ldots, u_m)v_k \end{aligned} \quad (i = 1, 2, \ldots, m;\ j, k = 1, 2, \ldots, 2(n-m)). \tag{2.14}$$

The equations (2.14) are the equations of the small oscillations of the nonholonomic system about the manifold (surface O_m) of equilibrium states. It follows from (2.14) that, since the quantities $v_1, v_2, \cdots, v_{2(n-m)}$ are small, the variables $u_1, u_2, \cdots, u_m$ are slowly varying functions of the time. The solution of the system of equations (2.14) can be found by the method of successive approximations, the constant values $u_i = u_i^0$ being taken as the zeroth approximation for u_i. Substituting these values into the right-hand sides of equations (2.14) in order to obtain approximate values of v_j, we obtain the equations

$$\frac{dv_j}{dt} = b_{jk}(u_1^0, \ldots, u_m^0)v_k. \tag{2.15}$$

In determining the solution of equations (2.15), we arrive at the characteristic equation

$$\Delta = \mathrm{Det}\|\delta_{jk}p - b_{jk}\| = 0. \tag{2.16}$$

Let $p_1, p_2, \cdots, p_{2(n-m)}$ be the roots of equation (2.16), which, for the sake of simplicity, we shall assume are different and nonvanishing. Then, the solution for v_j can be written in the form

$$v_j = c_k \Delta_j(p_k) \exp(p_k t), \tag{2.17}$$

where c_k are arbitrary constants determined by the initial conditions and $\Delta_j(p_k)$ are the minors of the determinant Δ [the value of $\Delta_j(p_k)$ can be obtained by substituting the root p_k into the minor of the determinant Δ formed by removing the jth column and the last row of the determinant].

Substituting the solution (2.17) into the first group of equations (2.14), we obtain the equations for the determination of the functions $u_i(t)$. After integration, these equations yield

$$u_i = \frac{1}{p_k} c_k \Delta_i(p_k) e^{p_k t} + d_i, \tag{2.18}$$

where d_i are arbitrary constants determined by the initial conditions. Then, substituting the expression (2.18) into the right-hand side of equation (2.14), we obtain equations for the determination of the subsequent approximations. These equations again split into two groups. The equations for the determination of v_k are now a system of $2(n-m)$ linear differential equations with exponential coefficients. Continuing this process further, one can obtain the solution of the system of equations (2.14) with any desired degree of accuracy.

If we are interested in the question of the stability of the manifold of equilibrium states, there is no need to find the exact solution of the system of equations (2.14). As follows from what we have said, it is sufficient for this purpose to investigate the behavior of the functions $v_i(t)$ in a small neighborhood of the surface O_m. However, in the first approximation, the behavior of these functions is determined by the roots of the characteristic equation (2.16). If the real parts of all the roots p_k $[k = 1, 2, \cdots, 2(n-m)]$ are negative, the functions $v_j(t)$ represent either an exponential decay or an oscillatory process with decreasing amplitude. Therefore, if the representative point is situated in a small neighborhood of the surface O_m of equilibrium states, it will tend to the surface O_m as $t \to +\infty$. In this case, the manifold of asymptotic states is said to be asymptotically stable. If the roots p_k include even one with positive real part, the manifold of equilibrium states is unstable.

We should mention that in order to obtain the characteristic equation (2.16), whose roots determine the stability of the manifold of equilibrium states, there is no need to go over to the variables u_i and v_j. This transition was made with a view to elucidating the significance of the vanishing

roots of the characteristic equation. It is easily seen that the characteristic equation (2.16) can be obtained from (2.7) by neglecting the vanishing roots, whose number is usually equal to the number of dimensions of the manifold of equilibrium states.[6] It follows from (2.14) that the vanishing roots of the characteristic equation are not connected with a critical case in the theory of small oscillations but are due to the existence of the m-dimensional manifold of equilibrium states. Therefore, in investigating the stability of this manifold, the vanishing roots may be simply omitted. As a result, we obtain the characteristic equation (2.16), to which one can apply all the known stability criteria, such as the Routh-Hurwitz criterion, the D-decomposition method, etc. The stability conditions that are found enable one to distinguish stability regions on the surface O_m and determine the behavior of the boundaries of the stability regions as the values of the physical parameters are varied.

We should mention that the process of linearization of the equations of motion of a nonholonomic system in the neighborhood of a certain point $(u_1^0, \cdots, u_m^0)$ of the surface O_m is exactly the same as the process described in subsection 1 of this section. However, an important difference between the equations that are obtained in this case and those found in subsection 1 is the fact that the point u_1^0 may now be any point of the surface O_m. Therefore, the quantities $u_1^0, u_2^0, \cdots, u_m^0$ in the coefficients a_{ij} and b_{ij} of equations (2.14) must be regarded as additional parameters, together with the physical parameters of the system.

Thus, the linearized equations enable one to investigate the stability of each point of the surface of equilibrium states and, by the same token, distinguish the stability regions on the surface O_m, whereas the linearized equations of subsection 1 enable one to determine the stability at only the single point $(0, 0, \cdots, 0)$. An estimate of the solution of equations (2.14) enables one to draw conclusions about the asymptotic behavior of the system in a small neighborhood of the surface O_m.

3. Theorem on the asymptotic stability of the surface of equilibrium states of a nonholonomic system. As we have already seen, it is only meaningful to study the stability of the equilibrium states of a nonholonomic system with respect to small deviations from the surface

[6] The case when the number of vanishing roots of the characteristic equation is greater than the dimension of the manifold of equilibrium states must be regarded as a singular (critical) case.

O_m. Regarding the variables $u_1, u_2, \cdots, u_m$ for the time being as parameters, it is natural to consider the second group of equations (2.14) independently of the first group. The characteristic polynomial of this auxiliary system is given by the expression (2.16).

Let us assume that in some region G of values $u_1^0, \cdots, u_m^0$ an equilibrium state $v_1 = v_2 = \cdots = v_{2(n-m)} = 0$ of the system of equations

$$\frac{dv_j}{dt} = b_{jk}(u_1^0, \ldots, u_m^0)\, v_k \tag{2.19}$$

is asymptotically stable, so that

$$\|v\| < M\|v^0\| e^{-\sigma t}, \tag{2.20}$$

where $\sigma > 0$, $0 < M < \infty$, and v_j^0 are the initial values of the variables v_j. We then have the following theorem on the asymptotic stability of the manifold of equilibrium states of the nonholonomic system.

THEOREM. *Let the initial values $u_1^0, \cdots, u_m^0$, $v_1^0, \cdots, v_{2(n-m)}^0$ be such that the values $u_1^0, \cdots, u_m^0$ lie inside the region G of asymptotic stability of the equations* (2.19), *and let the quantities $v_1^0, \cdots, v_{2(n-m)}^0$ be sufficiently small; then, by virtue of the equations of motion of the nonholonomic system*

$$\frac{du_i}{dt} = G_i(u, v), \quad \frac{dv_j}{dt} = H_j(u, v) \tag{2.21}$$

the limiting relations

$$\lim_{t\to\infty} v_j(t) = 0, \qquad \lim_{t\to\infty} u_i(t) = u_i^*,$$

where $u_i^* \subset O_m$, hold, but, in general, $u_i^* \neq u_i^0$. At the same time, the variables $v_j(t)$ satisfy the estimate

$$\|v(t)\| < M'\|v^0\| e^{-\sigma' t}, \tag{2.22}$$

where $0 < \sigma' < \sigma$ and $0 < M' < +\infty$.

PROOF. We write down the equations (2.21) in the form

$$\begin{aligned} \frac{du_i}{dt} &= \{a_{ij}(u_1, \ldots, u_m) + \Delta a_{ij}\}\, v_j, \\ \frac{dv_j}{dt} &= \{b_{jk}(u_1, \ldots, u_m) + \Delta b_{jk}\}\, v_k, \end{aligned} \tag{2.23}$$

where $|\Delta a_{ij}| < \epsilon$ and $|\Delta b_{ij}| < \epsilon$, provided the inequalities

$$\|u - u^0\| < \delta, \qquad \|v\| < \delta(\epsilon), \tag{2.24}$$

where δ is sufficiently small, are satisfied. As long as the inequalities (2.24) hold with $\delta \leqq \delta^*$, one can give the following estimate for the solution of equations (2.23):[7]

$$\|v(t)\| < M' \|v^0\| e^{-\sigma' t}, \tag{2.25}$$

from which it follows that

$$\|\ddot{u}\| < N \|v^0\| e^{-\sigma' t}$$

and, therefore,

$$\|u(t) - u^0\| < \frac{N}{\sigma'} \|v^0\|. \tag{2.26}$$

Suppose that at the initial instant of time the conditions (2.24) are satisfied with $\delta = \delta^*/2$ and

$$\|v^0\| < \min\left(\frac{\delta^*}{2M}, \frac{\delta^* \sigma'}{2N}, \frac{\delta^*}{2}\right). \tag{2.27}$$

By virtue of the uniformly continuous dependence of the solution on the time, the conditions (2.24) for $\delta = \delta^*$ are satisfied for a certain interval $\Delta t_0 > \tau > 0$. During this interval of time Δt_0, the estimates (2.25) and (2.26) hold. By virtue of these inequalities and the inequality (2.27), the quantity $v(\Delta t_0)$ satisfies the inequalities (2.24), in which $\delta = \delta^*/2$. Repeating this argument, we conclude that these inequalities are also satisfied during the interval of time $\Delta t_0 + \Delta t_1 + \cdots + \Delta t_s > (s+1)\tau$, i.e., for all t, from which the assertion of the theorem follows.

---

[7] For, according to [325], there exists for the system of equations (2.19) a positive definite quadratic form $V = C_{kj} v_k v_j$ such that

$$(*) \qquad \alpha \|v\|^2 < V < \beta \|v\|^2 \qquad (\alpha, \beta > 0)$$

and for which the inequalities

$$(**) \qquad \frac{dV}{dt} < -\gamma \|v\|^2 < -\frac{\gamma}{\alpha} V$$

hold for all sufficiently small variations of the coefficients b_{jk}. From (*) and (**), we obtain

$$V < (V)_{t=0} e^{-\frac{\gamma}{\alpha} t}$$

and, hence,

$$\alpha \|v\|^2 < \beta \|v^0\|^2 e^{-\frac{\gamma}{\alpha} t} \quad \text{for} \quad \|v\| < \left(\frac{\beta}{\alpha}\right)^{1/2} \|v^0\| e^{-\frac{\gamma}{2\alpha} t},$$

as was asserted.

4. Stability with respect to small continuously acting perturbations. If the manifold of equilibrium states of a nonholonomic system is asymptotically stable, we obtain the following theorem on the behavior of the system in the neighborhood of the manifold for small continuously acting perturbations.

THEOREM. *In a region G of asymptotic stability of equilibrium states one can, for any $\epsilon > 0$, find a $\delta > 0$ such that for all continuously acting perturbations less than δ the phase point does not, as long as its u components are situated in G, leave the ϵ-neighborhood of the set of equilibrium states; and one can always find arbitrarily small continuously acting perturbations for which the phase point moves on the surface of equilibrium states along any preassigned curve in the region G.*

It follows from this theorem that the connected branch of the set of equilibrium states is stable with respect to sufficiently small continuously acting perturbations when all its points are asymptotically stable, and is unstable if this connected branch contains unstable points.

PROOF. In the presence of continuously acting perturbations, the equations of motion near the manifold O_m of equilibrium states can be written in the form

$$\begin{aligned} \frac{du_i}{dt} &= a_{ij}(u_1, \ldots, u_m)\, v_j + O(\|v\|^2) + \omega_i(t), \\ \frac{dv_j}{dt} &= b_{jk}(u_1, \ldots, u_m)\, v_k + O(\|v\|^2) + \delta_j(t). \end{aligned} \tag{2.28}$$

In order to prove the first assertion of stability with respect to external perturbations, we shall use a Ljapunov function constructed for the second group of equations, in which the variables $u_1, u_2, \cdots, u_m$ are regarded as parameters. In the region of asymptotic stability G, there exists a positive quadratic form V with coefficients depending on u_1, $u_2, \cdots, u_m$, such that for certain α and β satisfying $0 < \alpha < \beta < +\infty$

$$\alpha \|v\|^2 < V < \beta \|v\|^2$$

and, by virtue of equations (2.19),

$$\frac{dV}{dt} = -\|v\|^2, \quad \sum_k \left|\frac{\partial V}{\partial u_k}\right| < a\|v\|^2 \ (a < +\infty).$$

For $v_1 = \cdots = v_{2(n-m)} = 0$, i.e., on the surface of equilibrium states, $V = 0$ and, by virtue of equations (2.28),

$$\frac{dV}{dt} \leqslant -\|v\|^2 + A\|v\|^3 + B\delta\|v\|, \tag{2.29}$$

where A and B are certain constants and $\delta = \sup\{|\omega_i(t)|, |\delta_j(t)|\}$. It follows from (2.29) that for $2B\delta < \|v\| < 1/2A$

$$dV/dt < 0.$$

This means that for sufficiently small δ the phase point $(u_1, \cdots, u_m, v_1, \cdots, v_{2(n-m)})$, whose motion is governed by equations (2.28), does not, as long as the variables $u_1, \cdots, u_m$ lie in the asymptotic stability region G, leave the $2\delta B$ neighborhood of the surface of equilibrium states. This proves the first assertion of the theorem.

In order to prove the second assertion, it is sufficient to show that one can choose arbitrarily small perturbations for which the quantity u_1 increases continuously and the remaining variables $u_2, u_3, \cdots, u_m$ remain unchanged. Let $\delta_j(t) = 0$ $[j = 1, 2, \cdots, 2(n-m)]$, $\dot{u}_i = 0$ $(i = 2, \cdots, m)$, and $\dot{u}_1 = \omega_1 > 0$; it then follows from the second group of equations (2.28) that

$$\|v\| < M\|v^0\|e^{-\sigma' t},$$

and, from the first m equations, we can find the desired perturbations $\omega_1(t)$ and $\omega_j(t)$:

$$\omega_1(t) = \omega_1 - a_{1j}v_j - O(\|v\|^2), \quad \omega_j(t) = -a_{jk}v_k - O(\|v\|^2).$$

5. Examples of the investigation of the stability and small oscillations of nonholonomic systems about the equilibrium states. We shall now consider some examples that illustrate the theory expounded in this section.

1. *Čaplygin's sleigh on an inclined plane.* Let us consider the motion of a rigid body parallel to an inclined plane. Let the body be supported on the inclined plane by three legs, two of which are absolutely smooth and the third is fitted with a semicircular knife edge, so that the third leg cannot be displaced in the direction perpendicular to the plane of the knife edge. Consider the case when the projection of the center of gravity C of the body onto the inclined plane lies on the straight line perpendicular to the knife edge passing through the point K of contact of the knife edge and the plane (Figure 5.7). The generalized coordinates are the Cartesian coordinates x and y of the point K on the inclined

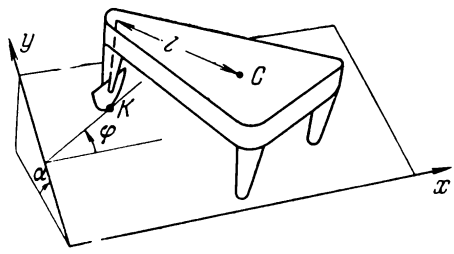


FIGURE 5.7

plane and the angle φ of rotation of the body about a straight line perpendicular to the inclined plane.

An expression for the Lagrange function is

$$L = \frac{m}{2}[(\dot{x} + l\dot{\varphi}\cos\varphi)^2 + (\dot{y} + l\dot{\varphi}\sin\varphi)^2 + k^2\dot{\varphi}^2] - mg\sin\alpha(y - l\cos\varphi),$$

where m is the mass of the body; k is the radius of gyration; α is the angle of the inclined plane; l is the distance from the projection of the center of mass onto the inclined plane to the point K; and g is the acceleration of the force of gravity. We introduce the dissipation function

$$F = \frac{m}{2}[h(\dot{x}^2 + \dot{y}^2) + h_1\dot{\varphi}^2],$$

where $h \geqq 0$ and $h_1 \geqq 0$ are the coefficients of the viscous friction of sliding and pivoting, respectively. The nonholonomic constraint can be expressed by the equation

$$\dot{y} - \dot{x}\tan\varphi = 0. \tag{2.30}$$

The equations of motion of the body are

$$\begin{aligned} &\frac{d}{dt}(\dot{x} + l\dot{\varphi}\cos\varphi) + h\dot{x} + \lambda\tan\varphi = 0, \\ &\frac{d}{dt}(\dot{y} + l\dot{\varphi}\sin\varphi) + h\dot{y} + g\sin\alpha - \lambda = 0, \\ &\frac{d}{dt}[l(\dot{x}\cos\varphi + \dot{y}\sin\varphi) + (l^2 + k^2)\dot{\varphi}] + h_1\dot{\varphi} + gl\sin\alpha\sin\varphi = 0. \end{aligned} \tag{2.31}$$

From the equations of motion (2.30) and (2.31), we obtain the equilibrium equations

$$\lambda\tan\varphi = 0, \quad \lambda = g\sin\alpha, \quad \sin\varphi = 0, \tag{2.32}$$

from which it follows that the equilibrium states of the holonomic system form two planes $\varphi = 0$ and $\varphi = \pi$ in the configuration space (x, y, φ). The manifold of equilibrium states has two dimensions despite the fact that there is only a single nonholonomic constraint equation. The increase in the number of dimensions by unity arises because the first and third equations of (2.32) are dependent. Setting $x = x^0 + \xi$, $\varphi = \varphi^0 + \zeta$, $\lambda = \lambda^0 + \theta$, $y = y^0 + \eta$, where x^0, y^0, φ^0, and λ^0 are the equilibrium values of the variables satisfying equations (2.32), we linearize equations (2.30) and (2.31) with respect to the small quantities ξ, η, ζ, and θ:

$$\dot{\eta} = 0,$$
$$\ddot{\xi} + h\dot{\xi} \pm l\ddot{\zeta} + \zeta g \sin\alpha = 0,$$
$$\ddot{\eta} + h\dot{\eta} - \theta = 0,$$
$$\pm l\ddot{\xi} + (l^2 + k^2)\ddot{\zeta} + h_1\dot{\zeta} \pm \zeta gl \sin\alpha = 0,$$

where the upper sign refers to the plane $\varphi = 0$ and the lower sign to the plane $\varphi = \pi$. The characteristic equation is

$$p^2\{k^2p^3 + [h(l^2 + k^2) + h_1]p^2 + hh_1p \pm hgl \sin\alpha\} = 0.$$

The two vanishing roots are due to the fact that the manifold of equilibrium states is two-dimensional. The stability of the manifold is determined by the roots of the equation

$$k^2p^3 + [h(l^2 + k^2) + h_1]p^2 + h_1hp \pm hgl \sin\alpha = 0. \tag{2.33}$$

It follows directly from (2.33) that the plane $\varphi = \pi$ is always unstable. Using the Routh-Hurwitz criterion, we find that the plane $\varphi = 0$ is stable only if

$$(l^2 + k^2)h > \frac{k^2gl \sin\alpha}{h_1} - h_1.$$

The boundary and region of stability in the plane of the parameters h and h_1 is shown in Figure 5.8, in which $h_1^* = k\sqrt{gl \sin\alpha}$.

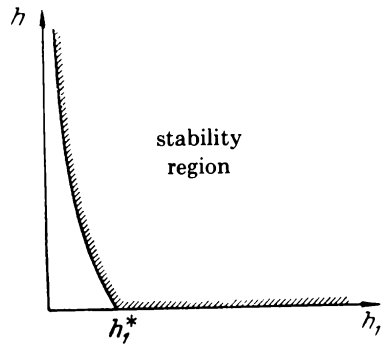


FIGURE 5.8

Thus, depending on the values of the parameters h and h_1, the plane $\varphi = 0$ is either completely stable or completely unstable. If it is unstable, the representative point will, if it is displaced from the plane, either not return to the plane at all or execute oscillations about it with increasing amplitude. In the second case, the trajectory of the point K of contact of the body and the inclined plane has the form shown in Figure 5.9a. If the plane $\varphi = 0$ is stable, the trajectory of the point K has the form shown in Figure 5.9b.

In accordance with the theorem on the asymptotic stability of the manifold of equilibrium states of a nonholonomic system, the representative point will, if it is displaced from a stable surface of equilibrium

states, return to the surface, but, in general, to a different point of it. This means that, having been perturbed at the point (1) in Figure 5.9b, the body arrives at a new equilibrium position, for example, at point (2) of Figure 5.9b.

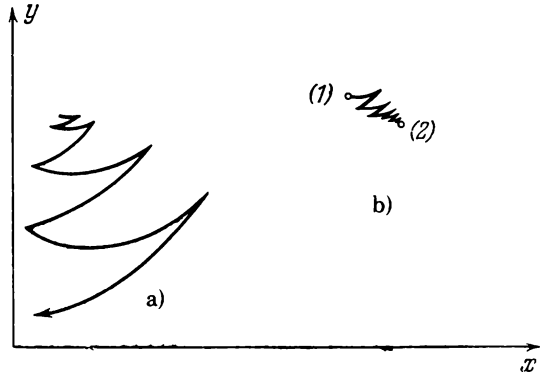


FIGURE 5.9

It is immediately clear from (2.33) that the characteristic equation (2.33) acquires a vanishing root if $l = 0$. In accordance with the foregoing, we have a critical case when $l = 0$. In order to investigate the stability, we must allow for the nonlinear terms in the equations of motion, which, when $l = 0$, can be written in the form

$$\begin{aligned} &\ddot{x} + h\dot{x} - \lambda \tan \varphi = 0, \quad \ddot{y} + h\dot{y} + \lambda + g \sin \alpha = 0, \\ &k^2\ddot{\varphi} + h_1\dot{\varphi} = 0, \qquad\qquad \dot{y} - \dot{x} \tan \varphi = 0. \end{aligned} \tag{2.34}$$

This system of equations can be easily investigated in the case $h_1 = 0$. Let us first consider this case and then study the influence of the coefficient h_1.

The case $h_1 = 0$. It follows from the third equation of (2.34) that $\dot{\varphi} = \omega = \text{const}$ and, hence, $\varphi = \omega t + \varphi_0$, where φ_0 is the initial angle. Eliminating the variables y and λ, we find

$$\ddot{x} + \dot{x}\,[h + \omega \tan(\omega t + \varphi_0)] + g \sin \alpha \sin(\omega t + \varphi_0) \cos(\omega t + \varphi_0) = 0. \tag{2.35}$$

We introduce the notation

$$u = \dot{x}, \quad \varphi = \omega t + \varphi_0, \quad a = \frac{h}{\omega}, \quad b = \frac{g \sin \alpha}{\omega}$$

and write equation (2.35) in the form

$$\frac{du}{d\varphi} + (a + \tan \varphi)\, u + b \sin \varphi \cos \varphi = 0.$$

The solution of this equation is

$$u = Ce^{-a\varphi}\cos\varphi + \frac{b}{a^2+1}(\cos^2\varphi - a\sin\varphi\cos\varphi),$$

where C is an arbitrary constant determined by the initial conditions. Reverting to the variable x and using the last equation of (2.34), we obtain

$$\begin{aligned}\omega\frac{dx}{d\varphi} &= Ce^{-a\varphi}\cos\varphi + \frac{b}{a^2+1}(\cos^2\varphi - a\sin\varphi\cos\varphi),\\ \omega\frac{dy}{d\varphi} &= Ce^{-a\varphi}\sin\varphi + \frac{b}{a^2+1}(\sin\varphi\cos\varphi - a\sin^2\varphi).\end{aligned} \tag{2.36}$$

Integrating these equations, we obtain a parametric equation of the trajectory of the motion of the point K:

$$\begin{aligned}\omega x &= C_1 + \frac{Ce^{-a\varphi}}{a^2+1}(\sin\varphi - a\cos\varphi)\\ &\quad + \frac{b}{a^2+1}\left(\frac{\varphi}{2} + \frac{1}{4}\sin 2\varphi + \frac{a}{4}\cos 2\varphi\right),\\ \omega y &= C_2 - \frac{Ce^{-a\varphi}}{a^2+1}(\cos\varphi + a\sin\varphi)\\ &\quad - \frac{b}{a^2+1}\left(\frac{a}{2}\varphi + \frac{1}{4}\cos 2\varphi - \frac{a}{4}\sin 2\varphi\right).\end{aligned} \tag{2.37}$$

We can determine the constants C, C_1, and C_2 from the initial conditions: for $t = 0$, $\varphi = x = y = 0$, $\dot{x} = \dot{x}_0$, and $\dot{\varphi} = \omega$. Substitution of these values into (2.36) and (2.37) gives

$$C = \dot{x}_0 - \frac{b}{a^2+1}, \quad C_1 = \frac{a}{(a^2+1)^2}[\dot{x}_0(a^2+1) - b] - \frac{ab}{4(a^2+1)},$$

$$C_2 = \frac{1}{(a^2+1)^2}[\dot{x}_0(a^2+1) - b] + \frac{b}{4(a^2+1)}.$$

If we introduce the complex plane $\zeta = x + iy$, the motion of the point of contact K satisfies the law

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3,$$

where

$$\zeta_1 = \frac{b(1-ai)}{2(a^2+1)}t, \quad \zeta_2 = \frac{\left(\dot{x}_0 - \frac{b}{a^2+1}\right)(a+i)}{\omega(a^2+1)}[1 - e^{(i\omega - h)t}],$$

$$\zeta_3 = \frac{b(i-a)}{4\omega(a^2+1)}[1 - e^{2i\omega t}].$$

The trajectory of this motion can be readily obtained graphically by addition of the vectors ζ_1, ζ_2, and ζ_3.

Thus, the integration of the equations of motion (2.34) in the case $h_1 = 0$ shows that, for any initial conditions for which $\varphi_0 \neq 0$, the body moves an arbitrarily large distance from the initial position. At the same time, the motion of the body is made up as follows. 1) A displacement with constant velocity along the straight line which forms an angle $\delta = -\tan^{-1}(h/\omega)$ with the horizontal; 2) a damped rotation with angular velocity ω; and 3) a rotation with angular velocity 2ω.

Consequently, the equilibrium state is always unstable if $h_1 = 0$.

The case $h_1 \neq 0$. If we now take into account the damping $h_1 \neq 0$, it follows from the third equation of (2.34) that the angle φ varies in accordance with the law

$$\varphi = \varphi_0 + \frac{\omega k^2}{h_1}\left(1 - e^{-\frac{h_1}{k^2}t}\right),$$

where φ_0 and ω are the initial values of the angle φ and the angular velocity $\dot{\varphi}$. Since the quantity $\varphi_0 + \omega k^2/h_1 = \lim_{t\to\infty}\varphi$ can be arbitrary and stable equilibrium of the body is possible only for $\varphi = 0$, we conclude that the equilibrium position $\varphi = 0$ is unstable in the case $h_1 \neq 0$ as well. We may mention in passing that there exists a set of perturbations (whose measure is zero) for which the system again returns to a state of rest, namely, when $\lim_{t\to\infty}\varphi = 2\pi n$, where n is any integer.

2. *Sphere on a rough horizontal plane.* Suppose that an inhomogeneous sphere whose central ellipsoid of inertia is an ellipsoid of revolution rolls and pivots on a rough horizontal plane. We shall consider the case when the center of mass does not coincide with the geometrical center O_1 of the sphere and the symmetry axis of the ellipsoid of inertia passes through the geometrical center of the sphere. The configuration of the body can be specified by the following five generalized coordinates: the Cartesian coordinates x and y of the point K of contact of the sphere and the plane, the Résal angles θ and ψ (Figure 5.10), and the angle φ of rotation of the body about the axis of dynamical symmetry $O_1\zeta$. The axis $O_1\xi$ is always parallel to the xOz plane. The cosines of the angles between the axes $Oxyz$ and $O_1\xi\eta\zeta$ are given in the following table:

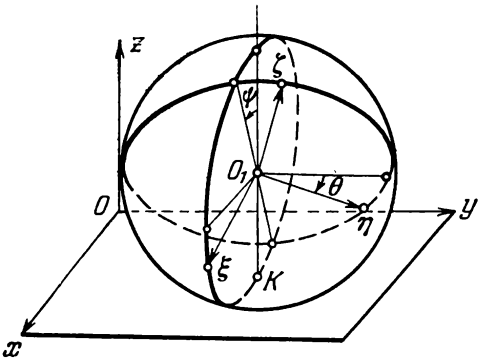


FIGURE 5.10

	ξ	η	ζ
x	$\cos\psi$	$\sin\psi\sin\theta$	$\sin\psi\cos\theta$
y	0	$\cos\theta$	$-\sin\theta$
z	$-\sin\psi$	$\cos\psi\sin\theta$	$\cos\psi\cos\theta$

The condition of rolling without sliding can be written in the form

$$\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}_{0k} = 0.$$

Substituting

$$\boldsymbol{v}_0\,(\dot{x},\ \dot{y},\ 0),\ \boldsymbol{r}_{0k}\,(0,\ 0,\ -R),$$
$$\boldsymbol{\omega}\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\psi\cos\theta,\ \dot{\psi} - \dot{\varphi}\sin\theta,\ -\dot{\theta}\sin\psi + \dot{\varphi}\cos\psi\cos\theta),$$

we obtain the equations of the nonholonomic constraints

$$\dot{x} - R\dot{\psi} + R\dot{\varphi}\sin\theta = 0,\ \dot{y} + R\dot{\varphi}\sin\psi\cos\theta + R\dot{\theta}\cos\psi = 0. \tag{2.38}$$

An expression for the Lagrange function is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + ml\,[\dot{x}(\dot{\theta}\sin\psi\sin\theta - \dot{\psi}\cos\psi\cos\theta) + \dot{y}\dot{\theta}\cos\theta]$$
$$+ \frac{1}{2}(A + ml^2)(\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta) + \frac{1}{2}C(\dot{\varphi} - \dot{\psi}\sin\theta)^2 + mgl\cos\psi\cos\theta,$$

where m is the mass of the body; l is the distance between the center of the sphere and its center of mass; C is the axial moment of inertia of the body; A is the central equatorial moment of inertia of the body; and g is the acceleration of the force of gravity.

We introduce a dissipation function by means of the expression

$$F = \frac{1}{2}h(\omega_x^2 + \omega_y^2) + \frac{1}{2}h_1\omega_z^2,$$

where h and h_1 are the coefficients of viscous friction of rolling and pivoting, respectively. Substituting into this expression the components of the angular velocity ω and using equations (2.38), we obtain

$$F = \frac{1}{2}\left[\frac{h}{R^2}(\dot{x}^2 + \dot{y}^2) + h_1(\dot{\varphi}\cos\psi\cos\theta - \dot{\theta}\sin\psi)^2\right].$$

The equations of motion with undetermined multipliers are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda_i a_{ik} - \frac{\partial F}{\partial \dot{q}_k} \quad (k = 1,\ 2,\ \ldots,\ 5),$$

where

$$q_1 = x,\quad q_2 = y,\quad q_3 = \theta,\quad q_4 = \psi,\quad q_5 = \varphi.$$

We obtain:

$$\frac{d}{dt}\left[m\dot{x} + ml\left(\dot{\theta}\sin\psi\sin\theta - \psi\cos\psi\cos\theta\right)\right] + \frac{h\dot{x}}{R^2} = \lambda_1,$$

$$\frac{d}{dt}\left[m\dot{y} + ml\dot{\theta}\cos\theta\right] + \frac{h}{R^2}\dot{y} = \lambda_2,$$

$$\begin{aligned}\frac{d}{dt}\left[ml\left(\dot{x}\sin\psi\sin\theta + \dot{y}\cos\theta\right) + A_0\dot{\theta}\right] - h_1\sin\psi\left(\varphi\cos\theta\cos\psi - \dot{\theta}\sin\psi\right)\\ - ml\left[\dot{x}\left(\dot{\theta}\sin\psi\cos\theta + \psi\cos\psi\sin\theta\right) - \dot{y}\dot{\theta}\sin\theta\right]\\ + A_0\psi^2\sin\theta\cos\theta + C\psi\cos\theta(\varphi - \psi\sin\theta)\\ + mgl\cos\psi\sin\theta = R\lambda_2\cos\psi,\end{aligned} \tag{2.39}$$

$$\begin{aligned}\frac{d}{dt}\left[A_0\psi\cos^2\theta - ml\dot{x}\cos\psi\cos\theta - C\sin\theta\left(\varphi - \psi\sin\theta\right)\right]\\ - ml\dot{x}\left(\dot{\theta}\cos\psi\sin\theta + \psi\sin\psi\cos\theta\right)\\ + mgl\sin\psi\cos\theta = -R\lambda_1,\end{aligned}$$

$$\begin{aligned}C\frac{d}{dt}\left(\varphi - \psi\sin\theta\right) + h_1\cos\psi\cos\theta\left(\dot{\varphi}\cos\psi\cos\theta - \dot{\theta}\sin\psi\right)\\ = R\lambda_1\sin\theta - R\lambda_2\sin\psi\cos\theta,\end{aligned}$$

where $A_0 = A + ml^2$. It follows from the equations of motion (2.39) and (2.38) that the following relations are satisfied in an equilibrium state of the sphere:

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \cos\psi\sin\theta = 0, \quad \sin\psi\cos\theta = 0. \tag{2.40}$$

These determine a three-dimensional surface of equilibrium states in the five-dimensional configuration space $(x, y, \theta, \psi, \varphi)$: 1) $\theta = 0$, $\psi = 0$, 2) $\theta = \pi$, $\psi = \pi$, 3) $\theta = \pi$, $\psi = 0$, 4) $\theta = 0$, $\psi = \pi$. There, the surfaces 1 and 2 and also 3 and 4 must be identified, since they represent identical configurations. We should mention that the values $\theta = \psi = \pi/2$, which also satisfy equations (2.40), must be disregarded since they are related to a singular choice of the system of angles and do not reflect any equilibrium state of the sphere. In order to find the stability conditions of the surface $\theta = \psi = 0$, we linearize the equations of motion (2.38) and (2.39) in the neighborhood of an equilibrium state of the surface. Denoting by primed quantities the small deviations of the corresponding quantities from their equilibrium values, we obtain

$$m\ddot{x}' - ml\ddot{\psi}' + \frac{h}{R^2}\dot{x}' = \lambda_1', \quad m\ddot{y}' + ml\ddot{\theta}' + \frac{h}{R^2}\dot{y}' = \lambda_2',$$

$$ml\ddot{y}' + A\ddot{\theta}' + mgl\theta' = R\lambda_2', \quad -ml\ddot{x}' + A_0\ddot{\psi}' + mgl\psi' = -R\lambda_1',$$

$$C\dot{\varphi}' + h_1\dot{\varphi}' = 0, \quad \dot{x}' - R\psi' = 0, \quad \dot{y}' + R\dot{\theta}' = 0.$$

Hence, we find the characteristic equation

$$(2.41) \qquad p^3(Cp + h_1)\{[A + m(R - l)^2]p^2 + hp + mgl\}^2 = 0.$$

The three-fold vanishing root of equation (2.41) is due to the three dimensions of the manifold of equilibrium states. The stability of the manifold is determined by the roots of the equation

$$(2.41') \qquad (Cp + h_1)\{[A + m(R - l)^2]p^2 + hp + mgl\} = 0,$$

from which it follows that, for $h > 0$ and $h_1 > 0$, the manifold $\theta = \psi = 0$ is always stable.

The characteristic equation of the manifold $\theta = \pi$, $\psi = 0$ can be obtained from (2.41) by replacing l by $-l$. For $l > 0$, the free term of equation (2.41′) is negative and, hence, the surface $\theta = \pi$, $\psi = 0$ is always unstable.

3. *Top in a spherical cup.* Let us consider the motion of an axisymmetric body bounded below by a spherical surface of radius R which can roll without sliding in a spherical cup of radius R_1. The center of gravity of the body is situated at a distance l from the center of its spherical surface. In accordance with the notation of Figure 5.11, the generalized coordinates are the angles θ, ψ, φ, θ_1, and ψ_1. The Lagrange function is

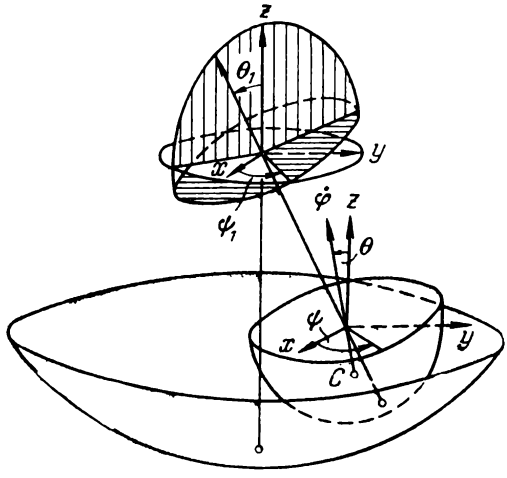


FIGURE 5.11

$$L = \frac{m}{2}(R_1 - R)^2(\dot\theta_1^2 + \dot\psi_1^2 \sin^2\theta_1) + \frac{1}{2}(A + ml^2)(\dot\theta^2 + \dot\psi^2 \sin^2\theta)$$
$$+ ml(R_1 - R)\{\psi\dot\psi_1 \sin\theta \sin\theta_1 \cos(\psi - \psi_1)$$
$$+ \dot\theta_1\dot\theta[\cos\theta \cos\theta_1 \cos(\psi - \psi_1) + \sin\theta \sin\theta_1]$$
$$+ \dot\theta\dot\psi_1 \cos\theta \sin\theta_1 \sin(\psi - \psi_1) - \dot\psi\dot\theta_1 \sin\theta \cos\theta_1 \sin(\psi - \psi_1)\}$$
$$+ \frac{1}{2}C(\dot\varphi + \dot\psi\cos\theta)^2 + mg[(R_1 - R)\cos\theta_1 - l\cos\theta],$$

where m is the mass of the body; C is the axial moment of inertia; A is the central equatorial moment of inertia; and g is the acceleration due to gravity. Assuming that the mechanical energy is dissipated as a result of viscous frictions we introduce the dissipation function

$$F = \frac{1}{2} h \left(\frac{R_1}{R} - 1\right)^2 (\dot{\theta}_1^2 + \dot{\psi}_1^2 \sin^2 \theta_1) + \frac{1}{2} h_1 \{\dot{\psi} \cos \theta_1 - \dot{\theta} \sin \theta_1 \sin (\psi - \psi_1) + \dot{\varphi} [\sin \theta \sin \theta_1 \cos (\psi - \psi_1) + \cos \theta \cos \theta_1]\}^2,$$

where $h \geqq 0$ and $h_1 \geqq 0$ are the coefficients of viscous friction of rolling and pivoting, respectively. The condition of rolling without sliding leads to two nonholonomic constraint equations

$$\begin{aligned} &(R_1 - R)\,\dot{\psi}_1 \sin \theta_1 + R\dot{\theta} \cos \theta_1 \sin (\psi - \psi_1) + R\dot{\psi} \sin \theta_1 \\ &\quad + R\dot{\varphi} [\cos \theta \sin \theta_1 - \sin \theta \cos \theta_1 \cos (\psi - \psi_1)] = 0, \\ &(R_1 - R)\,\dot{\theta}_1 + R \cos (\psi - \psi_1) + R\dot{\varphi} \sin \theta \sin (\psi - \psi_1) = 0. \end{aligned} \tag{2.42}$$

Writing down the equations of motion of the body, for which the generalized forces are $Q_\beta = -\partial F / \partial \dot{q}_\beta$, we obtain the following equations of equilibrium:

$$\begin{aligned} &\lambda_1 \sin \theta_1 = 0, \\ &mgl \sin \theta = R\lambda_1 \cos \theta_1 \sin (\psi - \psi_1) + R\lambda_2 \cos (\psi - \psi_1), \\ &mg \sin \theta_1 = \lambda_2, \quad \lambda_1 \sin \theta_1 = 0, \\ &\lambda_1 [\cos \theta \sin \theta_1 - \sin \theta \cos \theta_1 \cos (\psi - \psi_1)] \\ &+ \lambda_2 \sin \theta \sin (\psi - \psi_1) = 0. \end{aligned} \tag{2.43}$$

It follows from equations (2.43) that the surface of equilibrium states of the system is determined by the equations

$$\psi = \psi_1, \quad l \sin \theta = R \sin \theta_1 \tag{2.44}$$

and is three-dimensional.

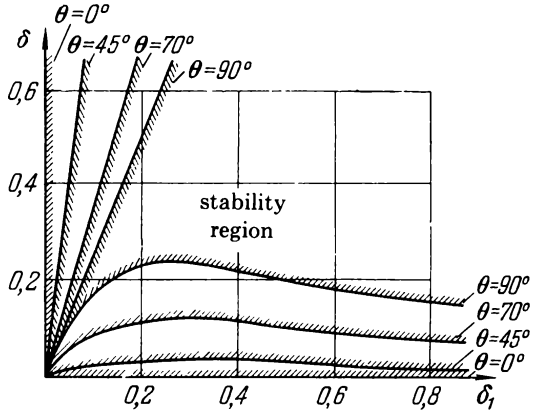


FIGURE 5.12

The increase in the number of dimensions of the manifold of equilibrium states by unity is due to the fact that the first and third equations of the system (2.43) are dependent.

We introduce dimensionless quantities:

$$\tau = t\sqrt{\frac{g}{R}}, \quad \alpha = \frac{A}{mR^2}, \quad \beta = \frac{C}{mR^2},$$
$$\gamma = \frac{l}{R}, \quad \rho = \frac{R_1}{R}, \quad \delta = \frac{h}{mR\sqrt{gR}}$$

and linearize the equations of motion in the neighborhood of the equilibrium states (2.44). A primed quantity will denote the deviation of the corresponding variable from its equilibrium value. Eliminating the undetermined multipliers, we obtain the following linearized equations:

$$[\alpha + \gamma^2 - \gamma\cos(\theta - \theta_1)]\,\ddot{\theta}' + \gamma\theta'\cos\theta - (\rho - 1)[1 - \gamma\cos(\theta - \theta_1)]\,\ddot{\theta}_1' - \delta(\rho - 1)\ddot{\theta}_1' - (\rho - 1)\,\theta_1'\cos\theta_1 = 0,$$

$$(\alpha\sin^2\theta + \beta\cos^2\theta)\,\ddot{\psi}' + \delta_1\psi_1'\cos^2\theta_1 - \delta(\rho - 1)\,\psi_1'\sin^2\theta_1 + \beta\ddot{\varphi}'\cos\theta + \delta_1\dot{\varphi}'\cos\theta_1\cos(\theta - \theta_1) = 0,$$

$$\beta\ddot{\varphi}' + \delta_1\dot{\varphi}'\cos^2(\theta - \theta_1) + (\rho - 1)\,\dot{\psi}_1'\sin\theta_1\sin(\theta - \theta_1) + \delta\psi'\sin\theta_1 \times \sin(\theta - \theta_1) + \gamma\psi_1'\sin^2\theta + [\beta\cos\theta + \gamma\sin\theta\sin(\theta - \theta_1)]\,\dot{\psi}' + \delta_1\psi'\cos\theta_1\cos(\theta - \theta_1) - \gamma\psi'\sin^2\theta = 0,$$

$$\dot{\psi}_1'(\rho - 1)\sin\theta_1 + \dot{\psi}'\sin\theta_1 - \dot{\varphi}'\sin(\theta - \theta_1) = 0,$$
$$(\rho - 1)\dot{\theta}_1' + \dot{\theta}' = 0.$$

Here, in accordance with (2.44), we have the relation $\gamma\sin\theta = \sin\theta_1$ which is satisfied in the equilibrium state. The characteristic equation of the system reduces to the form

$$p^3\,[a_0p^2 + \delta(\rho - 1)\,p + a_1](b_0p^3 + b_1p^2 + b_2p + b_3) = 0,$$

where

$$\begin{aligned}
a_0 &= (\rho - 1)\,[\alpha + 1 - 2\gamma\cos(\theta - \theta_1) + \gamma^2],\\
a_1 &= \cos\theta_1 + \gamma(\rho - 1)\cos\theta,\\
b_0 &= (\rho - 1)\,(\alpha\beta + ba^2),\\
b_1 &= (\rho - 1)\,\{b\,[\beta + (\alpha - \beta + \gamma^2)\sin(\theta - \theta_1)] + \\
&\qquad + \delta_1\,[\alpha + (\beta - \alpha)\sin^2(\theta - \theta_1) + a^2\cos^2\theta_1]\},\\
b_2 &= \delta\delta_1(\rho - 1) + \beta\rho\gamma\cos\theta + ab,\\
b_3 &= \delta_1\,[\rho\gamma\cos\theta_1\cos(\theta - \theta_1) + a\cos^2\theta_1] - \delta(\rho - 1)\,a\sin^2\theta
\end{aligned} \tag{2.45}$$

and we have introduced the notation $a = \cos\theta_1 - \gamma\cos\theta$, $b = \alpha\sin^2\theta + \beta\cos^2\theta$. For the values of the parameters $\rho > 1$, $0 < \gamma < 1$, the stability region of the manifold of equilibrium states of the system is determined by the inequalities

$$b_3 > 0, \quad b_1 b_2 - b_0 b_3 > 0. \tag{2.46}$$

In Figure 5.12, we show the boundaries of the stability region on the (δ_1, δ) plane constructed for a honogeneous hemisphere ($\alpha = 0.26$; $\beta = 0.4$; $\gamma = 0.375$), which rolls without sliding in a spherical cup. The calculation is made for the case when the radius of the hemisphere is four times less than the radius of the cup.

It follows from the expressions (2.45) that, as $\rho \to \infty$, when the spherical cup is transformed into a plane, the stability conditions (2.46) are satisfied for all values $\gamma > 0$. In particular, all equilibrium states of the hemisphere on a horizontal plane are stable (in the sense of the theorem on asymptotic stability; see §3).

4. *Kerkhoven-Wythoff problem.* This problem is considered by Whittaker in his book "Analytical dynamics" in the following formulation. A heavy homogeneous body which has the form of a hemisphere rests on a rough horizontal plane, its convex side pointing downward. A second hemisphere is placed on the upper plane of the first hemisphere, the point of contact coinciding with the center of the first hemisphere. Determine the small oscillations and establish the stability conditions of the system about this equilibrium position.

Linearizing the equations of motion in the neighborhood of the equilibrium position (which he obviously assumed to be isolated), Whittaker obtained the condition of conservative stability of the system in the form

$$9Rm > 40R_1 m_1, \tag{2.47}$$

where R and m are the radius and mass of the lower hemisphere and R_1 and m_1 are the same for the upper hemisphere.

Below, we shall consider the problem from the point of view of the stability theory of equilibrium states of nonholonomic systems expounded in subsections 2 – 4. In doing so, we shall show, in particular, that 1) the system possesses a six-dimensional manifold of equilibrium states; 2) without allowance for energy dissipation, the number of vanishing roots of the characteristic equation exceeds the number of dimensions of the manifold of equilibrium states by two.

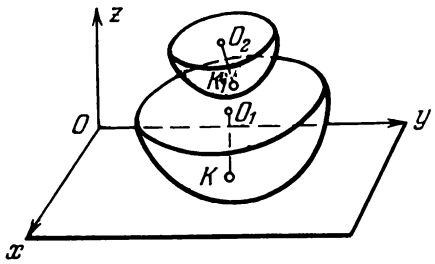


FIGURE 5.13

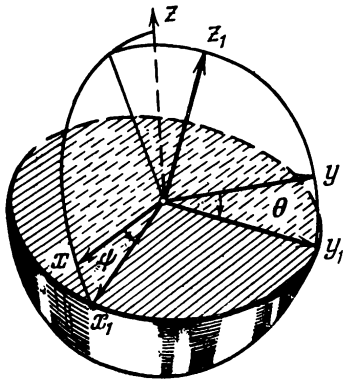


FIGURE 5.14

Thus, we consider the system described above, consisting of two hemispheres. In accordance with Figure 5.13, we introduce three systems of Cartesian coordinates: $Oxyz$, a fixed system Σ whose $z = 0$ plane coincides with the rough horizontal plane on which the lower hemisphere rolls; $O_1x_1y_1z_1$, a moving system Σ_1 with origin at the center of the lower hemisphere, O_1z_1 axis perpendicular to the diametral plane of the lower hemisphere, and O_1x_1 axis always situated in the vertical plane parallel to the xOz plane; $O_2x_2y_2z_2$, a moving system Σ_2 analogous to system Σ_1, but associated with the upper hemisphere. The configuration of the hemispheres is determined by ten generalized coordinates: the Cartesian coordinates x and y of the point K of contact of the lower hemisphere and the horizontal plane; the Résal angles θ and ψ, which fix the position of the Σ_1 axes (Figure 5.14); the angle φ of rotation of the lower hemisphere about the O_1z_1 axis; the Cartesian coordinates x_1 and y_1 of the point K_1 of contact of the hemispheres; the Résal angles θ_1 and ψ_1, which fix the positions of the Σ_2 axes relative to the system Σ; and the angle φ_1 of rotation of the upper hemisphere about the O_2z_2 axis. The cosines of the angles between the axes of the coordinate systems Σ, Σ_1, and Σ_2 are given in the tables:

	x_1	y_1	z_1
x	$\cos\psi$	$\sin\theta\sin\psi$	$\cos\theta\sin\psi$
y	0	$\cos\theta$	$-\sin\theta$
z	$-\sin\psi$	$\sin\theta\cos\psi$	$\cos\theta\cos\psi$

	x_2	y_2	z_2
x	$\cos\psi_1$	$\sin\theta_1\sin\psi_1$	$\cos\theta_1\sin\psi_1$
y	0	$\cos\theta_1$	$-\sin\theta_1$
z	$-\sin\psi_1$	$\sin\theta_1\cos\psi_1$	$\cos\theta_1\cos\psi_1$

	x_2	y_2	z_2
x_1	$\cos(\psi_1 - \psi)$	$\sin\theta_1 \sin(\psi_1 - \psi)$	$\cos\theta_1 \sin(\psi_1 - \psi)$
y_1	$-\sin\theta \sin(\psi_1 - \psi)$	$\sin\theta \sin\theta_1 \cos(\psi_1 - \psi) + \cos\theta \cos\theta_1$	$\sin\theta \cos\theta_1 \cos(\psi_1 - \psi) - \cos\theta \sin\theta_1$
z_1	$-\cos\theta \sin(\psi_1 - \psi)$	$\cos\theta \sin\theta_1 \cos(\psi_1 - \psi) - \sin\theta \cos\theta_1$	$\cos\theta \cos\theta_1 \cos(\psi_1 - \psi) + \sin\theta \sin\theta_1$

The condition of rolling without sliding of the lower hemisphere on the horizontal plane is $v_{O_1} + \omega \times r_{O_1 k} = 0$, where v_{O_1} is the velocity of the point O_1 and ω is the instantaneous angular velocity of the lower hemisphere. Substituting

$$v_{O_1}\{\dot{x}, \dot{y}, 0\}_\Sigma, \quad r_{O_1 k}\{0, 0, -R\}_\Sigma,$$

$$\omega\{\dot{\theta}\cos\psi + \dot{\varphi}\cos\theta\sin\psi, \psi - \dot{\varphi}\sin\theta, \dot{\varphi}\cos\theta\cos\psi - \dot{\theta}\sin\psi\}_\Sigma$$

into this relation, we obtain the equations of the nonholonomic constraints

$$\text{(2.48)} \quad \dot{x} - R\psi + R\dot{\varphi}\sin\theta = 0, \quad \dot{y} + R\dot{\theta}\cos\psi + R\dot{\varphi}\cos\theta\sin\psi = 0.$$

The condition of rolling without sliding of the upper hemisphere on the diametral plane of the lower hemisphere can be expressed by the relation

$$v_{O_2} + \omega_1 \times r_{O_2 k} = v_{O_1} + \omega \times r_{O_1 k_1},$$

where v_{O_2} is the velocity of the point O_2 and ω_1 is the instantaneous angular velocity of the upper hemisphere. Substituting

$$v_{O_2} = v_{O_1} + \dot{r}_{O_1 O_2}, \quad r_{O_1 k_1} = r_{O_1 O_2} + r_{O_2 k_1}$$

into this relation, we obtain

$$\dot{r}_{O_1 O_2} + (\omega_1 - \omega) \times r_{O_1 O_2} - \omega \times r_{O_1 O_2}$$

or, in the Σ_1 system,

$$\tilde{\dot{r}}_{O_1 O_2} + (\omega' - \omega) \times r_{O_1 O_2} + (\omega_1 - \omega) \times r_{O_1 k_1} = 0,$$

where $\tilde{\dot{r}}_{O_1 O_2}$ is the derivative of the vector $r_{O_1 O_2}$ referred to the Σ_1 system rotating with the instantaneous angular velocity ω'. Substituting into this relation

$$\tilde{\dot{r}}_{O_1 O_2}\{\dot{x}_1, \dot{y}_1, 0\}_{\Sigma_1}, \quad \omega\{\dot{\theta}, \psi\cos\theta, \dot{\varphi} - \psi\sin\theta\}_{\Sigma_1},$$
$$(\omega' - \omega)\{0, 0, -\dot{\varphi}\}_{\Sigma_1},$$

$$
\begin{aligned}
&(\omega_1 - \omega)\{\dot\theta_1 \cos(\psi_1 - \psi) + \dot\varphi \cos\theta_1 \sin(\psi_1 - \psi) - \dot\theta,\\
&\qquad - \dot\theta_1 \sin\theta \sin(\psi_1 - \psi) + \dot\psi_1 \cos\theta\\
&+ \dot\varphi_1 [\sin\theta \cos\theta_1 \cos(\psi_1 - \psi) - \cos\theta \sin\theta_1] - \dot\psi \cos\theta,\\
&\qquad - \dot\theta_1 \cos\theta \sin(\psi_1 - \psi) - \dot\psi_1 \sin\theta\\
&+ \dot\varphi_1 [\cos\theta \cos\theta_1 \cos(\psi_1 - \psi) + \sin\theta \sin\theta_1] - \dot\varphi + \dot\psi \sin\theta\}_{\Sigma_1},
\end{aligned}
\tag{2.49}
$$

we obtain two further equations of nonholonomic constraints

$$
\begin{aligned}
&\dot x_1 + y_1\dot\varphi + R_1\dot\psi \cos\theta + R_1\dot\theta_1 \sin\theta \sin(\psi_1 - \psi)\\
&- R_1\dot\psi_1 \cos\theta - R_1\dot\varphi_1 [\sin\theta \cos\theta_1 \cos(\psi_1 - \psi) - \cos\theta \sin\theta_1] = 0,\\
&\dot y_1 - x_1\dot\varphi - R_1\dot\theta + R_1\dot\theta_1 \cos(\psi_1 - \psi) + R_1\dot\varphi_1 \cos\theta_1 \sin(\psi_1 - \psi) = 0
\end{aligned}
\tag{2.50}
$$

An expression for the Lagrange function is

$$
\begin{aligned}
L = {}& \frac{m}{2}(\dot x^2 + \dot y^2) + ml[\dot x(\dot\theta \sin\psi \sin\theta - \dot\psi \cos\psi\cos\theta) + \dot y\dot\theta \cos\theta]\\
&+ \frac{1}{2}(A + ml^2)(\dot\theta^2 + \dot\psi^2 \cos^2\theta) + \frac{1}{2}C(\dot\varphi - \dot\psi \sin\theta)^2\\
&+ \frac{m_1}{2}(v_x^2 + v_y^2 + v_z^2) + \frac{1}{2}A_1(\dot\theta_1^2 + \dot\psi_1^2 \cos^2\theta_1)\\
&+ \frac{1}{2}C_1(\dot\varphi_1 - \dot\psi_1 \sin\theta_1)^2 + mgl \cos\theta \cos\psi
\end{aligned}
$$

where m and m_1 are the masses of the lower and upper hemispheres; C and C_1 are the axial moments of inertia; A and A_1 are the equatorial moments of inertia; l and l_1 are the distances from the center of mass to the equatorial plane of each of the hemispheres; R and R_1 are the radii of the spherical surfaces; and

$$
\begin{aligned}
v_x = {}& \dot x + \dot x_1 \cos\psi + \dot y_1 \sin\theta \sin\psi + \dot\psi(-x_1 \sin\psi + y_1 \sin\theta \cos\psi\\
&+ R_1 \cos\theta \cos\psi) + \dot\theta(y_1 \cos\theta \sin\psi - R_1 \sin\theta \sin\psi) + l_1\dot\theta_1 \sin 0_1 \sin\psi_1\\
&- l_1\dot\psi_1 \cos\theta_1 \cos\psi_1,\\
v_y = {}& \dot y + \dot y_1 \cos\theta - \dot\theta(y_1 \sin\theta + R_1 \cos\theta) + l_1\dot\theta_1 \cos\theta_1,\\
v_z = {}& -\dot x_1 \sin\psi + \dot y_1 \sin\theta \cos\psi - \dot\psi(x_1 \cos\psi + y_1 \sin\theta \sin\psi\\
&+ R_1 \cos\theta \sin\psi) + \dot\theta(y_1 \cos\theta \cos\psi - R_1 \sin\theta \cos\psi)\\
&+ l_1\dot\theta_1 \sin\theta_1 \cos\psi_1 + l_1\dot\psi_1 \cos\theta_1 \sin\psi_1.
\end{aligned}
$$

We introduce the dissipation function

$$
F = \frac{1}{2}[h(\omega_x^2 + \omega_y^2 + \tilde\omega_{x_1}^2 + \tilde\omega_{y_1}^2) + h_1(\omega_z^2 + \tilde\omega_{z_1}^2)],
$$

where h and h_1 are the coefficients of viscous friction of rolling and

pivoting; ω_x, ω_y, and ω_z are the projections of the vector $\boldsymbol{\omega}$ onto the Σ axes; and $\tilde{\omega}_{x_1}$, $\tilde{\omega}_{y_1}$, and $\tilde{\omega}_{z_1}$ are the projections of the vector $\boldsymbol{\omega}_1 - \boldsymbol{\omega}$ onto the Σ_1 axes. Expressing these projections in terms of the generalized coordinates and the velocities and using the expressions (2.48), (2.49), and (2.50), we obtain

$$
\begin{aligned}
F = {} & \frac{1}{2}\frac{h}{R^2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{h}{R_1^2}[(\dot{x}_1 + y_1\dot{\varphi})^2 + (\dot{y}_1 - x_1\dot{\varphi})^2] \\
& + \frac{1}{2}h_1(\dot{\varphi}\cos\psi\cos\theta - \dot{\theta}\sin\psi)^2 \\
& + \frac{1}{2}h_1\{\dot{\theta}_1\cos\theta\sin(\psi_1 - \psi) + \dot{\psi}_1\sin\theta \\
& - \dot{\varphi}_1[\cos\theta\cos\theta_1\cos(\psi_1 - \psi) + \sin\theta\sin\theta_1] + \dot{\varphi} - \dot{\psi}\sin\theta\}^2.
\end{aligned} \tag{2.51}
$$

The equations with undetermined multipliers are

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda_i a_{ik} - \frac{\partial F}{\partial \dot{q}_k} \qquad (k = 1, 2, \ldots, 10), \tag{2.52}
$$

where

$$
\begin{aligned}
& q_1 = x,\ q_2 = y,\ q_3 = x_1,\ q_4 = y_1,\ q_5 = \theta, \\
& q_6 = \psi,\ q_7 = \varphi,\ q_8 = \theta_1,\ q_9 = \psi_1,\ q_{10} = \varphi_1.
\end{aligned}
$$

Eliminating the undetermined multipliers λ_i from equations (2.52), and introducing the notation[8)]

$$
\varepsilon_q \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \frac{\partial F}{\partial \dot{q}_k}
$$

we write the resulting equations in the form

$$
\begin{aligned}
& \varepsilon_0 - \varepsilon_y R\cos\psi + \varepsilon_{y_1}R_1 + (ml\sin\theta + m_1y_1\cos\theta)g\cos\psi = 0, \\
& \varepsilon_\psi + \varepsilon_x R - \varepsilon_{x_1}R_1\cos\theta \\
& \qquad + [ml\cos\theta\sin\psi - m_1(x_1\cos\psi + y_1\sin\theta\sin\psi)]g = 0, \\
& \varepsilon_{0_1} - \varepsilon_{x_1}R_1\sin\theta\sin(\psi_1 - \psi) - \varepsilon_{y_1}R_1\cos(\psi_1 - \psi) \\
& \qquad + (l_1\sin\theta_1 - R_1\sin\theta)m_1g\cos\psi_1 = 0, \\
& \varepsilon_{\psi_1} + \varepsilon_{x_1}R_1\cos\theta + (l_1\cos\theta_1\sin\psi_1 - R_1\cos\theta\sin\psi)m_1g = 0, \\
& \varepsilon_\varphi - \varepsilon_x R\sin\theta - \varepsilon_y R\cos\theta\sin\psi - y_1\varepsilon_{x_1} \\
& \qquad + x_1\varepsilon_{y_1} + (y_1\sin\psi + x_1\sin\theta\cos\psi)m_1g = 0, \\
& \varepsilon_{\varphi_1} + \varepsilon_{x_1}R_1[\sin\theta\cos\theta_1\cos(\psi_1 - \psi) - \cos\theta\sin\theta_1] \\
& \qquad - \varepsilon_{y_1}R_1\cos\theta_1\sin(\psi_1 - \psi) \\
& \qquad - (\sin\theta\cos\theta_1\sin\psi_1 - \cos\theta\sin\theta_1\sin\psi)m_1gR_1 = 0.
\end{aligned} \tag{2.53}
$$

[8)] By the same token, we separate the terms in the differential equations (2.52) which contain derivatives of the generalized coordinates with respect to the time as factors.

We replace the last two equations of the system (2.53) by other equations, which are combinations of these equations and the remaining equations of the system (2.53). For this purpose, we multiply the first equation of (2.53) by $-\cos\theta\sin\psi$, the second by $\sin\theta\cos\psi$, and the penultimate equation by $\cos\psi$, and add the resulting expressions. Proceding in the same way, we multiply the third equation of (2.53) by $-\cos\theta_1\sin\psi_1$, the fourth by $\sin\theta_1\cos\psi_1$, and the last by $\cos\psi_1$ and again sum the resulting equations. As a result, we obtain the equations

$$-\varepsilon_\theta^0\cos\theta\sin\psi+\varepsilon_\psi^0\sin\theta\cos\psi+\varepsilon_\varphi^0\cos\psi=0,$$
$$-\varepsilon_{\theta_1}^0\cos\theta_1\sin\psi_1+\varepsilon_{\psi_1}^0\sin\theta_1\cos\psi_1+\varepsilon_{\varphi_1}^0\cos\psi_1=0,$$

where

$$\varepsilon_\theta^0=\varepsilon_\theta-\varepsilon_y R\cos\psi+\varepsilon_{y_1}R_1,\quad \varepsilon_\psi^0=\varepsilon_\psi+\varepsilon_x R-\varepsilon_{x_1}R_1\cos\theta,$$
$$\varepsilon_{\theta_1}^0=\varepsilon_{\theta_1}-\varepsilon_{x_1}R_1\sin\theta\sin(\psi_1-\psi)-\varepsilon_{y_1}R_1\cos(\psi_1-\psi),$$
$$\varepsilon_{\psi_1}^0=\varepsilon_{\psi_1}+\varepsilon_{x_1}R_1\cos\theta,$$
$$\varepsilon_\varphi^0=\varepsilon_\varphi-\varepsilon_x R\sin\theta-\varepsilon_y R\cos\theta\sin\psi-y_1\varepsilon_{x_1}+x_1\varepsilon_{y_1},$$
$$\varepsilon_{\varphi_1}^0=\varepsilon_{\varphi_1}+\varepsilon_{x_1}R_1[\sin\theta\cos\theta_1\cos(\psi_1-\psi)-\cos\theta\sin\theta_1]$$
$$-\varepsilon_{y_1}R_1\cos\theta_1\sin(\psi_1-\psi).$$

Thus, the equations of motion of the system can be expressed in the form

$$\begin{aligned}
&\varepsilon_\theta^0+(ml\sin\theta+m_1y_1\cos\theta)g\cos\psi=0,\\
&\varepsilon_\psi^0+[ml\cos\theta\sin\psi-m_1(x_1\cos\psi+y_1\sin\theta\sin\psi)]g=0,\\
&\varepsilon_{\theta_1}^0+(l_1\sin\theta_1-R_1\sin\theta)m_1g\cos\psi_1=0,\\
&\varepsilon_{\psi_1}^0+(l_1\cos\theta_1\sin\psi_1-R_1\cos\theta\sin\psi)m_1g=0,\\
&-\varepsilon_\theta^0\cos\theta\sin\psi+\varepsilon_\psi^0\sin\theta\cos\psi+\varepsilon_\varphi^0\cos\psi=0,\\
&-\varepsilon_{\theta_1}^0\cos\theta_1\sin\psi_1+\varepsilon_{\psi_1}^0\sin\theta_1\cos\psi_1+\varepsilon_{\varphi_1}^0\cos\psi_1=0,\\
&\dot x-R\dot\psi+R\dot\varphi\sin\theta=0,\\
&\dot y+R\dot\theta\cos\psi+R\dot\varphi\cos\theta\sin\psi=0,\\
&\dot x_1+y_1\dot\varphi+R_1\dot\psi\cos\theta+R_1\dot\theta_1\sin\theta\sin(\psi_1-\psi)-R_1\dot\psi_1\cos\theta\\
&-R_1\dot\varphi_1[\sin\theta\cos\theta_1\cos(\psi_1-\psi)-\cos\theta\sin\theta_1]=0,\\
&\dot y_1-x_1\dot\varphi-R_1\dot\theta+R_1\dot\theta_1\cos(\psi_1-\psi)\\
&+R_1\dot\varphi_1\cos\theta_1\sin(\psi_1-\psi)=0.
\end{aligned}\tag{2.54}$$

These equations form a complete system for the determination of the ten quantities x, y, x_1, y_1, θ, ψ, θ_1, ψ_1, φ, φ_1 as functions of the time. Assuming that all the derivatives with respect to the time in equations (2.54) vanish, we obtain a system of four independent equations

$$
\begin{aligned}
ml\sin\theta + m_1 y_1\cos\theta &= 0,\\
ml\cos\theta\sin\psi - m_1(x_1\cos\psi + y_1\sin\theta\sin\psi) &= 0,\\
l_1\sin\theta_1 - R_1\sin\theta &= 0,\\
l_1\cos\theta_1\sin\psi_1 - R_1\cos\theta\sin\psi &= 0,
\end{aligned}
\tag{2.55}
$$

which are satisfied by the generalized coordinates in an equilibrium state. It follows that the equilibrium states form a six-dimensional manifold in the 16-dimensional phase space.

We now write down the characteristic equation for the determination of the stability conditions of the surface O_6 of the manifold of equilibrium states. In order to simplify the calculations, we shall consider a small neighborhood of the surface O_6 around the point $x = y = x_1 = y_1 = \varphi = \varphi_1 = 0$, assuming that the coordinates $x, y, \cdots, \varphi$, and φ_1 and also the coefficients of viscous friction of rolling h and pivoting h_1 are small quantities.

We linearize the equations of motion (2.54) in a small neighborhood of the point $x = y = \cdots = \varphi_1 = 0$ of the surface O_6, omitting terms containing products of small quantities in the coefficients of the linearized equations. Using primed quantities to denote the deviations of the generalized coordinates from their equilibrium values, we obtain

$$
\begin{aligned}
a_0\ddot{x}' + \frac{h}{R}\dot{x}' + m_1R\ddot{x}_1' - \frac{h}{R_1}\dot{x}_1' - m_1g x_1' + a_1\ddot{\psi}' \\
- mgl\psi' - m_1l_1R\ddot{\psi}_1' - C\theta\ddot{\varphi}' = 0,\\
-a_0\ddot{y}' - \frac{h}{R}\dot{y}' - m_1R\ddot{y}_1' + \frac{h}{R_1}\dot{y}_1' + m_1gy_1' + a_1\ddot{\theta}' \\
+ mgl\theta' - m_1l_1R\ddot{\theta}_1' = 0,\\
a_2\ddot{x}' + a_2\ddot{x}_1' + \frac{h}{R_1}\dot{x}_1' + a_2R_1\ddot{\psi}' - gm_1R_1\psi' \\
+ a_3\ddot{\psi}_1' + gm_1l_1\psi_1' - C_1\theta_1\ddot{\varphi}_1' = 0,\\
-a_2\ddot{y}' - a_2\ddot{y}_1' - \frac{h}{R_1}\dot{y}_1' + a_2R_1\ddot{\theta}' - gm_1R_1\theta' + a_3\ddot{\theta}_1' + gm_1l_1\theta_1' = 0,\\
-m_1y_1\ddot{x}_1 + m_1x_1\ddot{y}_1' + (a_4 - C)\theta\ddot{\psi}' + m_1l_1y_1\ddot{\psi}_1' - a_4\psi\ddot{\theta}' \\
+ m_1l_1x_1\ddot{\theta}_1' + C\ddot{\varphi}' + 2h_1\dot{\varphi}' - h_1\dot{\varphi}_1' = 0,\\
(A - C)\theta_1\ddot{\psi}_1' - A_1\psi\ddot{\theta}_1' - h_1\dot{\varphi}' + C_1\ddot{\varphi}_1' + h_1\dot{\varphi}_1' = 0,\\
\dot{x}' - R\dot{\psi}' + R\theta\dot{\varphi}' = 0,\\
\dot{y}' + R\dot{\theta}' + R\psi\dot{\varphi}' = 0,\\
\dot{x}_1' + y_1\dot{\varphi}' + R_1\dot{\psi}' - R_1\dot{\psi}_1' + R_1(\theta_1 - \theta)\dot{\varphi}_1' = 0,\\
\dot{y}_1' - x_1\dot{\varphi}' - R_1\dot{\theta}' + R_1\dot{\theta}_1' + R_1(\psi_1 - \psi)\dot{\varphi}_1' = 0,
\end{aligned}
$$

where

$$a_0 = m(R-l) + m_1 R,\ a_1 = A + m_1 R_1 R - ml(R-l),$$
$$a_2 = m_1(R_1 - l_1),\ a_3 = A_1 + m_1 l_1 (R_1 - l_1),\ a_4 = A + ml^2 + mlR_1.$$

Substituting a solution in the form $x' = x_0' e^{pt}$, $x_1' = x_{10}' e^{pt}, \cdots, \varphi_1' = \varphi_{10}' e^{pt}$ into these equations, we obtain the characteristic equation

$$p^6 \begin{vmatrix} Q_1 & Q_2 & . & . & Q_5 & Q_6 & . & . & -C\theta p^2 & . \\ Q_3 & Q_4 & . & . & Q_7 & Q_8 & . & . & . & -C_1\theta_1 p^2 \\ . & . & -Q_1 & -Q_2 & . & . & Q_5 & Q_6 & . & . \\ . & . & -Q_3 & -Q_4 & . & . & Q_7 & Q_8 & . & . \\ . & -m_1 y_1 p & . & m_1 x_1 p & (a_4 - C)\theta p & m_1 l_1 y_1 p & -a_4\psi p & m_1 l_1 x_1 p & Cp + 2h_1 & -h_1 \\ . & . & . & . & . & (A-C)\theta_1 p & . & -A_1\psi_1 p & -h_1 & C_1 p + h_1 \\ 1 & . & . & . & -R & . & . & . & R\theta & . \\ . & 1 & . & . & R_1 & -R_1 & . & . & y_1 & R_1(\theta_1 - \theta) \\ . & . & 1 & . & . & . & R & . & R\psi & . \\ . & . & . & 1 & . & . & -R_1 & R_1 & -x_1 & R_1(\psi_1 - \psi) \end{vmatrix} = 0, \tag{2.56}$$

in which

$$Q_1 = a_0 p^2 + \frac{h}{R} p,\quad Q_2 = m_1 R p^2 - \frac{h}{R_1} p - m_1 g,\quad Q_3 = a_2 p^2,$$
$$Q_4 = a_2 p^2 + \frac{h}{R_1} p,\quad Q_5 = a_1 p^2 + mgl,\quad Q_6 = -m_1 l_1 R p^2,$$
$$Q_7 = a_2 R_1 p^2 - g m_1 R_1,\quad Q_8 = a_3 p^2 + g m_1 l_1.$$

It is immediately clear from the form of the characteristic equation that there are six vanishing roots due to the six dimensions of the manifold of equilibrium states. The remaining roots, which determine the stability of the manifold, satisfy equation (2.56) after the factor p^6 has been omitted. In a sufficiently small neighborhood of the point $x = y = \cdots = \varphi_1 = 0$ of the surface O_6, this equation has the form

$$[b_0 p^4 + b_1 p^2 + b_2 + hp(b_3 p^2 + b_4)]^2 \cdot [CC_1 p^2 + (C + 2C_1) h_1 p + h_1^2] = 0. \tag{2.57}$$

Here, the coefficients b_i are expressed in terms of the physical parameters by means of the relations

$$\begin{aligned} b_0 &= BB_1 + m_1 A_1 R^2, \\ b_1 &= g[B_1(ml + m_1 R_1) + m_1 l_1 (B + m_1 R^2) + \\ &\qquad + 2m_1^2 R R_1 (R_1 - l_1)], \\ b_2 &= m_1 g^2 [mll_1 - m_1 R_1 (R_1 - l_1)], \\ b_3 &= B + 2B_1 + m_1 R^2 + 2m_1 R(R_1 - l_1), \\ b_4 &= g[ml + m_1(2l_1 - R_1)], \\ B &= A + m(R-l)^2,\quad B_1 = A_1 + m_1(R_1 - l_1)^2 \end{aligned} \tag{2.58}$$

and do not depend on the generalized coordinates of the system. Assuming $h = h_1 = 0$ in (2.57), we obtain the case considered by Whittaker. It follows from (2.57) that, in this case, we obtain the characteristic equation

$$p^2(a_0 p^4 + a_1 p^2 + a_2) = 0, \tag{2.59}$$

which has two further vanishing roots; we therefore obtain a critical case of the theory of stability.

If we neglect the factor p^2, the resulting equation

$$a_0 p^4 + a_1 p^2 + a_2 = 0, \tag{2.60}$$

where the coefficients a_0, a_1, and a_2 are calculated for homogeneous hemispheres, coincides with Whittaker's equation given on p. 225 of his book [**502**]. In accordance with (2.60), the condition of (conservative) stability for two homogeneous hemispheres reduces to the inequality

$$a_2 > 0. \tag{2.61}$$

It can be seen that the inequalities (2.61) and (2.47) have the same sign. It should however be noted that it is not completely correct to neglect the two vanishing roots in equation (2.59).

We now return to equation (2.57) for the case when $h \neq 0$ and $h_1 \neq 0$. In accordance with this equation, a sufficiently small neighborhood of the point $x = y = \cdots = \varphi_1 = 0$ of the surface O_6 of equilibrium states is asymptotically stable if the real part of all the roots of the equations

$$\begin{gathered} CC_1 p^2 + (C + 2C_1) h_1 p + h_1^2 = 0, \\ a_0 p^4 + h a_3 p^3 + a_1 p^2 + h a_4 p + a_2 = 0 \end{gathered} \tag{2.62}$$

are negative. For this to be the case, it is necessary that the following inequalities hold:

$$\begin{gathered} h_1 > 0,\ h a_3 > 0,\ a_1 > 0,\ h a_4 > 0, \\ a_2 > 0,\ a_1 a_3 a_4 - a_0 a_4^2 - a_2 a_3^2 > 0, \end{gathered} \tag{2.63}$$

since it is clear from the expressions (2.58) that $a_0 > 0$ always. In the case $R_1 > l_1$ (for a homogeneous hemisphere, this condition is satisfied), we have $a_1 > 0$, $a_3 > 0$. In addition, $a_4 > 0$, if $a_2 > 0$. Therefore, the condition of asymptotic stability reduces to the inequalities

$$\begin{gathered} h > 0,\ h_1 > 0,\ a_2 > 0, \\ a_1 a_3 a_4 - a_0 a_4^2 - a_2 a_3^2 > 0. \end{gathered} \tag{2.64}$$

The last inequality is certainly satisfied, as is easily seen, for $R \to \infty$ or for $R \to 0$ and $m \to 0$. If the last inequality of (2.64) is satisfied, the

introduction of viscous friction ($h > 0$, $h_1 > 0$) is sufficient to transform a conservatively stable system into an asymptotically stable system.

§3. Stability of steady motions of holonomic and nonholonomic systems

The steady motions of holonomic and nonholonomic systems form manifolds of certain dimensions in the phase space and configuration space. In the investigation of the stability of steady motion, we therefore have a situation similar to that described above for the equilibrium states of a nonholonomic system.

Let us consider first the steady motions of a holonomic system. Let $q_1, q_2, \cdots, q_n$ be the generalized coordinates of a holonomic system with the Lagrange function

$$L = L(q_1, q_2, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m, \dot{q}_{m+1}, \ldots, \dot{q}_n),$$

where $m < n$; n is the number of generalized coordinates, of which the last $n - m$ are cyclic.

Let us consider a system with incomplete dissipation, when the Rayleigh function

$$F = \frac{1}{2} \sum_{i,\, j=1}^{m} h_{ij}(q_1, \ldots, q_m)\, \dot{q}_i \dot{q}_j$$

does not contain the velocities corresponding to the cyclic coordinates. The motion of such a system is described by the equations

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial F}{\partial \dot{q}_j} \qquad (j = 1, 2, \ldots, m), \\ &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{m+k}} = 0 \qquad\qquad (k = 1, 2, \ldots, n - m). \end{aligned} \tag{3.1}$$

Introducing the notation $\omega_k = \dot{q}_{m+k}$, we write equations (3.1) in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial F}{\partial \dot{q}_j}, \quad \frac{d}{dt}\frac{\partial L}{\partial \omega_k} = 0. \tag{3.2}$$

Equations (3.2) describe the motion of the representative point in the $n + m$-dimensional phase space Φ, the quantities $q_1, q_2, \cdots, q_m$, $\dot{q}_1$, $\dot{q}_2, \cdots, \dot{q}_m$, $\omega_1, \omega_2, \cdots, \omega_{n-m}$ being plotted along the coordinate axes. By definition, a steady motion is a motion in which the noncyclic coordinates have a constant value, as also the velocities corresponding to the cyclic coordinates. Consequently, in a steady motion, we have

$$q_1 = \text{const}, \ \ldots, \ q_m = \text{const}, \ \dot{q}_1 = \ldots = \dot{q}_m = 0,$$
$$\omega_1 = \text{const}, \ \ldots, \ \omega_{n-m} = \text{const},$$

i.e., a steady motion is represented in the phase space Φ by an equilibrium state. Thus, the problem of the stability of steady motion reduces to the problem of the stability of equilibrium positions in the phase space Φ. The equations of equilibrium

$$\frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, m)$$

form a system of m equations for the n unknowns $q_1^0, \cdots, q_m^0, \omega_1^0, \cdots, \omega_{n-m}^0$. Hence, we immediately obtain in the general case a manifold of steady motions of $n - m$ dimensions. Therefore, in accordance with the results of §2, the characteristic equation of the system contains $n - m$ vanishing roots and the stability of the manifold of steady motions is determined by the remaining $2m$ roots of the characteristic equation. For, introducing the quantities $q_i = q_i^0 + \xi_i^0$, $\omega_k = \omega_k^0 + \eta_k$, writing down the equations of motion for the small deviations ξ_i and η_k from the steady values q_i^0 and ω_k^0, and linearizing equations (3.2) with respect to ξ_i and η_k, we obtain

$$\begin{aligned} &\left(\frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\right)_0 \ddot\xi_i + \left(\frac{\partial^2 F}{\partial \dot q_i \partial \dot q_j} - \frac{\partial^2 L}{\partial q_i \partial \dot q_j} + \frac{\partial^2 L}{\partial \dot q_i \partial q_j}\right)_0 \dot\xi_j \\ &\qquad - \left(\frac{\partial^2 L}{\partial q_i \partial q_j}\right)_0 \xi_j + \left(\frac{\partial^2 L}{\partial \dot q_i \partial \omega_k}\right)_0 \dot\eta_k - \left(\frac{\partial^2 L}{\partial q_i \partial \omega_k}\right)_0 \eta_k = 0, \\ &\left(\frac{\partial^2 L}{\partial \dot q_i \partial \omega_k}\right)_0 \ddot\xi_i + \left(\frac{\partial^2 L}{\partial q_i \partial \omega_k}\right)_0 \dot\xi_i + \left(\frac{\partial^2 L}{\partial \omega_l \partial \omega_k}\right)_0 \dot\eta_l = 0, \end{aligned} \tag{3.3}$$

from which it is clear that the characteristic equation of the system has $n - m$ vanishing roots. The conditions that the real parts of the remaining $2m$ roots of the characteristic equation (3.3) are negative enable one to determine the region of asymptotic stability of the manifold of steady motions and also to study the behavior of the boundaries of the stability region as the values of the physical parameters of the system are varied.

We now consider steady motions of nonholonomic systems, restricting ourselves to the case when the last l coordinates are not contained explicitly in the Lagrange function or the dissipation function. Let $q_1, q_2, \cdots, q_n$ be the generalized coordinates of a nonholonomic system with the Lagrange function

$$L = L(q_1, q_2, \cdots, q_{n-l}, \dot q_1, \dot q_2, \cdots, \dot q_n),$$

where $l < n$ and l is the number of coordinates which, for the sake of brevity, we shall call cyclic coordinates. We shall assume that the nonholonomic constraints can be expressed by equations of the form

$$a_{ij}(q_1, \ldots, q_{n-l})\,\dot{q}_j = 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n), \tag{3.4}$$

where $m < n$. We shall consider a system with incomplete dissipation, when the Rayleigh function

$$F = \frac{1}{2}\sum_{s,\,k=1}^{n-l} h_{sk}(q_1, \ldots, q_{n-l})\,\dot{q}_s\dot{q}_k \tag{3.5}$$

does not contain the velocities corresponding to the cyclic coordinates. The motion of such a system is described by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda_i a_{ik} - \frac{\partial F}{\partial \dot{q}_k} \qquad (k = 1, 2, \ldots, n-l), \tag{3.6}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{n-l+r}} = \lambda_i a_{i,\,n-l+r} \qquad (r = 1, 2, \ldots, l). \tag{3.7}$$

Introducing the quantities $\omega_r = \dot{q}_{n-l+r}$ and using (3.5), we can write the equations of motion (3.6), (3.7), and (3.4) in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + h_{sk}(q_1, \ldots, q_{n-l})\,\dot{q}_s - \frac{\partial L}{\partial q_k} = \lambda_i a_{ik} \tag{3.8}$$

$$(i = 1, 2, \ldots, m;\quad s, k = 1, 2, \ldots, n-l),$$

$$\frac{d}{dt}\frac{\partial L}{\partial \omega_r} = \lambda_i a_{i,\,n-l+r} \qquad (r = 1, 2, \ldots, l), \tag{3.9}$$

$$a_{ik}\dot{q}_k + a_{i,\,n-l+r}\omega_r = 0 \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n-l), \tag{3.10}$$

where $L = L(q_1, \cdots, q_{n-l}, \dot{q}_1, \cdots, \dot{q}_{n-l}, \omega_1, \cdots, \omega_l)$. These equations form a system of $n + m$ equations for the determination of the $n + m$ quantities $q_1, \cdots, q_{n-l}, \omega_1, \cdots, \omega_l, \lambda_1, \cdots, \lambda_m$, as functions of the time. In a steady motion, $q_k = q_k^0 = \text{const}$, $\dot{q}_k = 0$, $\omega_r = \omega_r^0 = \text{const}$. Substituting these relations into (3.8) – (3.10), we obtain the following equations of the steady motions of the nonholonomic system:

$$\frac{\partial L}{\partial q_k} + \lambda_i^0 a_{ik} = 0, \tag{3.11}$$

$$\lambda_i^0 a_{i,\,n-l+r} = 0, \tag{3.12}$$

$$a_{i,\,n-l+r}\omega_r^0 = 0$$

$$(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n-l;\ r = 1, 2, \ldots, l). \tag{3.13}$$

Equations (3.11) – (3.13) form a system of $n + m$ equations for the determination of the $n + m$ quantities $q_1^0, \cdots, q_{n-l}^0, \omega_1^0, \cdots, \omega_l^0, \lambda_1^0, \cdots, \lambda_m^0$. If the equations in the system (3.11) – (3.13) are not all independent, the number of quantities to be determined is greater than the number of equations. In this case, we shall have, in accordance with the foregoing,

a manifold of steady motions. It may be noted that at least one equation of the system (3.11) – (3.13) is not independent. For if each of the equations of (3.12) is multiplied by ω_r^0 $(r = 1, 2, \cdots, l)$, respectively, and the resulting equations are summed, the expression obtained vanishes identically by virtue of equations (3.13). Hence, the number of independent equations of the system (3.11) – (3.13) is at least one less than the total number of equations. Therefore, the states of steady motion of a nonholonomic system form a manifold of dimensions not less than one.

Thus, in the study of the stability of steady states of a nonholonomic system we have a situation analogous to that for the equilibrium states of a nonholonomic system. We can therefore apply the theory developed in §2. We shall now give some examples of the study of steady motions of holonomic and nonholonomic systems in order to illustrate our general considerations.

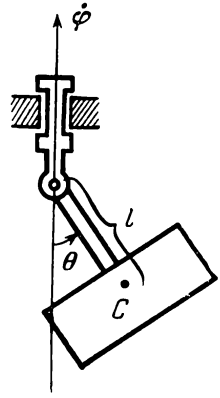


FIGURE 5.15

EXAMPLE 1. *Rotating plane pendulum.* Consider an axisymmetric body suspended from a hinge which is fixed in a freely rotating vertical bearing (Figure 5.15). We shall neglect the friction in the bearing and assume that the friction in the hinge is viscous. We shall study the inertial motion of the system. Let θ and φ be the generalized coordinates of the system; C, the axial moment of inertia of the body; A, the equatorial moment of inertia about the axis of suspension of the body; l, the distance from the axis of the hinge to the center of gravity of the body; m, the mass of the body; and g, the acceleration due to gravity. We now write down an expression for the Lagrange function L and the dissipation function F:

$$L = \frac{1}{2}[A(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + C\dot{\varphi}^2\cos^2\theta] + mgl\cos\theta,$$

$$F = \frac{1}{2}h\dot{\theta}^2,$$

where h is the coefficient of viscous friction in the hinge.

We introduce the dimensionless quantities

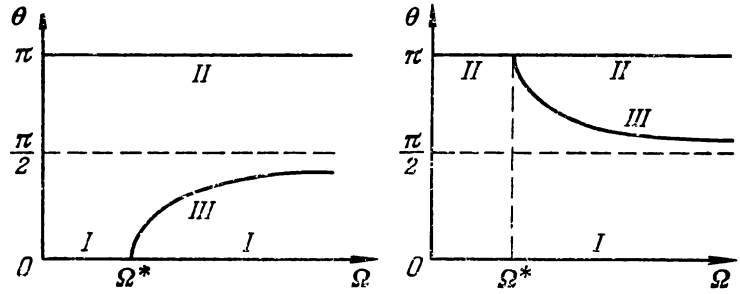
$$\tau = t\sqrt{\frac{mgl}{A}}, \quad \alpha = 1 - \frac{C}{A}, \quad \beta = \frac{h}{\sqrt{Amgl}},$$

where the parameter β is positive and the parameter α may take any values in the interval from -1 to $+1$. If $\omega = d\varphi/d\tau$, the equations of motion of the system can be written in the form

$$\frac{d^2\theta}{d\tau^2} + \beta\frac{d\theta}{d\tau} - \alpha\omega^2 \sin\theta\cos\theta + \sin\theta = 0,$$
$$\frac{d}{d\tau}[\omega(1 - \alpha\cos^2\theta)] = 0. \tag{3.14}$$

The equation of steady motions has the form

$$(\alpha\omega^2\cos\theta - 1)\sin\theta = 0. \tag{3.15}$$



a) $\alpha > 0$ b) $\alpha < 0$

FIGURE 5.16

It follows that the manifold of steady motions is one-dimensional and consists of three branches: I) $\theta = 0$, II) $\theta = \pi$, III) $\alpha\omega^2\cos\theta = 1$. Since $0 \leqq \theta \leqq \pi$ and $0 \leqq \omega^2 < +\infty$, we shall consider the first quadrant of the (Ω, θ) plane, where $\Omega = \omega^2$ (see Figure 5.16). For the investigation of the stability of the branches of steady motions, we write down the characteristic equation of the system

$$p\,[(p^2 + \beta p - \alpha\Omega\cos 2\theta + \cos\theta)(1 - \alpha\cos^2\theta) + \alpha^2\Omega\sin^2 2\theta] = 0. \tag{3.16}$$

The vanishing root of equation (3.16) arises because the manifold of steady motions is one-dimensional. The stability of this manifold is determined by the roots of the equation

$$(p^2 + \beta p - \alpha\Omega\cos 2\theta + \cos\theta)(1 - \alpha\cos^2\theta) + \alpha^2\Omega\sin^2 2\theta = 0. \tag{3.17}$$

Substituting $\theta = 0$ into this equation, we obtain the characteristic equation for branch I:

$$(1 - \alpha)(p^2 - \beta p - \alpha\Omega + 1) = 1.$$

The stability condition for branch I is: $1 - \alpha\Omega > 0$, $\beta > 0$. If $\alpha \leqq 0$, branch I is always stable. If $\alpha > 0$, branch I is stable only in the region $\Omega < 1/\alpha$.

Substituting the value $\theta = \pi$ into (3.17), we obtain the characteristic equation for branch II:

$$(1 - \alpha)(p^2 + \beta p - \alpha\Omega - 1) = 0.$$

The stability condition for branch II is: $\alpha\Omega + 1 < 0$, $\beta > 0$. It follows that if $\alpha \geqq 0$ branch II is instable and, if $\alpha < 0$, branch II is stable only in the region $\Omega > 1/|\alpha|$.

Substituting the relation $\alpha\Omega\cos\theta = 1$ into (3.17), we obtain the characteristic equation for branch III:

$$(1 - \alpha\cos^2\theta)\, p^2 + \beta(1 - \alpha\cos^2\theta)\, p + \frac{1 + 3\alpha\cos^2\theta}{\cos\theta}\sin^2\theta = 0.$$

The stability condition is

$$\frac{1 + 3\alpha\cos^2\theta}{\cos\theta} > 0, \quad \beta > 0.$$

Hence, branch III is always stable if $\alpha > 0$ and is unstable for $\alpha < 0$ if $|\alpha| \leqq 1/3$. For the values $1/3 < |\alpha| < 1$, branch III contains the stability region $1/|\alpha| < \Omega < \sqrt{3/|\alpha|}$. Outside this region, branch III is unstable.

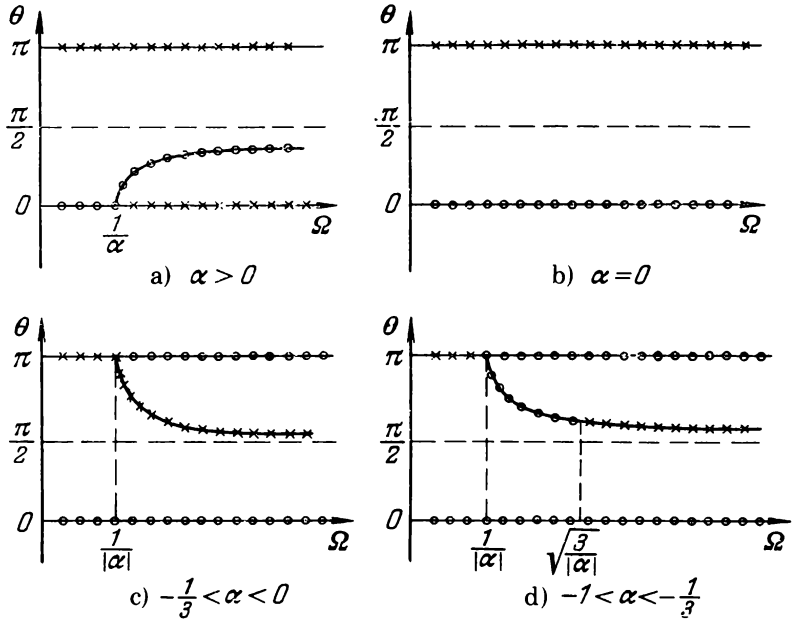


FIGURE 5.17

The results are shown in Figure 5.17, in which the open circles indicate stable steady motions and the crosses unstable motions. If $\beta = 0$, the stability is conservative.

Note. If we assume $\Omega = \text{const}$, we obtain the problem of the stability in the case of forced rotation (for example, by means of a motor of infinite power). In this case, we shall have only the first equation of (3.14), in which the quantity $\omega^2 = \Omega$ is, together with α, a given parameter. The equation of steady motions is still (3.15), but the stability conditions of the branches of steady motions are now somewhat different. The characteristic equation of the system has the form

$$p^2 + \beta p - \alpha\Omega \cos 2\theta + \cos\theta = 0.$$

The stability condition of the branches is

$$-\alpha\Omega \cos 2\theta + \cos\theta > 0, \quad \beta > 0.$$

Hence, the stability conditions for branches I and II remain unchanged, but for branch III we obtain

$$\frac{\sin^2\theta}{\cos\theta} > 0, \ \beta > 0,$$

i.e., if $\alpha > 0$, branch III is stable, as before, but if $\alpha < 0$ it is always unstable. This is the main difference between the dynamics of a system with forced rotation and the dynamics of the same system in free motion.

EXAMPLE 2. *Rotating double plane pendulum.* As a supplement to the foregoing example, consider a system in which there is a larger number of different branches of steady motion. This system is obtained from the foregoing by certain ramifications. Suppose that an axisymmetric body is fixed to a plane hinge rotating in a vertical bearing by means of a weightless rod at whose opposite end is another plane hinge (the plane of the hinges coinciding). Let l be the length of the rod; a, the distance from the axis of the hinge to the center of gravity of the body; C, the axial moment of inertia of the body; A, the equatorial moment of inertia of the body about the axis of the hinge; g, the acceleration due to gravity; φ, ψ, and θ the angles determining the configuration of the system (Figure 5.18). Neglecting the friction in the bearing and taking into account the viscous friction in the hinges, we now write down the equations of motion of the system. The Lagrange function is

$$L = \frac{1}{2}[ml^2\dot{\theta}^2 + 2mal\dot{\theta}\dot{\psi}\cos(\psi - \theta) + A\dot{\psi}^2 + \dot{\varphi}^2(ml^2\sin^2\theta + 2mal\sin\theta\sin\psi + A\sin^2\psi + C\cos^2\psi)] + mg(l\cos\theta + a\cos\psi).$$

Let the dissipation function have the form

$$F = \frac{h}{2}(\dot{\theta}^2 + \dot{\psi}^2).$$

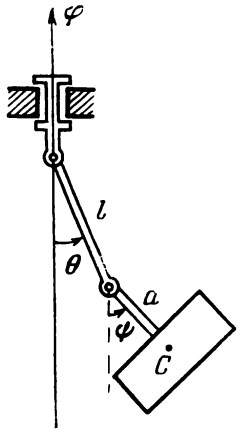


FIGURE 5.18

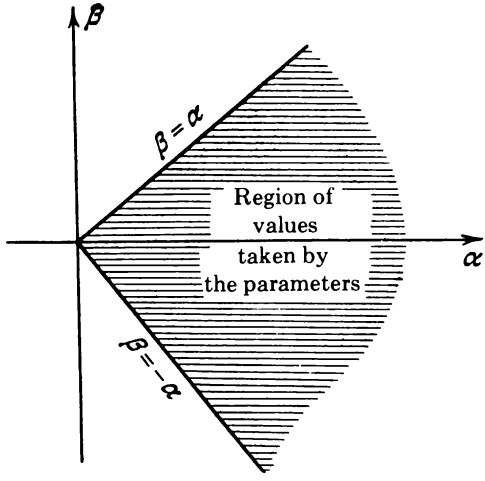


FIGURE 5.19

Then the equations of motion of the system are:

$$
\begin{aligned}
&ml^2\ddot{\theta} + h\dot{\theta} + mal\ddot{\psi}\cos(\psi-\theta) - mgl\dot{\psi}^2\sin(\psi-\theta)\\
&\qquad - ml\dot{\varphi}^2\cos\theta(l\sin\theta + a\sin\psi) + mgl\sin\theta = 0,\\
&A\ddot{\psi} + h\dot{\psi} + mal\ddot{\theta}\cos(\psi-\theta) + mal\dot{\theta}^2\sin(\psi-\theta)\\
&-\dot{\varphi}^2[lma\cos\psi\sin\theta + (A-C)\sin\psi\cos\psi] + mga\sin\psi = 0,\\
&\frac{d}{dt}\{\dot{\varphi}[m(l^2\sin^2\theta + 2al\sin\theta\sin\psi) + A\sin^2\psi + C\cos^2\psi]\} = 0.
\end{aligned}
\tag{3.18}
$$

Assuming $l > 0$ and $a > 0$, we introduce the dimensionless quantities

$$
\tau = t\sqrt{\frac{g}{a}},\quad \alpha = \frac{A}{ma^2},\quad \beta = \frac{A-C}{ma^2},\quad \gamma = \frac{l}{a},\quad \delta = \frac{h}{ml\sqrt{ga}},
$$

where, in accordance with the physical meaning of the parameters of the system, $\gamma \geqq 0$ and $\delta \geqq 0$ and the region of values taken by α and β is shown in Figure 5.19. Writing $\omega = \dot{\varphi}$, we obtain the equations of motion of the system in the dimensionless variables

$$
\begin{aligned}
&\gamma\ddot{\theta} + \delta\dot{\theta} + \ddot{\psi}\cos(\psi-\theta) - \dot{\psi}^2\sin(\psi-\theta)\\
&\qquad - \omega^2(\gamma\sin\theta + \sin\psi)\cos\theta + \sin\theta = 0,\\
&\alpha\ddot{\psi} + \gamma\delta\dot{\psi} + \gamma\ddot{\theta}\cos(\psi-\theta) + \gamma\dot{\theta}^2\sin(\psi-\theta)\\
&\qquad - \omega^2(\gamma\sin\theta + \beta\sin\psi)\cos\psi + \sin\psi = 0,\\
&\frac{d}{d\tau}[\omega(\gamma^2\sin^2\theta + 2\gamma\sin\theta\sin\psi + \alpha - \beta\cos^2\psi)] = 0.
\end{aligned}
\tag{3.19}
$$

Equations (3.19) describe the motion of the representative point in the phase space $(\theta, \dot{\theta}, \psi, \dot{\psi}, \omega)$. The equations of steady motion

(3.20) $\omega^2(\gamma \sin\theta + \sin\psi) = \tan\theta, \quad \omega^2(\gamma \sin\theta + \beta \sin\psi) = \tan\psi$

determine the values of θ, ψ, and ω, which vary in the intervals $0 \leqq \omega^2 < \infty$, $-\pi \leqq \psi \leqq \pi$, $0 \leqq \theta \leqq \pi$, where the planes $\psi = \pi$ and $\psi = -\pi$ must be identified. In accordance with these equations, the manifold of steady motions is one-dimensional and consists of eight branches:

I) $\theta = 0, \psi = 0$,
II) $\theta = 0, \psi = \pm\pi$,
III) $\theta = \pi, \psi = \pm\pi$,
IV) $\theta = \pi, \psi = 0$,
V) Curve (3.20), where $0 < \theta < \pi/2, -\pi/2 < \psi < \pi/2$,
VI) Curve (3.20), where $0 < \theta < \pi/2, \pi/2 < \psi < 3\pi/2$,
VII) Curve (3.20), where $\pi/2 < \theta < \pi, -\pi/2 < \psi < \pi/2$,
VIII) Curve (3.20), where $\pi/2 < \theta < \pi, \pi/2 < \psi < 3\pi/2$.

Branches I – IV are shown in Figure 5.20. The remaining branches are more complicated curves in space and we shall not investigate their behavior. The stability of branches I – VIII can be investigated by the method described previously.

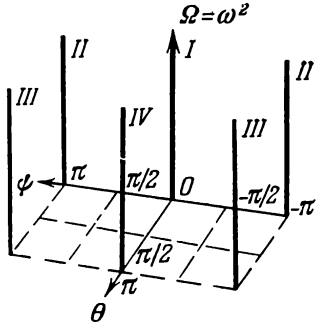


FIGURE 5.30

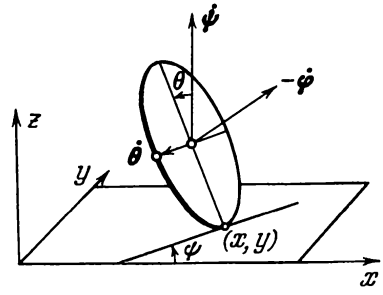


FIGURE 5.21

EXAMPLE 3. *Disc on a horizontal rough plane.* Suppose that a disc of mass m, whose radius is a, rolls without sliding on a horizontal plane. The configuration of this disc is determined by five generalized coordinates: the Cartesian coordinates x and y of the point of contact of the disc and the plane and the three Eulerian angles θ, ψ, and φ (Figure 5.21). The condition of rolling without sliding leads to two equations of nonholonomic constraints

(3.21) $$\dot{x} = a\dot{\varphi}\cos\psi, \quad \dot{y} = a\dot{\varphi}\sin\psi.$$

We now write down an expression for the Lagrange function L without allowance for the equations of the nonholonomic constraints (3.21):

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + ma[\dot{y}(\dot{\theta}\cos\theta\cos\psi - \psi\sin\theta\sin\psi) - \dot{x}(\dot{\theta}\cos\theta\sin\psi + \psi\sin\theta\cos\psi)] + \frac{1}{2}(A + ma^2)\dot{\theta}^2 + \frac{1}{2}(A\cos^2\theta + ma^2\sin^2\theta)\psi^2 + \frac{1}{2}C(\dot{\varphi} - \psi\sin\theta)^2 - mga\cos\theta.$$

Let us consider a system with incomplete dissipation, when the energy dissipation function has the form $F = h\dot{\theta}^2/2$. The equations of motion of the disc are

$$\begin{aligned}
&\frac{d}{dt}[m\dot{x} - ma(\dot{\theta}\cos\theta\sin\psi + \psi\sin\theta\cos\psi)] = \lambda_1,\\
&\frac{d}{dt}[m\dot{y} + ma(\dot{\theta}\cos\theta\cos\psi - \psi\sin\theta\sin\psi)] = \lambda_2,\\
&\frac{d}{dt}[ma(\dot{y}\cos\theta\cos\psi - \dot{x}\cos\theta\sin\psi) + (A + ma^2)\dot{\theta}]\\
&+ ma[\dot{y}(\dot{\theta}\sin\theta\cos\psi + \psi\cos\theta\sin\psi) - \dot{x}(\dot{\theta}\sin\theta\sin\psi\\
&\quad - \psi\cos\theta\cos\psi)] + h\dot{\theta} + \psi^2(A + ma^2)\sin\theta\cos\theta +\\
&\qquad + C\psi\cos\theta(\dot{\varphi} - \psi\sin\theta) - mga\sin\theta = 0,\\
&\frac{d}{dt}[(A\cos^2\theta + ma^2\sin^2\theta)\psi - ma(\dot{y}\sin\theta\sin\psi + \dot{x}\sin\theta\cos\psi)\\
&- C\sin\theta(\dot{\varphi} - \psi\sin\theta)] + ma[\dot{y}(\dot{\theta}\cos\theta\sin\psi + \psi\sin\theta\cos\psi)\\
&\qquad + \dot{x}(\dot{\theta}\cos\theta\cos\psi - \psi\sin\theta\sin\psi)] = 0,\\
&C\frac{d}{dt}(\dot{\varphi} - \psi\sin\theta) + a\lambda_1\cos\psi + a\lambda_2\sin\psi = 0.
\end{aligned} \tag{3.22}$$

Eliminating the quantities $\dot{x}$, $\dot{y}$, λ_1, and λ_2 from the system of equations (3.21) and (3.22) and introducing the notation

$$\Omega = \psi, \quad \omega = \dot{\varphi} - \psi\sin\theta, \tag{3.23}$$

we obtain the equations of motion of the disc in the form

$$\begin{aligned}
&(A + ma^2)\ddot{\theta} + h\ddot{\theta} + (C + ma^2)\Omega\omega\cos\theta\\
&\qquad + A\Omega^2\sin\theta\cos\theta - mga\sin\theta = 0,\\
&[2A\Omega\sin\theta + C\omega]\dot{\theta} - A\dot{\Omega}\cos\theta = 0,\\
&(C + ma^2)\dot{\omega} - ma^2\Omega\dot{\theta}\cos\theta = 0.
\end{aligned} \tag{3.24}$$

Equations (3.24) describe the motion of the representative point in the (θ, Ω, ω) space. A steady motion of the disc is represented in this space by an equilibrium state. It follows from (3.24) that the equilibrium states form the surface

$$(C + ma^2)\Omega\omega + A\Omega^2\sin\theta - mga\tan\theta = 0, \tag{3.25}$$

which is a two-dimensional manifold of steady motions of the disc. In the interval of values $0 \leqq \theta < \pi/2$, the surface (3.25) has the form

shown in Figure 5.22. On this surface, a point of the straight line $\theta = \Omega = 0$ corresponds to rolling of the disc along a straight line, the plane of the disc remaining in the vertical position. Points of the straight line $\theta = \omega = 0$ correspond to spinning of the disc about a fixed vertical diameter. All the remaining points of the surface (3.25) correspond to circular motions of the disc in which the point of contact of the disc and the horizontal plane describes a circle and the quantities θ, Ω, and ω have constant values satisfying the relations (3.25).

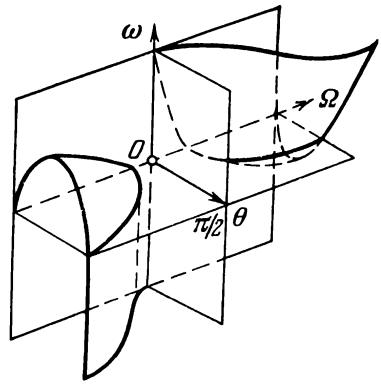


FIGURE 5.22

We now find the conditions of asymptotic stability of the manifold of steady motions (3.25). To this end, we set

$$\theta = \theta_0 + \theta', \quad \Omega = \Omega_0 + \Omega', \quad \omega = \omega_0 + \omega',$$

where θ_0, Ω_0, and ω_0 are the steady values, and linearize the equations of motion (3.24) with respect to the small quantities θ', Ω', and ω'. Omitting the subscript 0 of the quantities θ_0, Ω_0, and ω_0, we obtain

$$\begin{aligned}(A + ma^2)\,\ddot{\theta}' + h\dot{\theta}' - [(C + ma^2)\,\Omega\omega \sin\theta - A\Omega^2 \cos 2\theta \\ + mga\cos\theta]\,\theta' + [(C + ma^2)\,\omega\cos\theta + A\Omega \sin 2\theta]\,\Omega' \\ + (C + ma^2)\,\Omega\cos\theta\omega' = 0,\end{aligned}$$

$$(2A\Omega\sin\theta + C\omega)\,\dot{\theta}' - A\dot{\Omega}'\cos\theta = 0,$$

$$(C + ma^2)\,\dot{\omega}' - ma^2\Omega\dot{\theta}'\cos\theta = 0.$$

The characteristic equation has the form

$$p^2(p^2 + a_1 p + a_2) = 0, \tag{3.26}$$

where

$$a_1 = \frac{h}{A + ma^2}, \quad a_2 = \frac{1}{A + ma^2}\left\{\Omega^2\,[A\,(1 + 2\sin^2\theta + ma^2\cos^2\theta] + \Omega\omega\,(3C + ma^2)\sin\theta + \frac{C}{A}\,(C + ma^2)\,\omega^2 - mga\cos\theta\right\}.$$

The two vanishing roots of the characteristic equation (3.26) are due to the two dimensions of the manifold of steady motions of the disc. The condition of asymptotic stability reduces to the inequalities

$$h > 0, \tag{3.27}$$

$$\Omega^2 [A (1 + 2 \sin^2 \theta) + ma^2 \cos^2 \theta] + \Omega\omega (3C + ma^2) \sin \theta + \frac{C}{A} (C + ma^2) \omega^2 > mga \cos\theta. \tag{3.28}$$

According to (3.27), the coefficient of viscous friction must be positive. The inequality (3.28) enables us to determine the boundary of the stability region

$$\Omega^2 \frac{A (1 + 2 \sin^2 \theta) + ma^2 \cos^2 \theta}{mga \cos \theta} + \Omega\omega \frac{(3C + ma^2) \sin \theta}{mga \cos \theta} + \frac{C (C + ma^2)}{Amga \cos \theta} \omega^2 = 1. \tag{3.29}$$

For a fixed value of θ, we obtain an ellipse for the parameters Ω and ω, the semi-axes of the ellipse tending to zero as $\theta \to \pi/2$. For $\theta = 0$, we have

$$\frac{A + ma^2}{mga} \Omega^2 + \frac{C (C + ma^2)}{Amga} \omega^2 = 1,$$

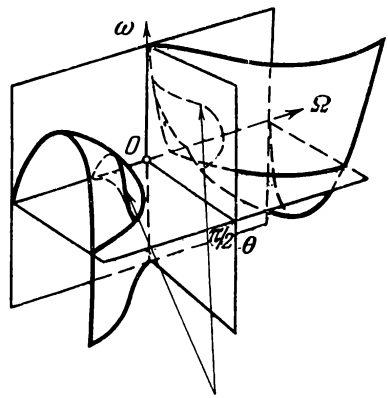
from which it follows that rectilinear rolling of the disc ($\theta = \Omega = 0$) is stable only for the values

$$\omega^2 > \omega_{cr}^2 = \frac{mgaA}{C (C + ma^2)}, \tag{3.30}$$

and spinning of the disc about a fixed vertical diameter ($\theta = \omega = 0$) is stable if

$$\Omega^2 > \Omega_{cr}^2 = \frac{mga}{A + ma^2}. \tag{3.31}$$

The intersection of the surface (3.29) with the surface (3.25) of steady motions determines the boundary of the stability region on the surface (3.25); the boundary is shown in Figure 5.23. In this case, asymptotic stability means that if an asymptotically stable steady motion is perturbed a transient regime of damped oscillations (or an exponentially decaying process) arises in the system resulting in the establishment of a steady motion which, in general, is different from the original motion. For example, a perturbation of rectilinear rolling of a disc leads eventually to a new steady motion in which the point of contact of the disc and the horizontal plane describes a circle of some radius $R \neq \infty$.



Boundary of stability region

FIGURE 5.23

The case $\dot{\varphi} = \omega = \text{const}$. Let us assume that the angular velocity $\dot{\varphi}$ of the rotation of the disc in its own plane remains constant. In this case, the quantity $\dot{\varphi} = \omega$, which occurs in the equations of the nonholonomic constraints and also in the expression for the Lagrange function, is not a coordinate but rather a parameter, together with the other physical parameters of the system. The equations of motion of the disc consist of the system of equations (3.21) and the system (3.22) without the last equation. Eliminating the velocities $\dot{x}$ and $\dot{y}$ from these equations and writing $\Omega = \dot{\psi}$, we obtain the equations of motion of the disc in the form

$$\begin{aligned} &(A + ma^2)\ddot{\theta} + h\dot{\theta} + (C + ma^2)\omega\Omega\cos\theta \\ &\quad - (C + ma^2 - A)\Omega^2 \sin\theta\cos\theta - mga\sin\theta = 0, \\ &\frac{d}{dt}\{[(C + ma^2)\sin^2\theta + A\cos^2\theta]\Omega - C\omega\sin\theta\} = 0. \end{aligned} \tag{3.32}$$

Equations (3.32) describe the motion of the representative point on the (Ω, θ) plane. In accordance with (3.32), the equilibrium states on this plane lie on the curve

$$(C + ma^2)\omega\Omega - (C + ma^2 - A)\Omega^2 \sin\theta - mga\tan\theta = 0. \tag{3.33}$$

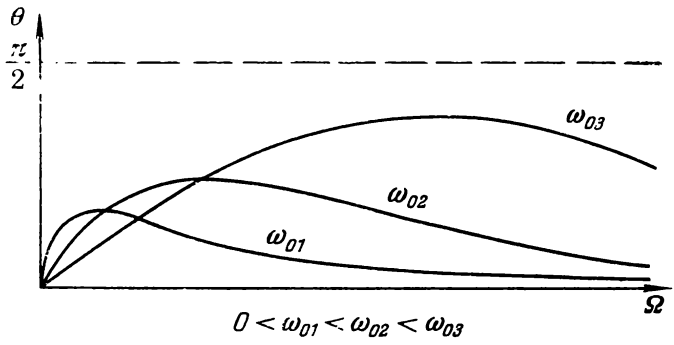


FIGURE 5.24

Thus, the steady motions of the disc form a one-dimensional manifold in this case. In Figure 5.24, we show the curve (3.33) constructed for several given values $\omega > 0$. For negative values of ω, the corresponding curves can be obtained by mirror reflection of the curves of Figure 5.24 with respect to the $\Omega = 0$ axis. We now find the conditions of asymptotic stability of the manifold of steady motions (3.33). To this end, we linearize equations (3.32) with respect to the deviations θ' and Ω' from the steady values θ and Ω:

$$(A + ma^2)\ddot{\theta}' + h\dot{\theta}' - [(C + ma^2)\,\omega\Omega \sin\theta + (C + ma^2 - A)\,\Omega^2 \cos 2\theta + mga \cos\theta]\,\theta' + [(C + ma^2)\,\omega \cos\theta - (C + ma^2 - A)\,\Omega \sin 2\theta]\,\Omega' = 0,$$

$$\frac{d}{dt}\{[(C + ma^2 - A)\,\Omega \sin 2\theta - C\omega \cos\theta]\theta' + [(C + ma^2)\sin^2\theta + A\cos^2\theta]\,\Omega'\} = 0. \tag{3.34}$$

The characteristic equation has the form

$$p(a_0 p^2 + a_1 p + a_2) = 0,$$

where

$$a_0 = (A + ma^2)[(C + ma^2)\sin^2\theta + A\cos^2\theta] > 0,$$

$$a_1 = h\,[(C + ma^2)\sin^2\theta + A\cos^2\theta] > 0,$$

$$a_2 = [(C + ma^2 - A)\,\Omega \sin 2\theta - C\omega \cos\theta][(C + ma^2 - A)\,\Omega \sin 2\theta - (C + ma^2)\,\omega \cos\theta] - [(C + ma^2)\sin^2\theta + A\cos^2\theta] \times [(C + ma^2)\,\omega\Omega \sin\theta + (C + ma^2 - A)\,\Omega^2 \cos 2\theta + mga \cos\theta].$$

The condition of asymptotic stability is

$$h > 0, \quad a_2 > 0. \tag{3.35}$$

It follows that the rectilinear motion $\theta = \Omega = 0$ is stable if

$$\omega^2 > \omega_{cr}^2 = \frac{mgaA}{C(C + ma^2)},$$

which is identical with the condition (3.30) for the stability of rolling of the disc when ω is not fixed.

The case $h = 0$. In this case, the disc is a conservative system. The roots of the characteristic equation (3.26) are, except for the vanishing roots, either real (and the motion of the disc is then unstable) or purely conjugate imaginary. In the second case, it is usually assumed that the system possesses conservative stability. However, from the point of view of the first approximation theory of stability (see §1) the second case is a so-called doubtful Ljapunov case and requires further investigation. This investigation was given in §2 of Chapter II.

CHAPTER VI

DYNAMICS OF NONHOLONOMIC SYSTEMS AND TECHNICAL PROBLEMS OF THE DIRECTIONAL STABILITY OF ROLLING SYSTEMS

§1. Theory of rolling of an elastic pneumatic tire. Equations of motion of vehicles with pneumatically tired wheels

The practical importance of the problem of the directional stability of a bicycle, motorcycle, automobile, or aircraft landing gear has led to the creation of a number of theories of the rolling of a pneumatically tired wheel. These theories range from Rocard's well-known hypothesis of transverse creep or drift to the comprehensive theory of rolling of an elastic pneumatic tire developed by M. V. Keldyš.

These theories are not related to one another and reflect the different approaches that have been taken to the study of the effect of deformability of a pneumatic tire when it rolls without slipping. At the same time, a simple analysis of the equations obtained on the basis of Keldyš's theory enables one not only to establish a connection between these theories but also to arrive at a natural generalization of the hypothesis of drift and also to determine the conditions when it is applicable.

In the study of the effects of deformability of a pneumatic tire on its rolling, certain simplifying assumptions are usually made. These enable one to describe the effect of deformation by a finite number of parameters. In the simplest theory, the deformation of a pneumatic tire is described by a single parameter. This is the basis, for example, of Rocard's theory, which is the expression of the so-called hypothesis of transverse creep.

1. Rocard's theory [387]. The phenomenon of transverse creep or drift with which we shall be concerned in this section can be readily observed in an experiment with an automobile. Suppose that the steering of the automobile is fixed in such a way that the front wheels are turned through a certain constant angle and the automobile is accelerated to a certain velocity. If the velocity of the automobile remains constant, it will describe a certain circle. Now suppose that the accelerator is further depressed; the automobile will describe a circle of larger radius,

although the angle through which the front wheels are turned remains the same. This phenomenon is connected with the transverse creep or drift of pneumatically tired wheels, the actual drift of the automobile depending on the relationship between the transverse drift for the front and rear wheels. It must be emphasized that the phenomenon of transverse creep is observed on a wheel which rolls without slipping, i.e., the effect is in no way connected with slipping of the tire.

Suppose that a transverse force acts on a rolling pneumatically tired wheel. The pneumatic tire then undergoes an elastic deformation (Figure 6.1; view from above). Suppose that it is displaced elastically without the orientation of its central plane being changed. Since the wheel is rolling at the same time, the successive areas of contact follow each other along a line that forms an angle ϵ with the plane of the wheel. In other words, the wheel rolls "askew" in a direction that forms an angle ϵ with its central plane, which coincides with the apparent direction of rolling. Rocard calls this phenomenon of elastic "slipping" in the transverse direction the transverse creep of the wheel or quasi-virage ("envirage" in Rocard's terminology). The angle of creep ϵ due to the transverse deformation of the tire is proportional to the magnitude of the deformation.

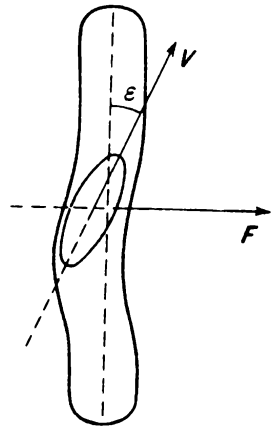


FIGURE 6.1

Suppose a wheel rolls with constant velocity V, its central plane always remaining parallel to the yOz plane. If we denote by F the transverse force acting in the direction of the Ox axis, by x the abscissa of the central plane of the axis and by x_1 the abscissa of the center of the plane of contact of the tire and the road, the equation of creep can be written in the form

$$\dot{x}_1 = \kappa V(x - x_1), \tag{1.1}$$

where κ is the creep coefficient. The relation (1.1) is the equation of a kinematic constraint, which is imposed on a pneumatically tired wheel when it rolls without slipping.

The parameter $\xi = x_1 = x$ characterizes the magnitude of the transverse deformation of the tire. Thus, allowance for the transverse creep of a pneumatically tired wheel leads us to the study of the deformation

of an elastic pneumatic tire. Simplifying the problem, one can idealize the phenomemon of creep in such a way that it remains within the framework of mechanics, i.e., it can be treated in the theory of rolling without slipping of absolutely rigid bodies. The idealization of the phenomenon proposed by Rocard is to assume that we are dealing with an absolutely rigid (nondeformable) wheel and endow it with the property of transverse creep. In such an idealization, the kinematic equation (1.1) disappears but the equations of motion take on a term connected with the presence of a transverse force F, which is proportional to the creep angle ϵ, i.e.,

$$F = -a\varepsilon = -\frac{a}{V}\dot{x}, \tag{1.2}$$

where the coefficient a is called the coefficient of resistance to creep. The hypothesis of creep, which is frequently used in applied problems, consists of using the relation (1.2) when one is deriving the equations of motion of a vehicle on pneumatically tired wheels.

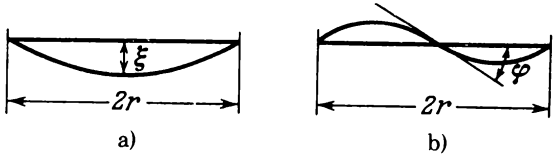


FIGURE 6.2

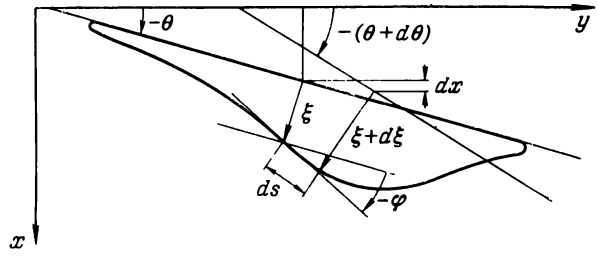


FIGURE 6.3

2. Greidanus's theory [177]. The theory of rolling of a pneumatic tire to which we now turn is more comprehensive than Rocard's theory, since it takes into account the torsional deformation of the tire as well as its transverse deformation.

Consider the rolling of a pneumatically tired wheel whose plane always remains in the vertical position (the plane of the road is horizontal). Let us find a mathematical formulation of the condition of rolling of the tire without slipping. To this end, we shall consider the

central line, which is the intersection of the central plane of the wheel with the outer edge of the undeformed tire. If the rim of the wheel is displaced sideward, the central line changes its form and its projection onto the plane of the road has the form of the curve shown in Figure 6.2a. If the rim of the wheel is turned through a small angle without any lateral displacement occurring, the projection of the central line onto the plane of the road has the form shown in Figure 6.2b. In the general case of deformation of the tire, the projection of the central line onto the horizontal plane has the form shown in Figure 6.3. The nature of the deformation of the tire determines the form of this curve. Greidanus's theory is based on the possibility of gauging the effect of deformation of the tire on the kinematics of its rolling from the form of the curve of Figure 6.3.

We shall call the locus of the successive positions on the road of the center of the area of contact of the tire, the rolling line. If a pneumatically tired wheel rolls without slipping, the projection of the central line of the deformed tire onto the plane of the road must have a common tangent and the same curvature as the rolling line at the center of the area of contact. The condition that the tangents coincide yields, in accordance with the notation of Figure 6.3, the relation

$$dx + (\xi + d\xi)\cos(\theta + d\theta) = -ds\sin(\theta + \varphi) + \xi\cos\theta,$$

which, for small values of ξ, θ, and φ, can be written in the form

$$dx + d\xi = -(\theta + \varphi)\,ds.$$

Substituting $s = Vt$ into this relation, we obtain the equation

$$\dot{x} + \xi + V\theta + V\varphi = 0. \tag{1.3}$$

The condition that the curvatures are equal can be written in the form

$$d(\theta + \varphi) = -ds/R, \tag{1.4}$$

where R is the radius of curvature of the rolling line. In the relation (1.4), the radius of curvature R depends on the magnitude of the transverse deflection ξ of the tire and the angle of twisting φ, i.e.,

$$1/R = f(\xi, \varphi). \tag{1.5}$$

The function $f(\xi, \varphi)$ vanishes for an undeformed tire when $\xi = \varphi = 0$. Therefore, expanding the right-hand side of the relation (1.5) in a Maclaurin series and restricting ourselves to linear terms, we obtain

(1.6) $$1/R = C_1\xi + C_2\varphi,$$

where the coefficients $C_1 = (\partial f/\partial\xi)_0$ and $C_2 = (\partial f/\partial\varphi)_0$ must be assumed given. These coefficients can be determined experimentally.

Substituting the expression (1.6) into the relation (1.4) and making the substitution $s = Vt$, we obtain the equation

(1.7) $$\dot{\theta} + \dot{\varphi} + C_1 V\xi + C_2 V\varphi = 0.$$

Equations (1.3) and (1.7) are equations of kinematic constraints and they impose certain restrictions on the possible motions of a pneumatically tired wheel when it rolls without slipping.

Greidanus's theory can be extended to the case of motions of an inclined wheel. To this end, equation (1.6) must be altered. Suppose that the angle between the plane of the wheel and the plane of the road is $(90° - \chi)$, where χ is a small quantity. The central line of the tire is then projected onto the plane of the road in the form of an ellipse with semi-axes r and $r\chi$, where r is the radius of the wheel. The radius of curvature of the projection at the point of contact of the wheel is r/χ. Greidanus therefore concluded that an additional term $\pm\chi/r$ must be added to the right-hand side of the relation (1.6), the sign being chosen in accordance with the signs of R and χ. If $\chi \neq 0$, the expression (1.6) is therefore replaced by

(1.8) $$1/R = C_1\xi + C_2\varphi \pm \chi/r.$$

This is the gist of Greidanus's theory.

Without going into the details of the other theories and hypotheses dealing with the rolling of an elastic pneumatic tire (of Metelicyn and others), we mention only that the phenomenon of creep which arises during the rolling of a pneumatically tired wheel with a fixed orientation of its central plane has been described in the papers of Ečeistov [149] and Čudakov [118, 119]. They explain this phenomenon by the occurrence of a transverse force and a moment of forces tending to turn the wheel in the direction in which it is inclined. For a small angle χ of inclination of the plane of the wheel, one can assume that the transverse force F and the moment M_χ are proportional to the angle χ, the proportionality coefficients being found experimentally.

We now turn to the theory of a pneumatic tire developed by Keldyš [228]. In this theory, a deformable pneumatic tire is also treated as a discrete model, but it is more consistent and comprehensive than the other similar theories.

3. Keldyš's theory. In Keldyš's theory [228], the deformation of a pneumatic tire is characterized by three parameters, the mass of the deformed part of the tire being assumed to be negligibly small. In addition, the energy dissipation associated with the deformation of the tire is neglected.

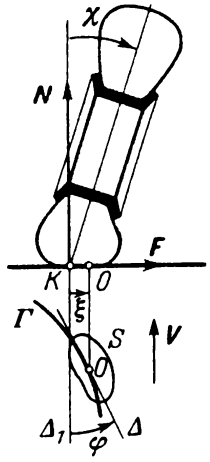


FIGURE 6.4

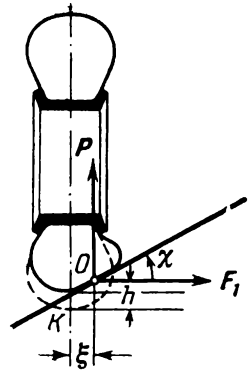


FIGURE 6.5

a) *Forces of reaction arising when a pneumatic tire rolls on a plane.* We shall first consider a pneumatically tired wheel at rest which is subjected to a constant vertical load N (Figure 6.4). Assuming that there is no slipping of the surface of contact of the tire relative to the road surface, we shall consider the small deformation of the tire when there are small displacements of the rim of the wheel.

Such a deformation can be characterized by three parameters ξ, φ, and χ, where ξ is the distance from the line Δ_1 of intersection of the diametral plane of the displaced rim of the wheel and the supporting plane (the road) to the center O of the area of contact; φ is the angle measured from the same line Δ_1 to the central line Δ of the area of contact; and χ is the angle of inclination of the rim of the wheel, measured from the vertical to the diametral plane of the wheel. Since the deformation of the tire is slight, the forces causing the deformation are related linearly to the deformation parameters. In order to express the reaction forces of the supporting plane on the tire in terms of the deformations, we invert the problem and regard the rim of the wheel as fixed and assume that the supporting plane moves (Figure 6.5). The deformation

of the pneumatic tire is then obtained as a result of a displacement of the supporting plane by an amount h in the vertical direction, corresponding to a compressive force N, a displacement ξ in the horizontal direction, and a rotation of the plane through the angles χ and φ about axes passing through the point O in the direction perpendicular to the drawing and in the vertical direction. Denoting the potential energy of deformation of the tire by $U = U(h, \xi, \varphi, \chi)$, we can reduce the forces, referred to the point O of the supporting plane, acting on the area of contact of the tire, to a vertical force P, a horizontal force F_1 and the moments M_1 and L_1 about the axes of the vertical and perpendicular planes of the drawing (Figure 6.5), the moduli of the vectors satisfying

$$P = \frac{\partial U}{\partial h}, \quad F_1 = \frac{\partial U}{\partial \xi}, \quad M_1 = \frac{\partial U}{\partial \varphi}, \quad L_1 = \frac{\partial U}{\partial \chi}.$$

The displacement h is finite and constant to within small second-order quantities in the small angles ξ, φ, and χ. Remembering that for $\xi = \varphi = \chi = 0$ the forces and moments vanish, i.e., $F_1 = M_1 = L_1 = 0$, we can write down the following expressions:

$$\begin{aligned} F_1 &= a_{11}\xi + a_{12}\chi + a_{13}\varphi, \\ L_1 &= a_{21}\xi + a_{22}\chi + a_{23}\varphi, \\ M_1 &= a_{31}\xi + a_{32}\chi + a_{33}\varphi, \end{aligned}$$

where a_{ik} are coefficients that depend on h or the force P and $a_{ik} = a_{ki}$ and $a_{11} > 0$, $a_{22} > 0$, and $a_{33} > 0$. Considering the deformation of the tire for $\varphi = \chi = 0$, we note that the symmetry of the tire implies $M_1 = 0$, and, hence,

$$a_{13} = a_{31} = 0.$$

Similarly, considering the deformation for $\varphi = \xi = 0$, we conclude by symmetry that $M_1 = 0$ and, hence,

$$a_{23} = a_{32} = 0.$$

Thus, the forces of reaction with which the supporting plane acts on the tire are, if referred to the center O of the area of contact of the tire, equivalent to the forces P and F_1 and the moments L_1 and M_1, where the force P is, to within the assumed accuracy, equal to the load on the tire and the quantities F_1, L_1, and M_1 are related to the parameters of the deformation of the tire by the equations

$$M_1 = a_{33}\varphi, \quad F_1 = a_{11}\xi + a_{12}\chi, \quad L_1 = a_{21}\xi + a_{22}\chi,$$

where $a_{12} = a_{21}$.

With a view to subsequent applications, we recalculate the reaction forces and moments acting on the wheel for another point of reference, namely, the point K of intersection of the diameter of greatest inclination of the central plane of the wheel and the supporting plane. We introduce the normal force of reaction $\boldsymbol{N}$ and the tangential force of reaction $\boldsymbol{F}$ of the supporting plane on the tire, the moment of the forces $\boldsymbol{L}$ about the axis perpendicular to the plane of the drawing in Figure 6.6 (the positive direction toward the reader) and the moment of the forces $\boldsymbol{M}$ about the axis normal to the supporting plane (the positive direction upward). In accordance with the notation of Figure 6.6, we have the following relations:

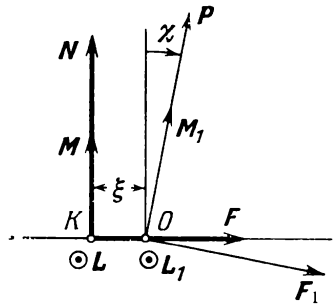


FIGURE 6.6

$$F = F_1 + P\chi, \quad M = M_1, \quad L = L_1 + P\xi.$$

Thus, when the reactions of the supporting plane on the tire are referred to the point K, they reduce to the normal reaction $\boldsymbol{N}$, the transverse force $\boldsymbol{F}$ and the moments $\boldsymbol{L}$ and $\boldsymbol{M}$, which are connected to the deformations of the tire by equations of the form

$$F = a\xi + d\chi, \quad L = d\xi + c\chi, \quad M = b\varphi,$$

in which $a = a_{11}$, $b = a_{33}$, and $d = a_{12} + N$.

For any given wheel, the quantities a, b, c, and d depend on the inner pressure p in the tire and the normal load N. The coefficients a and b remain finite if the normal load tends to zero, the coefficient b increasing with increasing N. This is connected with the fact that when N increases the area of contact of the tire, which remains at rest during the loading of the wheel, increases. The coefficient b increases for the same reason when the pressure p in the tire decreases. The quantities $d\xi$ and $c\chi$ give the moment of the normal reactions about the point K; they must therefore vanish when N vanishes. We assume

$$d = \sigma N, \qquad c = \rho N,$$

where the quantities σ and ρ also depend on N but have finite limits when N tends to zero. In order to establish the dependence of c on N, we consider the displacement of the wheel for $\chi \neq 0$, $\varphi = \xi = 0$. In this case, the moment L arises because, on the deformed tire, the resultant of the reactions of the supporting plane does not pass through the

center O of the undeformed surface of contact, which coincides in this case with the point K (Figure 6.4). The distance from the point K to the point of application of the resultant of the normal reactions of the supporting plane is, for small compressions, determined by the amount to which the tire rolls over to the side, and ρ is approximately equal to ρ_0, where ρ_0 is the radius of the transverse section of the undeformed tire. Therefore, for small values of N, we have $L = \rho N\chi$ $(\rho = \rho_0)$. As regards the coefficient σ, we mention only that it must obviously be positive: $\sigma > 0$. For if the rim of the wheel is displaced sideward by an amount ξ, the center of the resultant of the normal reactions cannot be displaced by a distance greater than ξ in the same direction. Therefore, for $\xi > 0$, the moment L about the point K is positive: $L = \sigma N\xi > 0$. By virtue of what we have said, we obtain finally

$$F = a\xi + \sigma N\chi, \quad L = \sigma N\xi + \rho N\chi, \quad M = b\varphi. \tag{1.9}$$

The exact determination of the constants a, b, σ, and ρ must be made experimentally.

In accordance with (1.9), the potential energy U of the deformed tire is

$$U(\xi, \varphi, \chi) = \frac{1}{2}(a\xi^2 + b\varphi^2 + \rho N\chi^2 + 2\sigma N\xi\chi), \tag{1.10}$$

from which the right-hand sides of the expressions (1.9) can be obtained by simple differentiation:

$$F = \frac{\partial U}{\partial \xi}, \quad L = \frac{\partial U}{\partial \chi}, \quad M = \frac{\partial U}{\partial \varphi}.$$

We introduce the notation $M = M_\theta$ and $L = -M_\chi$, where the positive direction of the moment M is into the drawing (Figure 6.4). Then, the forces and moments acting on the wheel reduce, after they are referred to the point K, to the normal force N, the transverse force F, and the moments M_χ and M_θ about the horizontal axis, whose positive direction forms a right-hand screw with increasing χ, and the vertical axis with its positive direction upward. At the same time,

$$F = \frac{\partial U}{\partial \xi}, \quad M_\chi = -\frac{\partial U}{\partial \chi}, \quad M_\theta = \frac{\partial U}{\partial \varphi},$$

where U is given by (1.10). Hence, we obtain

$$F = a\xi + \sigma N\chi, \quad M_\chi = -\sigma N\xi - \rho N\chi, \quad M_\theta = b\varphi. \tag{1.11}$$

Let us now consider the small deformations of the pneumatic tire of a wheel which rolls with a constant angular velocity and is subjected to a constant vertical load N. In the absence of a resistance to rolling,

the reactions of the supporting plane can again be reduced to a force $\boldsymbol{F}$ perpendicular to the plane of the wheel and moments $\boldsymbol{M}_\theta$ and $\boldsymbol{M}_\chi$ whose vectors lie in this plane. However, for a rolling wheel, the deformations of the tire are determined by the displacement of all the points of the surface of contact relative to the rim of the wheel and depend not only on the system of forces acting on the wheel at a given moment but also on the foregoing motion of the wheel. Strictly speaking, the very form of the surface of contact and the displacement of its individual points cannot be specified by a finite number of parameters. However, in this case too we shall introduce analogous deformation parameters ξ, φ, and χ which determine the deformation of the tire as a whole and associate forces with them by means of the dependence (1.11), neglecting local distortions of the surface of contact.

b) *Geometrical constraints arising during the rolling of a pneumatic tire on a plane.* Consider the material central line of the surface of a pneumatic tire lying in the diametral plane of the rim when the tire is not deformed. We shall call the point O (see Figure 6.4), the center of the section of this line that lies on the surface of contact of the tire, the *center of contact.* The displacement of the central line can be decomposed into the displacement ξ, the average rotation, and a symmetric deformation. We shall denote by φ the angle through which the surface of contact is turned with respect to the straight line Δ_1, the intersection of the diametral plane of the wheel and the supporting plane (Figure 6.4), and we shall call the straight line Δ that forms an angle φ with Δ_1 the axis of the surface of contact. We shall call the locus of the centers O of contact the rolling line of the tire or the load line. Strictly speaking, the assumption that the tire does not slip means that, at every moment of time, we know the entire section Γ of the rolling line which coincides with the section of the central line of the tire lying on the plane of contact. However, it would only be possible to treat the problem with allowance for the absence of slipping of all the points of the surface of contact by taking into account the infinite number of degrees of freedom of the tire. Since we characterize the deformation by means of the three parameters ξ, φ, and χ, the absence of slipping can only be taken into account for the surface of contact as a whole.

If the position of the wheel and the deformation of the tire and, in particular, the surface $S(t)$ of contact are known at a certain instant of time t, this information, together with the given angular velocity of the wheel, uniquely determines the section of the central line of the area of contact after an infinitesimally short interval of time dt has

elapsed (provided there is no slipping). The actual area S of contact also depends on the amount by which the wheel rolls over, which is determined by the displacement velocities of the rim perpendicular to the plane of the wheel. The center of the area of contact of the wheel moves along the tangent to the section of the line Γ and the axis of the surface of contact rotates through a certain angle determined uniquely by the deformation of the tire at the time t. The straight line Δ gives a certain average direction of the tangent to the section Γ at the time t, and we shall replace the requirement that the point O moves along the tangent to Γ by the requirement that the center of contact is displaced along the direction of the straight line Δ. The straight line Δ is then tangential to the rolling line and the angle through which the straight line Δ turns is determined by the curvature of the rolling line. Thus, the absence of slipping of the tire is taken into account by the following two conditions.

1. The tangent to the rolling line of the tire coincides with the axis of the surface of contact.

2. The curvature of the rolling line of the tire is uniquely determined by the deformation parameters ξ, φ, and χ.

We shall assume that the curvature is positive if the convex side of the rolling line points in the direction of positive ξ. Denoting by R the radius of curvature and taking into account the smallness of the deformation, we obtain

$$1/R = \alpha\xi - \beta\varphi - \gamma\chi, \tag{1.12}$$

where it is readily seen that the constants α, β, and γ are positive if the angles are measured in accordance with Figure 6.4.

If the deformation ξ, φ, and χ remains constant during the rolling of the tire, similarity considerations show that the rolling line is a circle whose radius depends on ξ, φ, and χ. This can also be deduced from the relations (1.12). Thus, if $\varphi = \chi = 0$, it is obvious that the center of the circle and the center of contact O are situated on different sides of the rim of the wheel (Figure 6.7a) and, consequently, $\alpha > 0$. If $\xi = \chi = 0$, the circle has the position shown in Figure 6.7b and, therefore, $\beta > 0$. Finally, if $\xi = \varphi = 0$, the position of the rolling line is shown in Figure 6.7c, a value $\gamma > 0$ corresponding to this position.

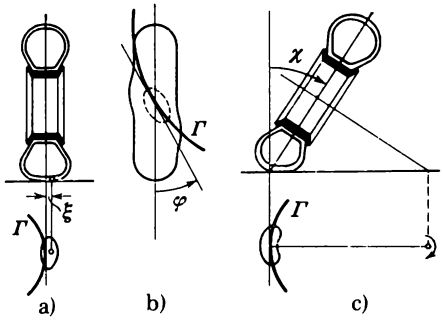


FIGURE 6.7

The order of magnitude of the coefficients α, β, and γ can be estimated by the following approximate consideration. When the tire is deformed, the material central line to which we referred at the beginning of this section is deformed. If $\beta = \gamma = 0$ and $\xi \neq 0$, the deflection of the projection of the central line onto the horizontal plane can be represented approximately as an arc of a circle (Figure 6.2a), from which it follows that $\alpha = 2/r^2$, where r is the radius of the wheel. If $\xi = \gamma = 0$ and $\beta \neq 0$, the projection of the deformed central line onto the horizontal plane can be represented as two arcs of circles (Figure 2.6b), from which it follows that $\beta = 2/r$.

Finally, the coefficient γ can be calculated from geometrical considerations if one takes into account that, in the case of rolling with a constant angle χ, the cone of normals to the section of the central line of the tire at the point O does not slip. Disregarding the compression of the tire, one can easily see from Figure 6.7c that $\gamma = 1/r$.

c) *Kinematic constraints arising when a pneumatic tire rolls on a plane.* The two conditions formulated above for the absence of slipping of a pneumatic tire when it rolls on a plane lead to two equations of kinematic constraints. For, denoting by x_1 the abscissa of the center of the area of contact of the tire and by θ the angle between the direction of the unperturbed motion (the Oy axis) and the path of the central plane of the wheel on the road, we obtain from the first of the two conditions of rolling of the tire without slipping the equation

$$dx_1/ds + \theta + \varphi = 0,$$

where $s = Vt$ is the distance traversed by the wheel during the time t. Further, bearing in mind that the curvature of the rolling line of the wheel is

$$\frac{1}{R} = -\frac{d^2x_1}{ds^2} = \frac{d(\theta + \varphi)}{ds},$$

we can write the second condition of rolling of the tire without slipping in the form

$$\frac{d\theta}{ds} + \frac{d\varphi}{ds} = \alpha\xi - \beta\varphi - \gamma\chi,$$

which follows directly from (1.12). Eliminating the variable s from the two equations obtained, we obtain the equations ($x_1 = x + \xi$)

$$\begin{aligned} \dot{x} + \dot{\xi} + V\theta + V\varphi &= 0, \\ \dot{\theta} + \dot{\varphi} - \alpha V\xi + \beta V\varphi + \gamma V\chi &= 0, \end{aligned} \tag{1.13}$$

which are the equations of the nonintegrable kinematic constraints imposed on the wheel when it rolls on the plane.

4. Equations of motion of a vehicle on pneumatically tired wheels. Using Keldyš's theory of rolling of an elastic pneumatic tire, we can now derive the equations of motion of a vehicle on m wheels with pneumatic tires when the vehicle makes small deviations from rectilinear motion with constant velocity V. We first introduce quantities characterizing the position of the ith wheel ($i = 1, 2, \cdots, m$). Let x_i be the coordinate of the point K_i of intersection of the diameter of greatest slope of the wheel and the xOy plane of the road; χ_i, the angle between the normal to the road and the central plane of the wheel; θ_i, the angle between the Oy axis and the path of the central plane of the wheel on the road; ξ_i, the lateral displacement of the center of the area of contact of the tire with respect to the point K_i; and φ_i, the angle through which the plane of contact is turned with respect to the path of the central plane of the wheel on the road (see Figure 6.4). The value of ξ_i characterizes the deformation of the tire resulting from a lateral displacement of the rim of the wheel and the value of φ_i the torsional deformation of the tire. The quantities χ_i have a dual role: they characterize the position of the rim of the ith wheel and, simultaneously, the deformation of the tire when the wheel is inclined. If we denote by $q_1, q_2, \cdots, q_n$ the generalized coordinates of the system without allowance for the deformation of the pneumatic tires of the wheels, the quantities x_i, θ_i, and χ_i ($i = 1, 2, \cdots, m$) are certain known functions of these coordinates. Since the parameter χ_i can be expressed in terms of $q_1, q_2, \cdots, q_n$, only the two quantities ξ_i and φ_i of the three that determine the position of the deformed part of the tire of the ith wheel are unknown. Thus, the generalized coordinates of a vehicle on m wheels with pneumatic tires are the quantities ξ_i, φ_i, and q_j ($i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$) and there are $n + 2m$ of them in all.

Let us assume that the tires do not slip and that the quantities ξ_i, φ_i, θ_i and χ_i are small. If these conditions are fulfilled, the constraints imposed upon the system are, in accordance with Keldyš's theory,

$$\begin{gathered} \dot{x}_i + \dot{\xi}_i + V\theta_i + V\varphi_i = 0, \\ \dot{\theta}_i + \varphi_i - \alpha_i V\xi_i + \beta_i V\varphi_i + \gamma_i V\chi_i = 0 \quad (i = 1, 2, \ldots, m), \end{gathered} \tag{1.14}$$

where α_i, β_i, and γ_i are constant coefficients.

Assuming that the constraints (1.14) are satisfied, we shall derive the equations of motion of the vehicle on pneumatically tired wheels by two methods. In the first, our starting point will be the principle of relaxation of constraints, which is well known in mechanics; in the second, we shall use the apparatus of the analytical mechanics of nonholonomic systems.

According to the relaxation principle, a mechanical system may be regarded as a free system if the action of the constraints imposed upon it are replaced by the forces of reaction of these constraints. In our case, the motion of the system is subject to the kinematic constraints (1.14) resulting from the rolling of the wheels without slipping. All these constraints are realized by forces of reaction which, referred to the point K_i (for the ith wheel) are equivalent to a normal force N_i, a transverse force F_i and moments M_{θ_i} and M_{χ_i} which, in accordance with (1.11), are given by

$$F_i = \frac{\partial U}{\partial \xi_i}, \quad M_{\chi i} = -\frac{\partial U}{\partial \chi_i}, \quad M_{\theta i} = \frac{\partial U}{\partial \varphi_i}, \tag{1.15}$$

where U is a sum of functions of the form (1.10):

$$U(\xi, \varphi, \chi) = \frac{1}{2}\sum_{i=1}^{m} (a_i \xi_i^2 + b_i \varphi_i^2 + \rho_i N_i \chi_i^2 + 2\sigma_i N_i \xi_i \chi_i). \tag{1.16}$$

We shall neglect the kinetic energy of deformation of the tire. The kinetic energy $T = T(q, \dot{q}, t)$ of the vehicle is then a function of the generalized coordinates $q_1, q_2, \cdots, q_n$, the generalized velocities $\dot{q}_1$, $\dot{q}_2, \cdots, \dot{q}_n$, and the time t. We shall denote by $Q_j = Q_j(q, \dot{q}, t)$ $(j = 1, 2, \cdots, n)$ the generalized forces, calculated with allowance for all the forces in the system. In calculating Q_j, we shall not take into account the forces of the deformation of the tires connected with the angle of inclination χ_i of the wheels, the lateral displacement ξ_i, and the angle φ_i of twisting of the tire.

After we have found the forces Q_j and the expression for the function U, which enable us to determine the forces of deformation of the tires by means of (1.15) and, hence, the reaction forces of the road on the wheels, we know all the forces acting on the system. For the system which is freed from the kinematic constraints (1.14), the latter being replaced by their corresponding reaction forces, the equations of motion can be written in the form of ordinary Lagrange equations of the second kind. Here, it is important to note that after the deformation forces of the ith tire have been referred to the point K_i, whose position is given in terms of the generalized coordinates q_j $(j = 1, 2, \cdots, n)$, the Lagrange equations of the second kind need only be written down for the coordinates q_i, since the equations of the kinematic constraints can be used to express the reaction forces. We shall now find expressions for the generalized forces $R_j(\xi, \varphi, \chi)$ of the reactions of the kinematic constraints due to the deformation of the tires. To this end, we write

down an expression for the virtual work of the forces and the moments (1.15):

$$\delta A = \sum_{i=1}^{m} (F_i \delta x_i + M_{\chi_i} \delta \chi_i + M_{\theta_i} \delta \theta_i)$$
$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{m} \left(\frac{\partial U}{\partial \xi_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial U}{\partial \varphi_i} \frac{\partial \theta_i}{\partial q_j} - \frac{\partial U}{\partial \chi_i} \frac{\partial \chi_i}{\partial q_j} \right) \right] \delta q_j = \sum_{j=1}^{n} R_j \delta q_j.$$

Hence,

$$R_j = \sum_{i=1}^{m} \left(\frac{\partial U}{\partial \xi_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial U}{\partial \varphi_i} \frac{\partial \theta_i}{\partial q_j} - \frac{\partial U}{\partial \chi_i} \frac{\partial \chi_i}{\partial q_j} \right). \tag{1.17}$$

Thus, the dynamical equations for a vehicle on wheels with pneumatic tires have the form

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j + R_j \quad (j = 1, 2, \ldots, n), \tag{1.18}$$

where the generalized forces R_j are given by the expressions (1.17). Since the forces R_j depend on ξ_i and φ_i, equations (1.14) must be added to the system of equations (1.18). We then have a complete system of equations for the determination of the $n + 2m$ quantities q_i, ξ_i, and φ_i as functions of the time.

Equations (1.14) and (1.18) are the desired equations of motion of a vehicle on wheels with pneumatic tires. It should be noted that, in such an approach to the derivation of the equations of motion of the system, equations (1.14) should not be regarded as the equations of the constraints of rolling of an elastic pneumatic tire, but rather as equations that determine the deformation parameters ξ_i and φ_i of the tire and, in terms of these parameters, the forces F_i and the moments M_{θ_i} and M_{χ_i} acting on the tire.

We now come to the second method of deriving the equations of motion.

It is readily seen that the relations (1.14) are equations of ideal nonholonomic constraints, since the relations (1.14) are nonintegrable in the general case because the bilinear covariants

$$d\delta\omega_{ik} - \delta d\omega_{ik} \quad (k = 1, 2)$$

of the Pfaffian forms

$$d\omega_{i1} = dx_i + d\xi_i + V(\theta_i + \varphi_i)\, dt,$$
$$d\omega_{i2} = d\theta_i + d\varphi_i + V(-\alpha_i \xi_i + \beta_i \varphi_i + \gamma_i \chi_i)\, dt$$

do not vanish (see Chapter I, §7). Further, in accordance with (1.14), the virtual variations of the generalized coordinates of the system satisfy the relations

$$\delta x_i + \delta\xi_i = 0, \quad \delta\theta_i + \delta\varphi_i = 0 \quad (i = 1, 2, \ldots, m). \tag{1.19}$$

The reaction forces of the road on the area of contact of the tire and the road do no virtual work. For, in accordance with Keldȳs's theory, these forces are, when referred to the center of the area of contact, equivalent to a resultant transverse force F_i directed along the Ox axis (the automobile moves along the Oy axis), a moment $\boldsymbol{M}_i$ about the axis perpendicular to the plane of the road, and a moment $\boldsymbol{L}_i$ about the longitudinal axis of the automobile. A virtual displacement of the area of contact consists of a translational displacement $\delta x_i' = \delta x_i + \delta\xi_i$ in the direction of the Ox axis and a rotational motion $\delta\vartheta_i = \delta\theta_i + \delta\varphi_i$ about the axis perpendicular to the plane of the road. An expression for the virtual work of the reaction forces of the road is

$$\delta A = \sum_{i=1}^{m} (F_i \delta x_i' + M_i \delta\vartheta_i).$$

Since the relations (1.19) hold, the virtual work of the reaction forces of the constraints vanishes:

$$\delta A = 0.$$

Thus, in writing down the equations of motion of the system, one can exploit the apparatus of the analytical mechanics of nonholonomic systems with ideal constraints.

According to the foregoing, the generalized coordinates of a vehicle on wheels with pneumatic tires are the quantities q_j, ξ_i, and φ_i ($j = 1, 2, \cdots, n$; $i = 1, 2, \cdots, m$). The system has the kinetic energy $T = T(q, \dot{q}, t)$ and potential energy U given by the expression (1.16). We introduce the operator

$$\mathscr{L}_q \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q},$$

where $L = T(q, \dot{q}, t) - U(q, \xi, \varphi)$ is the Lagrange function, and write down the fundamental d'Alembert-Lagrange equation

$$\sum_{j=1}^{n} (\mathscr{L}_{q_j} - Q_j)\,\delta q_j + \sum_{i=1}^{m} (\mathscr{L}_{\xi_i}\delta\xi_i + \mathscr{L}_{\varphi_i}\delta\varphi_i + \mathscr{L}_{\chi_i}\delta\chi_i) = 0. \tag{1.20}$$

In this sum, not all the variations of the generalized coordinates are independent, since they are related by equations (1.19).

Using these equations, we can replace the variations $\delta\xi_i$ and $\delta\varphi_i$ in equation (1.20) by the expressions

$$\delta\xi_i = -\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\delta q_j, \quad \delta\varphi_i = -\sum_{j=1}^{n} \frac{\partial \theta_i}{\partial q_j}\delta q_j.$$

In addition, we have

$$\delta\chi_i = \sum_{j=1}^{n} \frac{\partial \chi_i}{\partial q_j}\delta q_j.$$

Substituting these expressions into (1.20), we obtain

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j - \sum_{i=1}^{m}\left(\frac{\partial U}{\partial \xi_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial U}{\partial \varphi_i}\frac{\partial \theta_i}{\partial q_j} - \frac{\partial U}{\partial \chi_i}\frac{\partial \chi_i}{\partial q_j}\right)\right]\delta q_j = 0.$$

Since the variations δq_j are independent, this sum vanishes only if all the coefficients of the variations vanish, i.e., only if the expressions in the square brackets vanish. Hence, we obtain n equations, identical with equations (1.18).

The systems of equations (1.14) and (1.18) describe the motion of the representative points in the $2(n+m)$-dimensional phase space $(q_j, \dot{q}_j, \xi_i, \varphi_i)$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$.

5. Simplification of the equations of motion. The equations of motion (1.14) and (1.18) can be used to investigate the stability of vehicles on wheels with pneumatic tires. In the investigation of the stability of steady motions of a vehicle on wheels with pneumatic tires, the corresponding linearized equations are of $2(n+m)$th order, where n is the number of generalized coordinates of the vehicle and m is the number of independent wheels. However, even in relatively simple cases, the resulting characteristic equation is of a rather high order. From the point of view of practical applications, it is therefore of interest to discover cases in which the order of the characteristic equation can be reduced by means of certain simplifying assumptions. We shall now discuss cases when the order of the characteristic equation can be reduced by m or $2m$.

a) *The case of motion with high velocity.* Suppose that the velocity V of a vehicle on wheels with pneumatic tires is fairly high.[1] We introduce

[1] At the end of this subsection, we shall give a more precise definition of what is meant by high velocities.

the small parameter $\mu = 1/V$ and write the system of equations (1.14) in the form

$$(1.21)\quad \mu\frac{d\xi_i}{dt} = -\theta_i - \varphi_i - \frac{dx_i}{ds}, \qquad \mu\frac{d\varphi_i}{dt} = \alpha_i\xi_i - \beta_i\varphi_i - \gamma_i\chi_i - \frac{d\theta_i}{ds}.$$

Here, $s = Vt$ is the distance traversed by the vehicle during the time t; dx_i/ds is the angle of creep of the rim of the ith wheel; and $d\theta_i/ds$ is the curvature of the line described by the point K_i of the wheel on the road. Since x_i and θ_i can be expressed in terms of the generalized coordinates $q_1, q_2, \cdots, q_n$ of the system, the derivatives dx_i/ds and $d\theta_i/ds$ are certain functions of the generalized coordinates $q_1, q_2, \cdots, q_n$. Since μ is assumed to be fairly small, equations (1.21) are differential equations with a small coefficient of the highest derivative. As is well known [**10**], the phase space of such a system splits up into two regions when μ is very small: the region of rapid motions and the region of slow motions. It is immediately clear from equations (1.21) that the rapid variables in the phase space $(q_j, \dot{q}_j, \xi_i, \varphi_i)$ are ξ_i and φ_i. In the phase space, the region of slow motions forms a hypersurface G defined by the equations

$$(1.22)\qquad \theta_i + \varphi_i + \frac{dx_i}{ds} = 0, \quad \alpha_i\xi_i - \beta_i\varphi_i - \gamma_i\chi_i - \frac{d\theta_i}{ds} = 0.$$

Equations (1.22) can be obtained from (1.21) for $d\xi_i/dt = d\varphi_i/dt = 0$, and they determine the equilibrium states for the variables ξ_i and φ_i. In order to study the stability of the surface G with respect to the rapid motions, we turn to equations (1.21) and write down the characteristic equation for the rapid variables ξ_i and φ_i, assuming the remaining variables x_i, θ_i, and χ_i are constants:

$$(1.23)\qquad \prod_{i=1}^{m}(p^2 + \beta_i V p + \alpha_i V^2) = 0.$$

All the roots of equations (1.23) have negative real parts and the surface of slow motions G is therefore stable with respect to the rapid motions. Hence, for any initial conditions, the representative point in the phase space $(q_j, \dot{q}_j, \xi_i, \varphi_i)$ will, if the parameter μ is sufficiently small, "jump" onto the surface G, on which the motion will continue in accordance with equations (1.18) and (1.22). Thus, in the case of the motion of a vehicle on wheels with pneumatic tires with a very high velocity $V = \text{const}$, the variation of the variables ξ_i and φ_i can be regarded as rapid compared with the slow variation of the variables q_j and $\dot{q}_j$. Neglecting the time of the transient processes for the vari-

ables ξ_i and φ_i, one can assume that at every instant of time these variables take values satisfying equations (1.22). Eliminating the quantities φ_i from equations (1.22), we obtain the relation

$$\dot{x}_i + \frac{1}{\beta_i}\dot{\theta}_i = -\frac{\alpha_i}{\beta_i}V\xi_i + \frac{\gamma_i}{\beta_i}V\chi_i - V\theta_i \quad (i = 1, 2, \ldots, m), \tag{1.24}$$

which will be discussed below in subsection c.

Using equations (1.22), we can eliminate the quantities ξ_i and φ_i from the expressions (1.15). We then obtain

$$\begin{aligned} F_i &= \frac{\partial U}{\partial \xi_i} = \frac{a_{1i}}{V}\dot{\theta}_i - a_{2i}\theta_i - \frac{a_{2i}}{V}\dot{x}_i + a_{3i}\chi_i, \\ M_{\theta i} &= \frac{\partial U}{\partial \varphi_i} = -\frac{b_i}{V}\dot{x}_i - b_i\theta_i, \\ M_{\chi i} &= -\frac{\partial U}{\partial \chi_i} = \frac{b_{1i}}{V}\dot{\theta}_i + b_{2i}\theta_i + \frac{b_{2i}}{V}\dot{x}_i - b_{3i}\chi_i, \end{aligned} \tag{1.25}$$

where the positive coefficients a_{ki} and b_{ki} are related to Keldyš's parameters of the tire by the equations

$$\begin{aligned} a_{1i} &= \frac{a_i}{\alpha_i}, \quad a_{2i} = \frac{a_i\beta_i}{\alpha_i}, \quad a_{3i} = \frac{a_i\gamma_i}{\alpha_i} + \sigma_i N_i, \\ b_{1i} &= \frac{\sigma_i N_i}{\alpha_i}, \quad b_{2i} = \frac{\sigma_i N_i \beta_i}{\alpha_i}, \quad b_{3i} = N_i\left(\frac{\sigma_i\gamma_i}{\alpha_i} + \rho_i\right). \end{aligned} \tag{1.26}$$

These coefficients can be determined directly in an experiment.

Thus, in the case considered, the motion of the vehicle is described by equations (1.18), in which the generalized forces R_j can be calculated by means of the expressions (1.25). It is readily seen that we obtain a characteristic equation of $2n$th order. Consequently, compared with the original system, the order of the characteristic equation has been reduced by $2m$. We should like to point out that in this case the problem of deriving the equations of motion is considerably simplified and reduces to writing down Lagrange equations of the second kind (1.18).

To conclude, we should like to point out that equation (1.24) enables one to establish a relation that is important in applications. As is well known from the practice of automobile construction, the front wheels of an automobile are usually arranged with a slight outward inclination, i.e., with a so-called splay. This is in order to create an axial force which presses the hub of the wheel toward the inner bearing. However, according to our foregoing analysis, an inclination of the wheels leads to the appearance of creep forces during the motion of the automobile.

It follows from equations (1.24) that these forces can be eliminated by aligning the wheels with an appropriate "convergence". For, substituting into (1.24) the values $\dot{\theta} = \dot{x} = \xi = 0$ (which express the condition for the absence of creep and transverse deformation of the tire due to the presence of the transverse reaction forces of the road), we obtain the relation

$$\theta = \frac{\gamma}{\beta}\chi,$$

which determines the angle θ of "convergence" of the wheels as a function of the angle χ of their "splay".

b) *The case of large values of the kinematic parameters.* Suppose that the kinematic parameters α_i, β_i, and γ_i of the tires are fairly large. This case can be realized if the tires are fairly rigid. We introduce a small parameter μ such that

$$\mu\alpha_i = \alpha_i^0, \quad \mu\beta_i = \beta_i^0, \quad \mu\gamma_i = \gamma_i^0,$$

where α_i^0, β_i^0, and γ_i^0 are finite quantities, and write the second group of equations (1.14) in the form

$$\mu(\dot{\theta}_i + \varphi_i) = V\alpha_i^0\xi_i - V\beta_i^0\varphi_i - V\gamma_i^0\chi_i \qquad (i = 1, 2, \ldots, m). \tag{1.27}$$

If μ is fairly small, we again obtain a system of differential equations in which the derivative has a small coefficient. In the present case, the rapidly varying quantities in the phase space $(q_j, \dot{q}_j, \xi_i, \varphi_i)$ are the sums $\theta_i + \varphi_i$. As $\mu \to 0$, we obtain a surface of slow motions G_1 in the space $(q_j, \dot{q}_j, \xi_i, \varphi_i)$; the surface is defined by the equations $\alpha_i^0\xi_i - \beta_i^0\varphi_i - \gamma_i^0\chi_i = 0$ or, equivalently,

$$\alpha_i\xi_i - \beta_i\varphi_i - \gamma_i\chi_i = 0 \qquad (i = 1, 2, \ldots, m). \tag{1.28}$$

In order to determine the stability of the surface G_1 with respect to the rapid motions, we consider equations (1.27) and write down the characteristic equation for the "rapid" variables $(\theta_i + \varphi_i)$:

$$\prod_{i=1}^{m} (p + \beta_i V) = 0. \tag{1.29}$$

Since all the roots of equation (1.29) are negative ($V > 0$), the surface G_1 is stable with respect to the rapid motions. Consequently, for all initial values, the representative point in the phase space $(q_j, \dot{q}_j, \xi_i, \varphi_i)$ "jumps" onto the surface of slow motions, on which it then continues to move in accordance with equations (1.18), (1.28), and the first group of equations (1.14).

Using the relations (1.28), we eliminate the variables φ_i $(i = 1, 2, \cdots, m)$ from the remaining equations. As a result, the first group of equations (1.14) takes the form

$$\dot{x}_i + \xi_i + V\theta_i + V\frac{\alpha_i}{\beta_i}\xi_i - V\frac{\gamma_i}{\beta_i}\chi_i = 0 \quad (i = 1, 2, \ldots, m), \tag{1.30}$$

and the forces and moments acting on the ith wheel are determined by the equations

$$\begin{aligned} F_i &= a_i\xi_i + \sigma_i N_i\chi_i, \\ M_{\theta i} &= \varkappa_{1i} b_i\xi_i - \varkappa_{2i} b_i\chi_i \quad \left(\varkappa_{1i} = \frac{\alpha_i}{\beta_i}\right), \\ M_{\chi i} &= -\sigma_i N_i\xi_i - \rho_i N_i\chi_i \quad \left(\varkappa_{2i} = \frac{\gamma_i}{\beta_i}\right). \end{aligned} \tag{1.31}$$

Thus, in the case of fairly large values of the kinematic parameters of the tires, the motion of the vehicle is described by equations (1.30) and (1.18), in which the generalized forces R_j can be calculated by means of the expressions (1.31). In this case, we arrive at a characteristic equation of $(2n + m)$th order. Hence, compared with the original system, the order of the characteristic equations is reduced by m.

c) *Generalization of the hypothesis of creep and the condition of its applicability.* We now wish to show that the special cases considered above, in which the equations of motion of a vehicle on wheels with pneumatic tires are simplified, contain a generalization of the theory and hypothesis of creep of Rocard, Čudakov, Ečeistov, and others. To this end, we consider equations (1.24) and the expressions for the forces and the moments (1.25) obtained in the case of a fairly high velocity of the vehicle, and also equations (1.30) and the expressions for the forces and the moments (1.31) obtained in the case of fairly large values of the kinematic parameters of the tires. It is readily seen that in the special case $\theta_i = \chi_i \equiv 0$ equations (1.30) are identical with the kinematic equations (1.1) proposed by Rocard (it should be recalled that $x_1 = x + \xi$). It follows from the derivation of these equations that the creep coefficient κ_1 is related to Keldyš's kinematic parameters by the simple dependence $\kappa_1 = \alpha/\beta$. In the special case $\xi_i = \theta_i \equiv 0$, $\chi_i = \text{const} \neq 0$, equations (1.30) and, accordingly, (1.24) describe the phenomenon of creep studied in the papers of Ečeistov and Čudakov. In the special case under consideration, it follows from equations (1.24) and (1.30) that this phenomenon is characterized by a creep coefficient $\kappa_2 = \gamma/\beta$. Finally, in the special case when there is no deformation of the tire

($\xi_i = \varphi_i = 0$ and $\gamma_i = 0$), equations (1.24) and (1.30) are transformed into the equations of the nonholonomic constraints which hold when an absolutely rigid wheel rolls.

Rocard formulated his hypothesis of creep for an elastic wheel in the case $\chi = 0$. Consequently, Rocard's hypothesis of creep can only be used to study the motion of vehicles for which the axes of rotation of the wheels are always parallel to the plane of the road. For this reason, the applications of Rocard's hypothesis are rather restricted. For example, it does not encompass the motions of bicycles and motorcycles or the nose wheel of an aircraft landing gear in the case of an elastic strut, the front wheels of an automobile when allowance is made for the inclination of the axes of the kingpins, etc.

The foregoing analysis enables one to study all these cases. In addition, the expressions for the forces and moments (1.31) and (1.25) also generalize and make more precise the previously proposed formulas. Therefore, equations (1.24) and (1.34) and, accordingly, the expressions (1.25) and (1.31) are a generalization of the hypothesis of creep.

It follows from what we have said that all the hypotheses of creep can be obtained from Keldyš's theory if the transient processes in the tire are neglected. In the case of the motion of a vehicle with a fairly high velocity, the time τ_i of the transient processes in the ith tire is determined in accordance with the characteristic equation (1.23) by the expression

$$\tau_i = \operatorname{Re}\left\{2\left[\beta_i V\left(1 + \sqrt{1 - 4\frac{\alpha_i}{\beta_i}}\right)\right]^{-1}\right\}, \tag{1.32}$$

and, in the case of fairly large kinematic parameters α_i, β_i, and γ_i, by the expression

$$\tau_i = (\beta_i V)^{-1}. \tag{1.33}$$

Therefore, the condition for the applicability of one or other form of the generalized hypothesis of creep is

$$\tau \gg \tau_i \quad (i = 1, 2, \cdots, m), \tag{1.34}$$

where τ is the shortest time of the transient processes for the variables $q_1, q_2, \cdots, q_n$ and τ_i is determined by the expression (1.32) or, respectively, by (1.33). We should mention that if the inequality (1.34) is not satisfied for all $i = 1, 2, \cdots, m$, the simplified equations of motion can only be written down for those values of i for which the inequality (1.34) does hold.

§2. Stability of a bicycle and a motorcycle

The problem of the stability of a bicycle or a motorcycle is of great importance and it is therefore not surprising that it has attracted the attention of many authors, who have regarded the bicycle as a classical example of a nonholonomic system. In investigating the stability of rectilinear motion of a bicycle or motorcycle, it has been customary to consider a model with absolutely rigid wheels. On the assumption that the fork of the front wheel is vertical (zero stagger), such a treatment leads to the so-called elementary theory of a bicycle. In more recent investigations, the equations of the elementary theory of a bicycle have been used to study the stability of an automatically controlled bicycle.

In the present section, we shall discuss questions of the stability of rectilinear motion of a bicycle on rigid disc-like and toroidal wheels and also on wheels with pneumatic tires. We shall consider the kinematics of rolling of a bicycle, and we shall derive the equations of motion for different models of a bicycle and also study the dependence of the stability of a controlled and an uncontrolled bicycle on the relationships between the physical parameters. We shall also study the effects that lateral displacements of the rider have on the directional stability of a bicycle in different cases, for example, when the rider changes the inclination of the frame, the direction of the handlebars, the rate of change of the inclination of the frame or the direction of the handlebars etc.

Since bicycles and motorcycles are, from the dynamical point of view, machines of the same kind, we shall, in what follows, always speak of a bicycle although the treatment will also apply to a motorcycle.

1. Kinematics of a bicycle. Everyone knows how a bicycle is constructed (Figure 6.8).

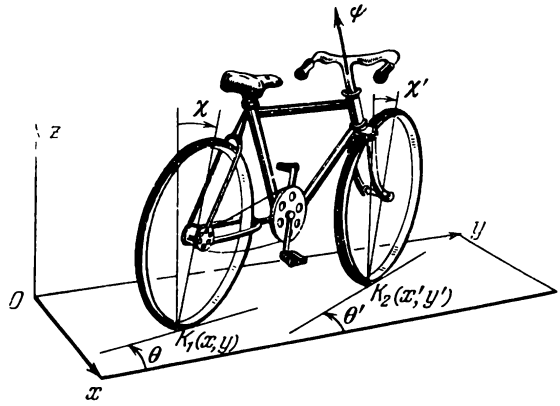


FIGURE 6.8

We shall assume that the rider is rigidly fixed to the bicycle and that the motion of his legs and the pedal and chain mechanism can be neglected.[2] If the elasticity of the tires is neglected, the idealized system obtained as a result of the above simplifications consists of four kinematically connected rigid bodies: the frame with the rider, the two wheels, and the fork of the front wheel.[3]

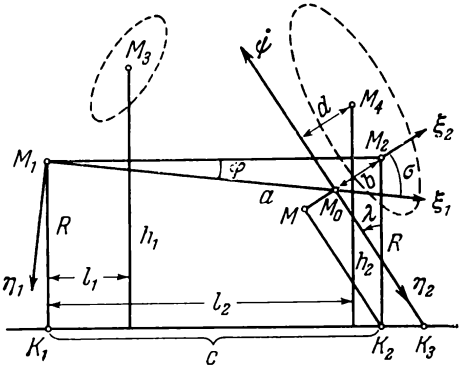


FIGURE 6.9

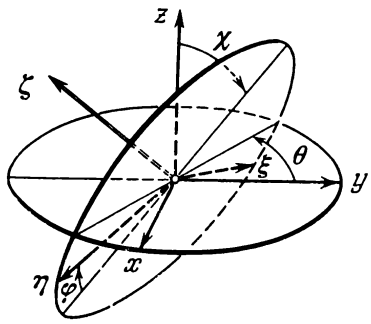


FIGURE 6.10

Disregarding for the time being the rotation of the wheels, which we shall first regard as thin rigid discs, in their own plane, we introduce generalized coordinates of the bicycle: the Cartesian coordinates x and y of the point of contact of the rear wheel and the road, which is assumed to be a horizontal plane; the angle θ between the direction of the plane of the frame and a fixed direction on the road (the Oy axis); the angle χ of inclination of the frame; and the angle ψ through which the fork of the front wheel is turned (Figure 6.8). In addition, we introduce the auxiliary variables x', y', θ', χ', and φ (see Figures 6.8 and 6.9). These variables are related to the generalized coordinates by equations that can easily be found. To this end, we introduce three frames of reference: a fixed system of coordinates $\Sigma(Oxyz)$ with the xOy plane coinciding with the plane of the road and Oz axis vertically upward; a moving system of coordinates $\Sigma_1(M_1\xi_1\eta_1\zeta_1)$ fixed to the frame with the origin at the center of the rear wheel; and a moving frame $\Sigma_2(M_0\xi_2\eta_2\zeta_2)$ attached to the fork of the front wheel with origin at the point M_0, which is the point of intersection of the steering column and the perpendicular onto the latter from the center

[2] In the case of a motorcycle, we shall accordingly neglect the motions connected with the operation of the motor and the mechanism that transfers the turning moment from the motor to the rear wheel.

[3] The mass of the fork of a bicycle is usually neglected; however, in the case of a motorcycle, the mass of the fork may be comparable with the mass of the front wheel, particularly if one takes into account the headlight, handlebars, shield, and brake mechanism attached to it.

M_2 of the front wheel. The coordinates of the point $M_1(x_1, y_1, z_1)_\Sigma$ can be expressed by the relations

$$x_1 = x + R\sin\chi\cos\theta,\quad y_1 = y + R\sin\chi\sin\theta,\quad z_1 = R\cos\chi, \tag{2.1}$$

where R is the radius of the wheels of the bicycle. Introducing the Eulerian angles θ, φ, and χ (Figure 6.10), we can express the cosines of the angles between the axes of the Σ and Σ_1 systems by means of the following table:

(2.2)

	x	y	z
ξ_1	$-\cos\varphi\sin\theta -$	$\cos\varphi\cos\theta -$	$-\sin\varphi\cos\chi$
	$-\sin\varphi\sin\chi\cos\theta$	$-\sin\varphi\sin\chi\sin\theta$	
η_1	$\sin\varphi\sin\theta -$	$-\sin\varphi\cos\theta -$	$-\cos\varphi\cos\chi$
	$-\cos\varphi\sin\chi\cos\theta$	$-\cos\varphi\sin\chi\sin\theta$	
ζ_1	$-\cos\chi\cos\theta$	$-\cos\chi\sin\theta$	$\sin\chi$

We now find the coordinates of the point $M_0(x_0, y_0, z_0)_\Sigma$. In the Σ_1 frame, we have $M_0(a, 0, 0)_{\Sigma_1}$. Hence, in accordance with (2.1) and (2.2), we find

$$\begin{aligned} x_0 &= x + R\sin\chi\cos\theta - a(\cos\varphi\sin\theta + \sin\varphi\sin\chi\cos\theta),\\ y_0 &= y + R\sin\chi\sin\theta + a(\cos\varphi\cos\theta - \sin\varphi\sin\chi\sin\theta),\\ z_0 &= R\cos\chi - a\sin\varphi\cos\chi. \end{aligned} \tag{2.3}$$

The cosines of the angles between the axes of the Σ_2 and Σ systems are given in the following table:

(2.4)

	x	y	z
ξ_2	$-\cos\psi[\cos(\varphi-\sigma)\sin\theta +$	$\cos\psi[\cos(\varphi-\sigma)\cos\theta -$	$-\cos\psi\sin(\varphi-\sigma)\cos\chi +$
	$+\sin(\varphi-\sigma)\sin\chi\cos\theta] -$	$-\sin(\varphi-\sigma)\sin\chi\sin\theta] -$	$+\sin\psi\sin\chi$
	$-\sin\psi\cos\chi\cos\theta$	$-\sin\psi\cos\chi\sin\theta$	
η_2	$\sin(\varphi-\sigma)\sin\theta -$	$-\sin(\varphi-\sigma)\cos\theta -$	$-\cos(\varphi-\sigma)\cos\chi$
	$-\cos(\varphi-\sigma)\sin\chi\cos\theta$	$-\cos(\varphi-\sigma)\sin\chi\sin\theta$	
ζ_2	$\sin\psi[\cos(\varphi-\sigma)\sin\theta +$	$-\sin\psi[\cos(\varphi-\sigma)\cos\theta -$	$\sin\psi\sin(\varphi-\sigma)\cos\chi +$
	$+\sin(\varphi-\sigma)\sin\chi\cos\theta] -$	$-\sin(\varphi-\sigma)\sin\chi\sin\theta] -$	$+\cos\psi\sin\chi$
	$-\cos\psi\cos\chi\cos\theta$	$-\cos\psi\cos\chi\sin\theta$	

The coordinates of the point $M_2(x_2, y_2, z_2)_\Sigma$ in the system Σ_2 are equal to $M_2(b, 0, 0)_{\Sigma_2}$. Hence, in accordance with (2.3) and (2.4), we obtain

$$
\begin{aligned}
x_2 &= x + R\sin\chi\cos\theta - a(\cos\varphi\sin\theta + \sin\varphi\sin\chi\cos\theta) \\
&\quad - b\{\cos\psi[\cos(\varphi-\sigma)\sin\theta + \sin(\varphi-\sigma)\sin\chi\cos\theta] \\
&\qquad + \sin\psi\cos\chi\cos\theta\}, \\
y_2 &= y + R\sin\chi\sin\theta + a(\cos\varphi\cos\theta - \sin\varphi\sin\chi\sin\theta) \\
&\quad + b\{\cos\psi[\cos(\varphi-\sigma)\cos\theta - \sin(\varphi-\sigma)\sin\chi\sin\theta] \\
&\qquad - \sin\psi\cos\chi\sin\theta\}, \\
z_2 &= R\cos\chi - a\sin\varphi\cos\chi - \\
&\quad - b[\cos\psi\sin(\varphi-\sigma)\cos\chi - \sin\psi\sin\chi].
\end{aligned} \tag{2.5}
$$

We now determine the position of the point M_2 by means of the auxiliary variables x', y', θ', and χ' (see Figure 6.8). In accordance with Figure 6.11, the table of the cosines of the angles between the axes of the $\Sigma'\colon M_2\xi'\eta'\zeta'$ system and the Σ system has the form

(2.6)

	x	y	z
ξ'	$-\sin\theta'$	$\cos\theta'$	0
η'	$-\sin\chi'\cos\theta'$	$-\sin\chi'\sin\theta'$	$-\cos\chi'$
ζ'	$-\cos\chi'\cos\theta'$	$-\cos\chi'\sin\theta'$	$\sin\chi'$

Hence, the coordinates of the point $M_2(x_2, y_2, z_2)_\Sigma$ are

$$
\begin{aligned}
x_2 &= x' + R\sin\chi'\cos\theta', \\
y_2 &= y' + R\sin\chi'\sin\theta', \\
z_2 &= R\cos\chi'.
\end{aligned} \tag{2.7}
$$

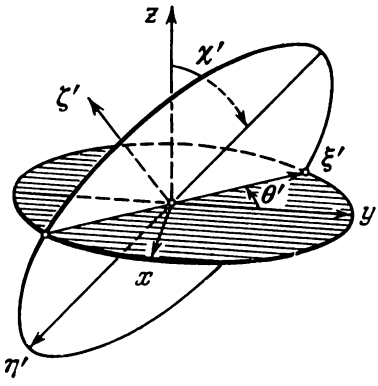


FIGURE 6.11

Equating the expressions (2.5) and (2.7) for the coordinates of the point M_2 and using the conditions of coincidence of the axes $M_2\zeta_2$ and $M_2\zeta'$, whose direction cosines are given by the last row of the tables (2.4) and (2.6), we obtain the desired relations:

$$
\begin{aligned}
x' + R\sin\chi'\cos\theta' &= x + R\sin\chi\cos\theta \\
&\quad - a(\cos\varphi\sin\theta + \sin\varphi\sin\chi\cos\theta) \\
&\quad - b\{\cos\psi[\cos(\varphi-\sigma)\sin\theta + \sin(\varphi-\sigma)\sin\chi\cos\theta] \\
&\qquad + \sin\psi\cos\chi\cos\theta\}, \\
y' + R\sin\chi'\sin\theta' &= y + R\sin\chi\sin\theta + \\
&\quad + a(\cos\varphi\cos\theta - \sin\varphi\sin\chi\sin\theta) + \\
&\quad + b\{\cos\psi[\cos(\varphi-\sigma)\cos\theta - \sin(\varphi-\sigma)\sin\chi\sin\theta] - \\
&\qquad - \sin\psi\cos\chi\sin\theta\},
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
R\cos\chi' &= R\cos\chi - a\sin\varphi\cos\chi - \\
&\quad - b\,[\cos\psi\sin(\varphi-\sigma)\cos\chi - \sin\psi\sin\chi], \\
\cos\chi'\sin\theta' &= \sin\psi\,[\cos(\varphi-\sigma)\cos\theta - \\
&\quad - \sin(\varphi-\sigma)\sin\chi\sin\theta] + \cos\psi\cos\chi\sin\theta, \\
\sin\chi' &= \cos\psi\sin\chi + \sin\psi\sin(\varphi-\sigma)\cos\chi.
\end{aligned}
$$

In the case of small deviations of the bicycle from a motion with a constant velocity V along the Oy axis, the relations (2.8) can be simplified. In this case, linearizing with respect to the small quantities θ, θ', χ, χ', and ψ, we obtain

$$
\begin{aligned}
&x' + R\chi' = x - a(\theta\cos\varphi + \chi\sin\varphi) + R\chi - \\
&\qquad - b\,[\theta\cos(\varphi-\sigma) + \chi\sin(\varphi-\sigma) + \psi], \\
&y' = y + a\cos\varphi + b\cos(\varphi-\sigma), \\
&\qquad a\sin\varphi + b\sin(\varphi-\sigma) = 0, \\
&\theta' = \psi\cos(\varphi-\sigma) + \theta, \quad \chi' = \chi + \psi\sin(\varphi-\sigma).
\end{aligned}
\tag{2.9}
$$

Hence, $\varphi = \text{const}$ and the expressions for the primed quantities in terms of the unprimed quantities take the form

$$
\begin{aligned}
&x' = x - c\theta + c_1\psi, \quad y' = y + c, \\
&\chi' = \chi - \psi\sin\lambda, \quad \theta' = \theta + \psi\cos\lambda,
\end{aligned}
\tag{2.10}
$$

where the coefficients

$$
\lambda = \sigma - \varphi, \quad c = a\cos\varphi + b\cos\lambda, \quad c_1 = R\sin\lambda - b \tag{2.11}
$$

are constructional parameters of the bicycle. In order to discover the physical meaning of the parameters, let us imagine the bicycle in the vertical position when, in addition, the planes of the front wheel and the frame coincide. In accordance with the notation of Figure 6.9, we have: $K_1K_2 = c$, the base length of the bicycle; λ, the angle between the steering column and the vertical; $c_1 = K_2K_3\cos\lambda$, a quantity proportional to the so-called stagger, by which the distance K_2K_3 is meant.[4] Since $R\sin\lambda = MM_2$ and, in ordinary constructions of a bicycle, the point M_0 lies inside the segment MM_2 (see Figure 6.9), the parameter c_1 is a positive quantity. The quantity λ is also usually positive.

[4] A distinction is made between "backward stagger" and "forward stagger". In the case of backward stagger, the point K_2 of contact of the wheel and the road lies behind the point K_3 of intersection of the axis of the steering column with the plane of the road. In the case of forward stagger (negative values of the parameter c_1), the point K_2 is in front of the point K_3.

We now express the condition of rolling of the wheels without slipping. The form of the equations of the kinematic constraints depends on the specific choice of the model of the wheel. The following cases are possible.

1) *Wheel as a thin absolutely rigid disc.* In this case, for the rear wheel we obtain

$$\dot{x} = -R\dot{\vartheta}_1 \sin\theta, \qquad \dot{y} = R\dot{\vartheta}_1 \cos\theta, \tag{2.12}$$

where $\dot{\vartheta}_1$ is the angular velocity of rotation of the rear wheel in its own plane. For the front wheel, we obtain similar relations:

$$\dot{x}' = -R\dot{\vartheta}_2 \sin\theta', \qquad \dot{y}' = R\dot{\vartheta}_2 \cos\theta', \tag{2.13}$$

where $\dot{\theta}_2$ is the angular velocity of rotation of the front wheel in its own plane. Substituting the expressions for the primed quantities into (2.13), we obtain the final equations of the nonholonomic constraints imposed on the motion of the front wheel. Equations (2.12) and (2.13) are four equations of nonholonomic constraints imposed on a system with seven generalized coordinates. It follows that, in this case, the bicycle has three degrees of freedom. If the rotation of the rear wheel is prescribed, the number of degrees of freedom is reduced to two.

We now find the form of the equations of nonholonomic constraints (2.12) and (2.13) in the case of small deviations of the bicycle from rectilinear motion along the Oy axis with constant velocity V. In this case, the system is subjected to a rheonomic constraint:

$$\vartheta_1 - \omega t = 0 \quad \left(\omega = \text{const}, \ \omega = \frac{V}{R}\right).$$

Linearizing equations (2.12) and (2.13) and using the relations (2.10), we obtain the integrable constraint equations

$$R\dot{\vartheta}_1 = R\dot{\vartheta}_2 = \dot{y} = \dot{y}' = V = \text{const}, \tag{2.14}$$

and also the equations of the nonholonomic constraints

$$\dot{x} + V\theta = 0, \quad c\dot{\theta} = c_1\psi + \psi V \cos\lambda. \tag{2.15}$$

It follows from equations (2.14) and (2.15) that the virtual variations of the generalized coordinates x, y, θ, χ, ψ, ϑ_1, and ϑ_2 are connected by the equations

$$\delta x = \delta y = \delta\vartheta_1 = \delta\vartheta_2 = 0, \quad c\delta\theta = c_1\delta\psi,$$

and, hence, the system has two degrees of freedom.

2. *The wheel as an absolutely rigid torus.* This model of a wheel can

be useful in the investigation of the motion of a motorcycle with tires pumped up very hard. In this case, the condition of rolling of the rear wheel without slipping reduces to the equations of nonholonomic constraints

$$\begin{aligned} &\dot{x} - r\dot{\chi}\cos\theta + \dot{\vartheta}_1 (R + r\cos\chi)\sin\theta = 0, \\ &\dot{y} - r\dot{\chi}\sin\theta - \dot{\vartheta}_1 (R + r\cos\chi)\cos\theta = 0. \end{aligned} \tag{2.16}$$

For suppose $R + r$ is the radius of the equatorial section of the torus, r is the radius of its meridianal section, and x and y are the coordinates of the point K of contact of the torus and the supporting plane. An increment of the angle χ by the amount $d\chi$ (subject to the condition $\theta = \text{const}$, $\vartheta_1 = \text{const}$) corresponds to a displacement of the point K through an interval $ds_1 = rd\chi$, and an increment of the angle ϑ_1 by the amount $d\vartheta_1$ (subject to the condition $\theta = \text{const}$, $\chi = \text{const}$) corresponds to a displacement of the point K through the interval $ds_2 = (R + r\cos\chi)d\vartheta_1$ (Figure 6.12). If the angle θ varies (and $\chi = \text{const}$ and $\vartheta_1 = \text{const}$), the point K does not move. Hence, we obtain equations (2.16). The condition of rolling without slipping of the front wheel of the bicycle leads to the equations

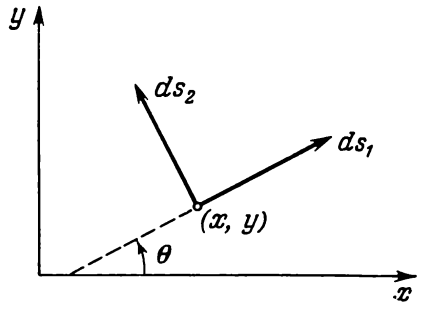


FIGURE 6.12

$$\begin{aligned} &\dot{x}' - r\dot{\chi}'\cos\theta' + \dot{\vartheta}_2 (R + r\cos\chi')\sin\theta' = 0, \\ &\dot{y}' - r\dot{\chi}'\sin\theta' - \dot{\vartheta}_2 (R + r\cos\chi')\cos\theta' = 0, \end{aligned} \tag{2.17}$$

in which the primed quantities are related to the unprimed quantities by means of equations (2.8).

In the case of small deviations of the bicycle from rectilinear motion along the Oy axis with constant velocity V, the second equations of (2.16) and (2.17) can be transformed into the integrable relations

$$\dot{y} = (R + r)\dot{\vartheta}_1 = \dot{y}' = (R + r)\dot{\vartheta}_2 = V, \tag{2.18}$$

and the first two equations of (2.16) and (2.17) take the form

$$\begin{aligned} &\dot{x} - r\dot{\chi} + V\theta = 0, \\ &c\dot{\theta} = (c_1 + r\sin\lambda)\psi + \psi V\cos\lambda. \end{aligned} \tag{2.19}$$

As $r \to 0$, equations (2.19) go over accordingly into equations (2.15). Thus, in the case when the wheels may be regarded as absolutely rigid tori, the bicycle also has two degrees of freedom.

3. *Wheel with a deformable pneumatic tire.* According to Keldyš's theory (see §1), the condition of rolling of the rear wheel without slipping in the case of small deviations of the bicycle from its motion along the Oy axis with constant velocity V leads to two equations of nonholonomic constraints:

$$\begin{aligned} &\dot{x} + \dot{\xi}_1 + V\theta + V\varphi_1 = 0, \\ &\dot{\theta} + \dot{\varphi}_1 - \alpha V\xi_1 + \beta V\varphi_1 + \gamma V\chi = 0, \end{aligned} \tag{2.20}$$

where ξ_1 is the transverse displacement of the center of the area of contact of the rear wheel with respect to the point K (see Figure 6.4); φ_1 is the angle through which the area of contact is twisted with respect to the rim of the wheel; and α, β, and γ are the kinematic deformation parameters of the tire.[5)]

For the front wheel, we have the similar equations

$$\begin{aligned} &\dot{x}' + \dot{\xi}_2 + V\theta' + V\varphi_2 = 0, \\ &\dot{\theta}' + \dot{\varphi}_2 - \alpha V\xi_2 + \beta V\varphi_2 + \gamma V\chi' = 0, \end{aligned}$$

where the kinematic coefficients α, β, and γ are assumed to be the same as those for the rear wheels. Finally, using the relations (2.10), we obtain

$$\begin{aligned} &\dot{x} - c\dot{\theta} + c_1\dot{\psi} + \dot{\xi}_2 + V\theta + V\psi\cos\lambda + V\varphi_2 = 0, \\ &\dot{\theta} + \dot{\psi}\cos\lambda + \dot{\varphi}_2 - \alpha V\xi_2 + \beta V\varphi_2 + \gamma V\chi - \gamma V\psi\sin\lambda = 0. \end{aligned} \tag{2.21}$$

Here, ξ_2 and φ_2 are the parameters of the lateral deformation and twisting of the tire of the front wheel of the bicycle. Thus, when allowance is made for the deformation of the tires, the configuration space of the bicycle $(x, \theta, \chi, \psi, \xi_1, \xi_2, \varphi_1, \varphi_2)$ is eight-dimensional and, in accordance with the equations of the nonholonomic constraints (2.20) and (2.21), the system has four degrees of freedom. We should mention that if allowance is made only for the lateral deformation of the tires, the number of degrees of freedom of the bicycle is still equal to four. For in this case the equations of the kinematic constraints are the equations of creep (1.1) derived with allowance for the angle θ through which the rim of the wheel is turned. For the rear wheel, we have

$$\dot{x} + \xi_1 + \kappa_1 V\xi_1 + V\theta = 0, \tag{2.22}$$

[5)] We recall that the quantity x that occurs in the first equation of (2.20) is the abscissa of the point of intersection of the diameter of greatest slope of the central plane of the wheel with pneumatic tire and the plane of the road. Thus, the role of the thin absolutely rigid disc is now played by the central plane of the wheel, the radius of this disc depending, in general, on the normal load N on the wheel. However, since the normal load is assumed to be constant, the radius of the disc is also constant.

and, for the front wheel,

$$\dot{x}' + \xi_2 + \kappa_1 V\xi_2 + V\theta = 0,$$

where κ_1 is the coefficient of creep, which we shall assume is the same for both wheels of the bicycle. Replacing the primed quantities by their expressions (2.10) in the last equation, we obtain

$$c\dot{\theta} = c_1\dot{\psi} + \psi V \cos\lambda + \dot{\xi}_2 - \dot{\xi}_1 + \varkappa_1 V(\xi_2 - \xi_1). \tag{2.23}$$

Equations (2.22) and (2.23) are the equations of the kinematic constraints in the case when the configuration space $(x, \theta, \chi, \psi, \xi_1, \xi_2)$ is six-dimensional and, in accordance with the equations of the kinematic constraints (2.22) and (2.23), the system has four degrees of freedom.

2. Equations of motion of a bicycle. In accordance with the results of our kinematic investigation, we shall derive the equations of motion of a bicycle for two models: a) A bicycle on absolutely rigid wheels; b) a bicycle on wheels with pneumatic tires.

In both models, we shall assume that, apart from the reaction forces of the supporting plane, whose virtual work vanishes, the bicycle is subjected only to the force of gravity. Under such conditions, the dynamics of the bicycle can be characterized by a Lagrange function L in which it is sufficient to take into account the linear and quadratic terms in the small quantities θ, ψ, and χ. At the same time, we shall assume that energy is dissipated in the system due to the presence of viscous friction in the steering column. In order to allow for this dissipation, we introduce the Rayleigh function

$$F = \tfrac{1}{2}\delta\dot{\psi}^2, \tag{2.24}$$

where δ is the coefficient of viscous friction. Since we are considering small deviations of the bicycle from rectilinear motion along a horizontal plane under a constant vertical load, we can ignore the spring suspension of the wheels.

a) *Bicycle on absolutely rigid wheels.* Let us begin our investigation with the case when the wheels of the bicycle can be regarded as thin, absolutely rigid discs.

We shall write down an expression for the kinetic energy T and the potential energy U_0 of the bicycle. The kinetic energy T consists of the kinetic energy T_1 of the rear part of the bicycle (the frame + rear wheel + rider) and the kinetic energy T_2 of the front part of the bicycle (front wheel + fork + handlebars + headlight). We introduce the

following notation for the parameters of the rear part of the bicycle: m_1, its mass; l_1 and h_1, the coordinates of the center of mass M_3 (see Figure 6.9); A_1 and B_1, the central moments of inertia about the horizontal axis and the mutually perpendicular axis which lie in the plane of the rear wheel; D_1, the centrifugal moment of inertia about these axes; C_1, the moment of inertia of the rear wheel about the axis of rotation in its own plane. We shall denote the parameters of the front part of the bicycle by the same letters with the subscript 2.

In accordance with Koenig's theorem, we have $T_1 = T_1' + T_1''$, where $T_1' = m_1 v_3^2/2$ is the kinetic energy of the translational motion with velocity v_3 of the center of mass M_3 and T_1'' is the kinetic energy of the rotational motion about the center of mass.

For the determination of $v_3^2 = \dot{x}_3^2 + \dot{y}_3^2 + \dot{z}_3^2$, we have

$$\begin{aligned} x_3 &= x - l_1 \sin\theta + h_1 \cos\theta \sin\chi, \\ y_3 &= y + l_1 \cos\theta + h_1 \sin\theta \sin\chi, \\ z_3 &= h_1 \cos\chi. \end{aligned} \tag{2.25}$$

Differentiating these relations and using (2.14), we obtain

$$v_3^2 = (\dot{x} - l_1\dot{\theta} + h_1\dot{\chi})^2 - 2V(l_1\theta - h_1\chi)\dot{\theta} + 2h_1V\theta\dot{\chi},$$

where small terms of third order and higher have been omitted. The projections of the instantaneous angular velocity ω onto the aforementioned axes are $\omega\{\dot{\chi}, -\dot{\theta}\cos\chi, \dot{\vartheta}_1 + \dot{\theta}\sin\chi\}$. Taking into account (2.14), we obtain $\omega\{\dot{\chi}, -\dot{\theta}, V/R + \chi\dot{\theta}\}$ in the approximation which we have adopted; hence,

$$T_1'' = \frac{1}{2}(A_1\dot{\chi}^2 - 2D_1\dot{\chi}\dot{\theta} + B_1\dot{\theta}^2) + \frac{V}{R}C_1\chi\dot{\theta}.$$

Thus, we obtain the following expression for T_1:

$$\begin{aligned} T_1 = \frac{1}{2}m_1[(\dot{x} - l_1\dot{\theta} + h_1\dot{\chi})^2 - 2V(l_1\theta - h_1\chi)\dot{\theta} + 2h_1V\theta\dot{\chi}] \\ + \frac{1}{2}(A_1\dot{\chi}^2 - 2D_1\dot{\chi}\dot{\theta} + B_1\dot{\theta}^2) + \frac{C_1V}{R}\chi\dot{\theta}. \end{aligned} \tag{2.26}$$

In calculating the kinetic energy T_2 of the front part of the bicycle, we proceed similarly: $T_2 = T_2' + T_2''$, where $T_2' = m_2 v_4^2/2$ is the kinetic energy of the translational motion of the front part of the bicycle with the velocity v_4 of the center of mass M_4 and T_2'' is the kinetic energy of the rotational motion about the point M_4. In order to determine the value of v_4^2, we shall first find the coordinates (x_4, y_4, z_4) of the point M_4. Suppose that the position of the point M_4 in the Σ_2 system intro-

duced in §2.1 is given by the quantities $M_4\{d, -d_1, 0\}_{\Sigma_2}$, where $d > 0$ and $d_1 > 0$. The coordinates x_4, y_4, and z_4 can then be represented in

$$x_4 = x_0 + x_\Sigma,\ y_4 = y_0 + y_\Sigma,\ z_4 = z_0 + z_\Sigma, \tag{2.27}$$

where x_0, y_0, and z_0 are given by the expressions (2.3) and x_Σ, y_Σ, and z_Σ are given, in accordance with Table 2.4, by the expressions

$$\begin{aligned} x_\Sigma &= d\cos\psi\,(-\cos\lambda\sin\theta + \sin\lambda\sin\chi\cos\theta) - d\sin\psi\cos\chi\cos\theta \\ &\quad + d_1\sin\lambda\sin\theta + d_1\cos\lambda\sin\chi\cos\theta, \\ y_\Sigma &= d\cos\psi\,(\cos\lambda\cos\theta + \sin\lambda\sin\chi\sin\theta) - d\sin\psi\cos\chi\sin\theta - \\ &\quad - d_1\sin\lambda\cos\theta + d_1\cos\lambda\sin\chi\sin\theta, \\ z_\Sigma &= d\cos\psi\cos\chi\sin\lambda + d\sin\psi\sin\chi + d_1\cos\lambda\cos\chi. \end{aligned}$$

Differentiating the relations (2.27) with respect to the time and squaring and adding them, we obtain

$$\begin{aligned} v_4^2 &= (\dot{x} + h_2\dot{\chi} - l_2\dot{\theta} - d\dot{\psi})^2 + 2V\,(h_2\chi - l_2\theta - d\psi)\,\dot{\theta} \\ &\quad + 2Vh_2\theta\dot{\chi} - 2Vd\,(\theta + \psi\cos\lambda)\,\psi, \end{aligned}$$

where small terms of third and higher order have been omitted and the parameters h_2 and l_2 and d and d_1 are related by the equations

$$\begin{aligned} l_2 &= a\cos\varphi - d_1\sin\lambda + d\cos\lambda, \\ h_2 &= R - a\sin\varphi + d_1\cos\lambda + d\sin\lambda, \\ d &= h_2\sin\lambda + (l_2 - c)\cos\lambda - c_1 \\ &\qquad (\varphi = \text{const}). \end{aligned} \tag{2.28}$$

It is readily seen that the parameter d is the distance of the center of mass of the front part of the bicycle from the steering column and that the quantities l_2 and h_2 are the coordinates of the point M_4 (see Figure 6.9).

We shall now derive an expression for the kinetic energy T_2'' of the rotational motion of the front part of the bicycle about the center of mass M_4. We note that the vector of the instantaneous angular velocity of rotation of the front part of the bicycle (without allowance for the rotation of the front wheel in its own plane) is made up of the angular velocities $\dot{\theta}$ and $\dot{\chi}$ of rotation with the frame and also the angular velocity $\dot{\psi}$ of rotation about the axis of the steering column. In accordance with the notation in Figure 6.13, in which the shaded region indicates the plane of the front wheel, the projections of the instantaneous angular velocity onto the horizontal axis and the mutually perpendicular axis lying in the plane of the front wheel and also onto the axis perpendicular to the plane of the front wheel are respectively: $\dot{\chi} - \dot{\psi}\sin\lambda + \text{small}$

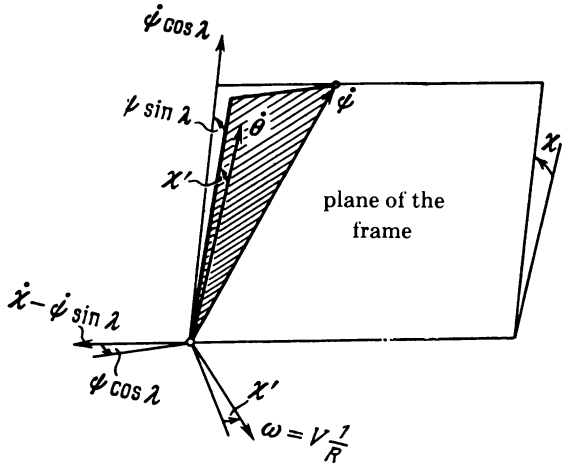


FIGURE 6.13

terms of second order and higher; $\dot{\theta} + \dot{\psi}\cos\lambda +$ terms of second order and higher; $V/R - \dot{\chi}\psi\cos\lambda + \dot{\theta}(\chi - \psi\sin\lambda) +$ small terms of third order and higher.

Consequently, the kinetic energy T_2'' can be written in the form

$$T_2'' = \frac{1}{2}\,[A_2(\dot{\chi} - \psi\sin\lambda)^2 + 2D_2(\dot{\chi} - \psi\sin\lambda)(\dot{\theta} + \psi\cos\lambda) + \\ + B_2(\dot{\theta} + \psi\cos\lambda)^2] - \frac{C_2V}{R}\,[\dot{\chi}\psi\cos\lambda - \dot{\theta}(\chi - \psi\sin\lambda)],$$

where the constant term and small terms of third order and higher have been omitted. Hence, we obtain the following expression for T_2:

$$T_2 = \frac{1}{2}\,m_2(\dot{x} + h_2\dot{\chi} - l_2\dot{\theta} - d\dot{\psi})^2 + m_2V\,(h_2\chi - l_2\theta \\ - d\psi)\,\dot{\theta} + m_2h_2V\theta\dot{\chi} - m_2dV\,(\theta + \psi\cos\lambda)\,\psi \\ + \frac{1}{2}\,[A_2(\dot{\chi} - \psi\sin\lambda)^2 + 2D_2(\dot{\chi} - \psi\sin\lambda)(\dot{\theta} + \psi\cos\lambda) \\ + B_2(\dot{\theta} + \psi\cos\lambda)^2] - \frac{C_2V}{R}\,[\dot{\chi}\psi\cos\lambda + \dot{\theta}(\psi\sin\lambda - \chi)]. \tag{2.29}$$

In the same approximation, the potential energy U_0 of the bicycle can be written in the form

$$U_0 = -\frac{1}{2}\,g\,[(m_1h_1 + m_2h_2)\,\chi^2 - 2m_2d\psi\chi + \psi^2m_2d\sin\lambda]. \tag{2.30}$$

For, by definition,

$$U_0 = m_1gz_3 + m_2gz_4,$$

where

$$z_3 = h_1 \cos\chi,\ z_4 = R\cos\chi - a\sin\varphi\cos\chi + d(\sin\lambda\cos\psi\cos\chi + \sin\psi\sin\chi) + d_1\cos\lambda\cos\chi.$$

Expanding these expressions in series in the small quantities ψ and χ and restricting ourselves to terms of second order, we obtain the expression (2.30). We should mention that, in accordance with the derivation of the expressions for T_1 and T_2 of the kinetic energy of the bicycle, the positive coefficients D_1 and D_2 reflect a distribution of mass in the front and rear parts of the bicycle for which the central ellipsoids of inertia have the positions shown in Figure 6.9 by the dashed curves.

Thus, we have the Lagrange function

$$L = T_1 + T_2 - U_0, \tag{2.31}$$

where T_1, T_2, and U_0 are given, respectively, by the expressions (2.26), (2.29), and (2.30).

Further, we introduce the operator

$$\mathcal{L}_q \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

and calculate $\mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{L}_\theta$, and $\mathcal{L}_\psi$. Using (2.31), we find

$$\begin{aligned}
\mathcal{L}_x &= m\ddot{x} - l\ddot{\theta} + h\ddot{\chi} - m_2 d\ddot{\psi},\\
\mathcal{L}_\chi &= h\ddot{x} - J_{12}\ddot{\theta} - V(C_1^0 + C_2^0)\dot{\theta} + J_1\ddot{\chi} - gh\chi - J_3\ddot{\psi}\\
&\qquad - \psi C_2^0 V\cos\lambda + gm_2 d\psi,\\
\mathcal{L}_\theta &= -l\ddot{x} + J_2\ddot{\theta} - J_{12}\ddot{\chi} + V(C_1^0 + C_2^0)\chi + J_4\ddot{\psi} - \psi C_2^0 V\sin\lambda,\\
\mathcal{L}_\psi &= -m_2 d\ddot{x} + J_4\ddot{\theta} + \dot{\theta}C_2^0 V\sin\lambda - J_3\ddot{\chi} + \dot{\chi}C_2^0 V\cos\lambda\\
&\qquad + gm_2 d\chi + J\ddot{\psi} - \psi g m_2 d\sin\lambda.
\end{aligned} \tag{2.32}$$

Here, we have introduced the following notation:

$$\begin{aligned}
&m = m_1 + m_2 \text{ is the mass of the complete system (bicycle + rider);}\\
&l = m_1 l_1 + m_2 l_2,\ h = m_1 h_1 + m_2 h_2,\\
&J = m_2 d^2 + A_2\sin^2\lambda - D_2\sin 2\lambda + B_2\cos^2\lambda,\\
&J_1 = m_1 h_1^2 + A_1 + m_2 h_2^2 + A_2,\ J_2 = m_1 l_1^2 + B_1 + m_2 l_2^2 + B_2,\\
&J_3 = m_2 h_2 d + A_2\sin\lambda - D_2\cos\lambda,\ J_{12} = m_1 h_1 l_1 + D_1 +\\
&\qquad + m_2 h_2 l_2 - D_2,\ J_4 = m_2 l_2 d - D_2\sin\lambda + B_2\cos\lambda,
\end{aligned} \tag{2.33}$$

and instead of the moments of inertia of the wheels C_1 and C_2 we have introduced the quantities

$$C_1^0 = C_1 R^{-1},\quad C_2^0 = C_2 R^{-1}, \tag{2.34}$$

where R is the radius of the wheel. In order to determine the physical meaning of the parameters (2.33), we consider a bicycle with the handlebars fixed so that the front wheel lies in the plane of the frame. We introduce coordinate axes with origin at the point K_1 of support of the rear wheel such that K_1x_1 is perpendicular to the central plane of the bicycle, in which K_1y_1 is horizontal and K_1z_1 vertically upward. Then $0, l/m, h/m$ are the coordinates of the center of mass of the complete system; J is the moment of inertia of the front part of the bicycle about the steering column; J_1 is the moment of inertia of the system about the K_1y_1 axis; J_2 is the moment of inertia of the system about the K_1z_1 axis; J_{12} is the centrifugal moment of inertia about these axes; and J_3 and J_4 are parameters having the dimensions of moments of inertia.

In accordance with the equations of the kinematic constraints (2.15), the dynamical equations of our model of a bicycle can be written in the form

$$\mathcal{L}_\chi = 0, \quad c\left(\mathcal{L}_\psi + \frac{\partial F}{\partial \dot{\psi}}\right) + c_1\mathcal{L}_\theta = 0. \tag{2.35}$$

Using the expressions (2.32), we can eliminate the variables x and θ from these equations by means of the equations of the nonholonomic constraints (2.15). The first equation of (2.35) then takes the form

$$a_0\ddot{\chi} - a_1\chi - a_2\ddot{\psi} - a_3 V\dot{\psi} + (a_4 - a_5V^2)\psi = 0, \tag{2.36}$$

where

$$\begin{aligned} &a_0 = cJ_1, \quad a_3 = (J_{12} + cC_2^0)\cos\lambda + c_1(h + C_1^0 + C_2^0), \\ &a_1 = cgh, \quad a_4 = cgm_2d, \\ &a_2 = cJ_3 + c_1J_{12}, \quad a_5 = (h + C_1^0 + C_2^0)\cos\lambda. \end{aligned} \tag{2.37}$$

For a bicycle of normal construction, all the coefficients $a_0, a_1, \cdots, a_5$ are positive. Equation (2.36), which corresponds to the coordinate χ, also contains the variable ψ, which characterizes the influence that the angle ψ through which the handlebars are turned has on the angle χ of inclination of the frame of the bicycle. In order to obtain a closed system of equations, one must write down the second equation of (2.35) and use the equations of the nonholonomic constraints (2.15). However, in an investigation of the stability of a bicycle fitted with a controlling device, it is sufficient to write down only equations (2.36) because the system of equations is completed by the additional relation between the variables ψ and χ provided by the prescribed law controlling the bicycle.

We now write down the second equation of (2.35). Using the expressions (2.32), (2.34), and the equations of the nonholonomic constraints (2.15), we obtain

$$b_0\ddot{\psi} + b_1 V\psi + (b_2V^2 - b_3)\psi - b_4\ddot{\chi} + b_5 V\dot{\chi} + b_6\chi = 0, \tag{2.38}$$

where

$$b_0 = cJ + 2c_1J_4 + \frac{c_1^2}{c}J_2,\quad b_2 = \left(m_2d + C_2^0\sin\lambda + \frac{l}{c}c_1\right)\cos\lambda,$$

$$b_1 = J_4\cos\lambda + c_1\left(m_2d + \frac{1}{2}J_2\cos\lambda\right) + \frac{l}{c}c_1^2 + c\delta_1, \tag{2.39}$$

$$b_3 = cgm_2d\sin\lambda, \qquad b_5 = cC_2^0\cos\lambda + c_1(C_1^0 + C_2^0),$$
$$b_4 = cJ_3 + c_1J_{12}, \qquad b_6 = cgm_2d$$

and we have introduced the new parameter

$$\delta_1 = \delta/V.$$

If the bicycle has a normal construction, all the coefficients $b_0, b_1, \cdots, b_6$ are positive.

Equations (2.36) and (2.38) are the equations of motion of an uncontrolled bicycle with rigid disc-like wheels. These equations can be integrated independently of the equations of the nonholonomic constraints, since under the assumptions made the bicycle is a nonholonomic Čaplygin system.

We now turn to the study of the model of a bicycle whose wheels can be regarded as absolutely rigid tori.

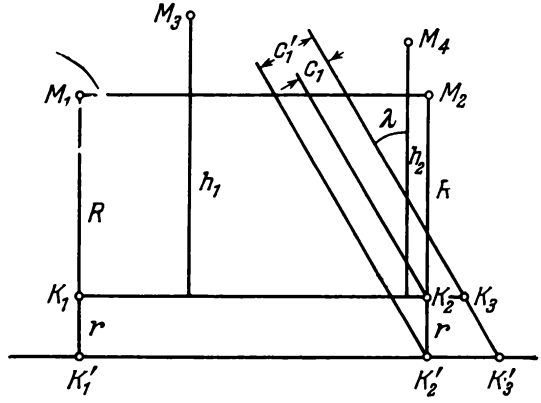


FIGURE 6.14

According to the equations of the kinematic constraints (2.19), the variations of the generalized coordinates x, χ, θ, and ψ of the model are related by the equations

$$\delta x = r\delta\chi, \quad c\delta\theta = c_1'\delta\psi, \tag{2.40}$$

where $c_1' = c_1 + r\sin\lambda$ is the analog of c_1 in the model with discs. In accordance with the notation in Figure 6.14, the quantity $c_1' = K_2'K_3'\cos\lambda$ is proportional to the stagger $K_2'K_3'$.

It follows from the relations (2.40) that the dynamical equations of a bicycle with rigid toroidal wheels can be written in the form

$$\mathcal{L}_\chi + r\mathcal{L}_x = 0, \quad c\left(\mathcal{L}_\psi + \frac{\partial F}{\partial\dot\psi}\right) + c_1'\mathcal{L}_\theta = 0. \tag{2.41}$$

We now turn to the expression for the Lagrange function (2.31) which was derived for the case $r = 0$. Considering the various stages in the derivation of the expressions for the kinetic energy T and the potential energy U_0 of the bicycle, we can readily see that the calculations are exactly the same in the case $r \neq 0$ because the quantity r adds a constant term to the z coordinate and this affects only the coefficients which have a factor $1/R$. This factor can now be replaced by $1/(R + r)$. Substituting the expressions (2.32) into (2.41) and eliminating the variables x and θ by means of the equations of the nonholonomic constraints (2.19), we obtain two equations of motion of a bicycle on rigid toroidal wheels. The first equation is

$$a_0'\ddot\chi - a_1'\chi - a_2'\ddot\psi - a_3'V\dot\psi + (a_4' - a_5'V^2)\psi = 0, \tag{2.42}$$

where

$$\begin{aligned} a_0' &= cJ_1', & a_3' &= (J_{12} + cC_2')\cos\lambda + c_1'(h' + C_1' + C_2'), \\ a_1' &= cgh, & a_4' &= cgm_2d, \\ a_2' &= cJ_3' + c_1'J_{12}', & a_5' &= (h' + C_1' + C_2')\cos\lambda, \end{aligned} \tag{2.43}$$

and the second equation is

$$b_0'\ddot\psi + b_1'V\dot\psi + (b_2'V^2 - b_3')\psi - b_4'\ddot\chi + b_5'V\dot\chi + b_6'\chi = 0, \tag{2.44}$$

where

$$\begin{aligned} b_0' &= cJ + 2c_1'J_4 + \frac{c_1'^2}{c}J_2, \quad b_2' = \left(m_2d + C_2'\sin\lambda + \frac{l}{c}c_1'\right)\cos\lambda, \\ b_1' &= J_4\cos\lambda + c_1'\left(m_2d + \frac{1}{c}J_2\cos\lambda\right) + \frac{l}{c}c_1'^2 + c\delta_1, \\ b_3' &= cgm_2d\sin\lambda, \; b_5' = cC_2'\cos\lambda + c_1'(C_1' + C_2'), \\ b_4' &= cJ_3' + c_1'J_{12}', \; b_6' = cgm_2d. \end{aligned} \tag{2.45}$$

In the expressions (2.43) and (2.45), we have $c_1' = c_1 + r\sin\lambda$ and the remaining primed quantities are given by formulas (2.33), in which

h_1 must be replaced by $h_1 + r$. The quantities C_1' and C_2' are related to the moments of inertia C_1 and C_2 by equations (2.34), in which R must be replaced by $R + r$. It is readily seen that the coefficients a_i' and b_i' defined by the expressions (2.43) and (2.45) go over into the corresponding coefficients a_i and b_i defined by the expressions (2.33) and (2.34) as $r \to 0$.

Thus, the only difference between the equations (2.42) and (2.44) of the motion of a bicycle on rigid toroidal wheels and the equations (2.36) and (2.38) of the motion of a bicycle on rigid disc-like wheels is in the values of certain coefficients. We may therefore conclude that allowance for the toroidal shape of the wheels has no qualitative influence on the dynamics of a bicycle with rigid wheels.

b) *Bicycle on wheels with pneumatic tires.* In order to write down the equations of motion of a bicycle with allowance for the elasticity of pneumatic tires, we use the general equations of motion of a vehicle on wheels with pneumatic types (1.18).

First of all, we write down an expression for the potential energy of the deformation of the tires of the bicycle. In accordance with (1.16), we have

$$2U = a(\xi_1^2 + \xi_2^2) + 2\sigma(N_1\xi_1\chi + N_2\xi_2\chi') + \rho(N_1\chi^2 + N_2\chi'^2) + b(\varphi_1^2 + \varphi_2^2). \tag{2.46}$$

Here, it is assumed that the coefficients of elasticity a, σ, ρ, and b have the same values for both wheels. However, the value of the normal load N_1 on the rear wheels is generally different from the normal load N_2 on the front wheel. Since the vertical load on the wheels is assumed to be constant, the values of N_1 and N_2 can be calculated from the equations of static equilibrium of the forces N_1 and N_2 and the forces of gravity $m_1 g$ and $m_2 g$. Hence,

$$N_1 = \frac{g}{c}[m_1(c - l_1) + m_2(c - l_2)], \quad N_2 = \frac{g}{c}(m_1 l_1 + m_2 l_2). \tag{2.47}$$

Writing down the dynamical equations of the bicycle (1.18) for the generalized coordinates x, θ, χ, and ψ, we obtain the following equations:

$$\mathcal{L}_x = R_x, \quad \mathcal{L}_\theta = R_\theta, \quad \mathcal{L}_\chi = R_\chi, \quad \mathcal{L}_\psi + \frac{\partial F}{\partial \dot{\psi}} = R_\psi. \tag{2.48}$$

Here, the generalized forces R_j defined by the expression (1.17) can be calculated by means of equations (2.14) and (2.46) and have the form

$$
\begin{aligned}
R_x &= \frac{\partial U}{\partial \xi_1} + \frac{\partial U}{\partial \xi_2} = a\xi_1 + a\xi_2 + \sigma(N_1 + N_2)\chi - \psi\sigma N_2 \sin\lambda, \\
R_\theta &= -c\frac{\partial U}{\partial \xi_2} + \frac{\partial U}{\partial \varphi_1} + \frac{\partial U}{\partial \varphi_2} = -ac\xi_2 - c\sigma N_2\chi + \psi c\sigma N_2 \sin\lambda \\
&\qquad + b\varphi_1 + b\varphi_2, \\
R_\chi &= -\frac{\partial U}{\partial \chi} - \frac{\partial U}{\partial \chi'} = -\sigma N_1\xi_1 - \sigma N_2\xi_2 - \rho(N_1 + N_2)\chi \\
&\qquad + \psi\rho N_2 \sin\lambda, \\
R_\psi &= c_1\frac{\partial U}{\partial \xi_2} + \frac{\partial U}{\partial \varphi_2}\cos\lambda + \frac{\partial U}{\partial \chi'}\sin\lambda = (ac_1 + \sigma N_2 \sin\lambda)\,\xi_2 \\
&+ (c_1\sigma + \rho\sin\lambda)\,N_2\chi - \psi(c_1\sigma + \rho\sin\lambda)N_2\sin\lambda + b\varphi_2\cos\lambda.
\end{aligned}
\tag{2.49}
$$

Substituting the expressions (2.32) and (2.49) into (2.48), we obtain the dynamical equations of a bicycle with allowance for the deformability of the tires:

$$
\begin{aligned}
&m\ddot{x} - l\ddot{\theta} + h\ddot{\chi} - \sigma(N_1 + N_2)\chi - m_2 d\ddot{\psi} + \psi\sigma N_2 \sin\lambda - a\xi_1 \\
&\qquad - a\xi_2 = 0, \\
&-l\ddot{x} + J_2\ddot{\theta} - J_{12}\ddot{\chi} + V(C_1^0 + C_2^0)\dot{\chi} + c\sigma N_2\chi + J_4\ddot{\psi} \\
&\qquad - \psi C_2^0 V \sin\lambda - \psi c\sigma N_2 \sin\lambda + ac\xi_2 - b\varphi_1 - b\varphi_2 = 0, \\
&h\ddot{x} - J_{12}\ddot{\theta} - V(C_1^0 + C_2^0)\dot{\theta} + J_1\ddot{\chi} + [gh - \rho(N_1 + N_2)]\chi \\
&- J_3\ddot{\psi} + (gm_2 d - \rho N_2 \sin\lambda)\psi - \psi C_2^0 V\cos\lambda + \sigma N_1\xi_1 \\
&\qquad + \sigma N_2\xi_2 = 0, \\
&-m_2 d\ddot{x} + J_4\ddot{\theta} + \dot{\theta}C_2^0 V\sin\lambda - J_3\ddot{\chi} + \dot{\chi}C_2^0 V\cos\lambda \\
&+ [gm_2 d - N_2(c_1\sigma + \rho\sin\lambda)]\chi + J\ddot{\psi} + \delta\psi \\
&- [gm_2 d - N_2(c_1\sigma + \rho\sin\lambda)]\psi\sin\lambda - (ac_1 + \sigma N_2\sin\lambda)\,\xi_2 \\
&\qquad - b\varphi_2\cos\lambda = 0.
\end{aligned}
\tag{2.50}
$$

Equations (2.50) together with (2.20) and (2.21) are then the equations of motion of a bicycle in which allowance is made for the deformability of the pneumatic tires.

For what follows, we introduce the new variables

$$
s = Vt, \quad u = \frac{dx}{ds} + \theta, \quad w = \frac{d\theta}{ds}, \quad \varepsilon = \frac{1}{V^2}, \quad \delta_1 = \frac{\delta}{V}
\tag{2.51}
$$

and, denoting by a dot the operation of differentiation with respect to the variable s, we write equations (2.20), (2.21), and (2.50) as a single system of equations

$$
\begin{aligned}
&u + \dot{\xi}_1 + \varphi_1 = 0,\\
&w + \gamma\chi - \alpha\xi_1 + \dot{\varphi}_1 + \beta\varphi_1 = 0,\\
&u - cw + c_1\dot{\psi} + \psi\cos\lambda + \dot{\xi}_2 + \varphi_2 = 0,\\
&w + \gamma\chi + \psi\cos\lambda - \psi\gamma\sin\lambda - \alpha\xi_2 + \dot{\varphi}_2 + \beta\varphi_2 = 0,\\
&m\dot{u} - l\dot{w} - mw + h\ddot{\chi} - \varepsilon\sigma(N_1 + N_2)\dot{\chi} - m_2d\dot{\psi}\\
&\qquad + \psi\varepsilon\sigma N_2\sin\lambda - \varepsilon a\xi_1 - \varepsilon a\xi_2 = 0,\\
&-l\dot{u} + J_2\dot{w} + lw - J_{12}\ddot{\chi} + (C_1^0 + C_2^0)\dot{\chi} + \varepsilon c\sigma N_2\chi + J_4\ddot{\psi}\\
&\qquad - \psi C_2^0\sin\lambda - \psi c\varepsilon\sigma N_2\sin\lambda + \varepsilon ac\xi_1 - \varepsilon b\varphi_1 - \varepsilon b\varphi_2 = 0,\\
&h\dot{u} - J_{12}\dot{w} - (h + C_1^0 + C_2^0)w + J_1\ddot{\chi} - \varepsilon[gh - \rho(N_1 + N_2)]\chi\\
&- J_3\ddot{\psi} - \psi C_2^0\cos\lambda + \varepsilon(gm_2d - \rho N_2\sin\lambda)\psi + \varepsilon\sigma N_1\xi_1\\
&\qquad + \varepsilon\sigma N_2\xi_2 = 0,\\
&-m_2d\dot{u} + J_4\dot{w} + (m_2d + C_2^0\sin\lambda)w + J_3\ddot{\chi} + \dot{\chi}C_2^0\cos\lambda\\
&+ \varepsilon[gm_2d - N_2(\sigma c_1 + \rho\sin\lambda)]\chi + J\ddot{\psi} + \delta_1\dot{\psi} - \varepsilon[gm_2d\\
&- N_2(\sigma c_1 + \rho\sin\lambda)]\psi\sin\lambda - \varepsilon(ac_1 + \sigma N_2\sin\lambda)\xi_2\\
&\qquad - \varepsilon b\varphi_2\cos\lambda = 0.
\end{aligned}
\tag{2.52}
$$

Equations (2.52) describe the motion of the representative point in a nine-dimensional phase space Φ, the quantities u, w, χ, $\dot{\chi}$, ψ, $\dot{\psi}$, ξ_1, ξ_2, φ_1, φ_2 being plotted along the axes of this phase space. There is an equilibrium state at the origin of the space Φ and it corresponds to the manifold of steady motions of the bicycle satisfying the condition

$$u \equiv dx/ds + \theta = 0.$$

We shall now find the equations of motion of a bicycle on wheels with pneumatic tires in special cases when the order of the differential equations is reduced. In accordance with the results of §1, let us consider two cases: 1) The bicycle has a fairly high velocity; 2) the kinematic parameters α, β, and γ are fairly large.

In both cases, the dynamical equations of the bicycle can also be written in the form (2.48). However, the generalized forces R_x, R_θ, T_χ, and R_ψ are now given by different expressions. Thus, in the case of a fairly high velocity of the bicycle, the forces must be calculated by means of equations (1.25). For the rear wheel,

$$
\begin{aligned}
&F = \frac{a_{11}}{V}\dot{\theta} - a_{21}\theta - \frac{a_{21}}{V}\dot{x} - a_{31}\chi,\quad M_0 = -\frac{b}{V}\dot{x} - b\theta,\\
&M_\chi = -\frac{b_{11}}{V}\dot{\theta} + b_{21}\theta + \frac{b_{21}}{V}\dot{x} - b_{31}\chi.
\end{aligned}
\tag{2.53}
$$

For the front wheel we have similar expressions:

$$
\begin{aligned}
&F' = \frac{a_{12}}{V}\dot{\theta}' - a_{22}\theta' - \frac{a_{22}}{V}\dot{x}' - a_{32}\chi', \quad M_{\theta'} = -\frac{b}{V}\dot{x}' - b\theta', \\
&M_{\chi'} = -\frac{b_{12}}{V}\dot{\theta}' + b_{22}\theta' + \frac{b_{22}}{V}\dot{x}' - b_{32}\chi',
\end{aligned} \tag{2.54}
$$

in which $a_{12} = a_{11}$ and $a_{22} = a_{21}$ [see (1.26)], because these coefficients do not depend on the magnitude of the normal load N on the wheel. Using the relations (2.10), we can express the primed variables in (2.54) in terms of the generalized coordinates x, θ, χ, and ψ:

$$
\begin{aligned}
F' &= -\frac{a_{21}}{V}\dot{x} + \frac{a_{11} + ca_{21}}{V}\dot{\theta} + \frac{1}{V}(a_{11}\cos\lambda - a_{21}c_1)\psi \\
&\quad - a_{21}\theta - (a_{21}\cos\lambda - a_{32}\sin\lambda)\psi - a_{32}\chi, \\
M_{\theta'} &= -\frac{b}{V}\dot{x} - \frac{bc}{V}\dot{\theta} - \frac{bc_1}{V}\psi - b\theta - \psi b\cos\lambda, \\
M_{\chi'} &= \frac{b_{22}}{V}\dot{x} - \frac{b_{12} + cb_{22}}{V}\dot{\theta} - \frac{1}{V}(b_{12}\cos\lambda - c_1 b_{22})\psi \\
&\quad + b_{22}\theta + (b_{22}\cos\lambda + b_{32}\sin\lambda)\psi - b_{32}\chi.
\end{aligned} \tag{2.55}
$$

In accordance with the expressions (2.53) and (2.55), we obtain

$$
\begin{aligned}
R_x &= F + F' = -\frac{2a_{21}}{V}\dot{x} + \frac{1}{V}(2a_{11} + ca_{21})\dot{\theta} + \frac{1}{V}(a_{11}\cos\lambda - c_1 a_{21})\psi \\
&\quad - 2a_{21}\theta - (a_{21}\cos\lambda - a_{32}\sin\lambda)\psi - (a_{31} + a_{32})\chi, \\
R_\chi &= M_\chi + M_{\chi'} = \frac{1}{V}(b_{21} + b_{22})\dot{x} - \frac{1}{V}(b_{11} + b_{12} + cb_{22})\dot{\theta} \\
&\quad - \frac{1}{V}(b_{12}\cos\lambda - c_1 b_{22})\psi + (b_{21} + b_{22})\theta + (b_{22}\cos\lambda + b_{32}\sin\lambda)\psi \\
&\quad - (b_{31} + b_{32})\chi, \\
R_\theta &= M_\theta + M_{\theta'} - cF' = -\frac{1}{V}(2b - ca_{21})\dot{x} - \frac{c}{V}(a_{11} - b + ca_{21})\dot{\theta} \\
&\quad - \frac{1}{V}[ca_{11}\cos\lambda + c_1(b - ca_{21})]\psi - (2b - ca_{21})\theta + ca_{32}\chi \\
&\quad - [(b - ca_{21})\cos\lambda + a_{32}\sin\lambda]\psi, \\
R_\psi &= c_1 F' + M_{\theta'}\cos\lambda - M_{\chi'}\sin\lambda = -\frac{1}{V}(c_1 a_{21} + b\cos\lambda \\
&\quad + b_{22}\sin\lambda)\dot{x} + \frac{1}{V}[bc\cos\lambda + (b_{12} + cb_{22})\sin\lambda + c_1(a_{11} + ca_{21})]\dot{\theta} \\
&\quad + \frac{1}{V}[(b_{12}\cos\lambda - c_1 b_{22})\sin\lambda - c_1 b\cos\lambda + c_1(a_{11}\cos\lambda - c_1 a_{21})]\psi \\
&\quad - (b\cos\lambda + b_{22}\sin\lambda + c_1 a_{21})\theta - [b\cos^2\lambda + (b_{22}\cos\lambda + b_{32}\sin\lambda)\sin\lambda \\
&\quad + c_1(a_{21}\cos\lambda - a_{32}\sin\lambda)]\psi - (c_1 a_{32} - b_{32}\sin\lambda)\chi.
\end{aligned}
$$

We now substitute these expressions into the right-hand sides of equations (2.48), whose left-hand sides are known [see (2.32)]. We then

perform the substitution (2.51) in the resulting equations and obtain the following system of differential equations, which describes the motion of a bicycle on wheels with pneumatic tires in the case of a fairly high velocity V:

$$
\begin{aligned}
&m\dot{u} + 2\varepsilon a_{21}u - l\dot{w} - [m + \varepsilon(2a_{11} + ca_{21})]w + h\ddot{\chi} \\
&\quad + \varepsilon(a_{31} + a_{32})\chi - m_2 d\ddot{\psi} - \varepsilon(a_{11}\cos\lambda - c_1 a_{21})\psi + \varepsilon(a_{21}\cos\lambda \\
&\qquad - a_{32}\sin\lambda)\psi = 0, \\
&-l\dot{u} + \varepsilon(2b - ca_{21})u + [l + \varepsilon c(a_{11} - b + ca_{21})]w + J_2\dot{w} \\
&\quad - J_{12}\ddot{\chi} + (C_1^0 + C_2^0)\dot{\chi} - \varepsilon ca_{32}\chi + J_4\ddot{\psi} - [C_2^0\sin\lambda \\
&\quad - \varepsilon(ca_{11}\cos\lambda + c_1(b - ca_{21}))]\psi + \varepsilon[(b - ca_{21})\cos\lambda \\
&\qquad + a_{32}\sin\lambda]\psi = 0, \\
&h\dot{u} - \varepsilon(b_{21} + b_{22})u - J_{12}\dot{w} - [h + C_1^0 + C_2^0 - \varepsilon(b_{11} + b_{12} + cb_{22})]w \\
&\quad + J_1\ddot{\chi} - \varepsilon(gh - b_{31} - b_{32})\chi - J_3\ddot{\psi} - [C_2^0\cos\lambda \\
&\quad - \varepsilon(b_{12}\cos\lambda - c_1 b_{22})]\psi + \varepsilon(gm_2 d - b_{22}\cos\lambda - b_{32}\sin\lambda)\psi = 0, \\
&-m_2 d\dot{u} + \varepsilon(c_1 a_{21} + b\cos\lambda + b_{22}\sin\lambda)u + J_4\dot{w} + \{m_2 d + C_2^0\sin\lambda \\
&\quad - \varepsilon[bc\cos\lambda + (b_{12} + cb_{22})\sin\lambda + c_1(a_{11} + ca_{21})]\}w - J_3\ddot{\chi} \\
&\quad + \dot{\chi}C_2^0\cos\lambda + \varepsilon(gm_2 d + c_1 a_{32} - b_{32}\sin\lambda)\chi + J\ddot{\psi} + \{\delta_1 \\
&\quad - \varepsilon[(b_{12}\cos\lambda - c_1 b_{22})\sin\lambda - c_1 b\cos\lambda + \dot{c}_1(a_{11}\cos\lambda - c_1 a_{21})]\}\psi \\
&\quad - \varepsilon[gm_2 d + c_1 a_{32}\sin\lambda - b\cos^2\lambda - (b_{22}\cos\lambda + b_{32}\sin\lambda)\sin\lambda \\
&\qquad - c_1 a_{21}\cos\lambda]\psi = 0.
\end{aligned}
\tag{2.56}
$$

Equations (2.56) describe the motion of the representative point in a six-dimensional phase space Φ, in which the coordinates are u, w, χ, $\dot{\chi}$, ψ, and $\dot{\psi}$.

We now turn to the second case, when the kinematic parameters α, β, and γ are fairly large. In this case, the values of F, M_θ, and M_χ are given by formulas (1.36). For the rear wheel

$$
F = a\xi_1 + \sigma N_1\chi, \quad M_\theta = b\varkappa_1\xi_1 - b\varkappa_2\chi, \quad M_\chi = -\sigma N_1\xi_1 - \rho N_1\chi, \tag{2.57}
$$

where $\kappa_1 = \alpha/\beta$ and $\kappa_2 = \gamma/\beta$ are the coefficients of creep. For the front wheel,

$$
F' = a\xi_2 + \sigma N_2\chi', \quad M_{\theta'} = b\varkappa_1\xi_2 - b\varkappa_2\chi', \quad M_{\chi'} = -\sigma N_2\xi_2 - \rho N_2\chi'
$$

or, in accordance with the relations (2.10),

$$
\begin{aligned}
F' &= a\xi_2 + \sigma N_2\chi - \psi\sigma N_2\sin\lambda, \quad M_{\theta'} = b\varkappa_1\xi_2 - b\varkappa_2\chi \\
&\quad + \psi b\varkappa_2\sin\lambda, \quad M_{\chi'} = -\sigma N_2\xi_2 - \rho N_2\chi + \psi\rho N_2\sin\lambda.
\end{aligned}
\tag{2.58}
$$

Substituting (2.58) and (2.57) into (1.17), we obtain

$$R_x = F + F' = a\xi_1 + a\xi_2 + \sigma (N_1 + N_2)\,\chi - \psi\sigma N_2 \sin\lambda,$$
$$R_\theta = M_\theta + M_{\theta'} - cF' = b\varkappa_1\xi_1 + (b\varkappa_1 - ac)\,\xi_2 - (2b\varkappa_2 + c\sigma N_2)\chi + (b\varkappa_2 + c\sigma N_2)\,\psi \sin\lambda,$$
$$R_\chi = M_\chi + M_{\chi'} = -\sigma N_1\xi_1 - \sigma N_2\xi_2 - \rho (N_1 + N_2)\,\chi + \psi\rho N_2 \sin\lambda,$$
$$R_\psi = c_1F' + M_{\theta'}\cos\lambda - M_{\chi'}\sin\lambda = (ac_1 + b\varkappa_1\cos\lambda + \sigma N_2 \sin\lambda)\,\xi_2 + (c_1\sigma N_2 + \rho N_2 \sin\lambda - b\varkappa_2\cos\lambda)\,\chi - [c_1\sigma N_2 + (\rho N_2 - b\varkappa_2)\cos\lambda]\,\psi \sin\lambda.$$

These expressions, together with (2.34), must be used in the derivation of the dynamical equations of a bicycle (2.48) in the case under consideration. Performing the substitution (2.51) in the resulting equations, we obtain

$$\begin{aligned}
&m\dot u - l\dot w - mw + h\ddot\chi - \varepsilon\sigma (N_1 + N_2)\,\chi - m_2 d\ddot\psi \\
&+ \psi\varepsilon\sigma N_2 \sin\lambda - \varepsilon a\xi_1 - \varepsilon a\xi_2 = 0, \\
&-l\dot u + J_2\dot w + lw - J_{12}\ddot\chi + (C_1^0 + C_2^0)\,\dot\chi + \varepsilon (2b\varkappa_1 \\
&+ c\sigma N_2)\,\chi + J_4\ddot\psi - \dot\psi C_2^0 \sin\lambda - \varepsilon (b\varkappa_2 + c\sigma N_2)\,\psi \sin\lambda \\
&- \varepsilon b\varkappa_1\xi_1 + \varepsilon (ac - b\varkappa_1)\,\xi_2 = 0, \\
&h\dot u - J_{12}\dot w - (h + C_1^0 + C_2^0)\,w + J_1\ddot\chi - \varepsilon [gh - \rho (N_1 \\
&+ N_2)]\,\chi - J_3\ddot\psi - \dot\psi C_2^0 \cos\lambda + \varepsilon (gm_2 d - \rho N_2 \sin\lambda)\,\psi \\
&+ \varepsilon\sigma N_1\xi_1 + \varepsilon\sigma N_2\xi_2 = 0, \\
&- m_2 d\dot u + J_4\dot w + (m_2 d + C_2^0 \sin\lambda)\,w - J_3\ddot\chi \\
&+ \dot\chi C_2^0 \cos\lambda + \varepsilon (gm_2 d + b\varkappa_2 \cos\lambda - c_1\sigma N_2 \\
&- \rho N_2 \sin\lambda)\,\chi + J\ddot\psi + \delta_1\dot\psi - \varepsilon [gm_2 d - c_1\sigma N_2 + (b\varkappa_2 \\
&- \rho N_2)\cos\lambda]\,\psi \sin\lambda - \varepsilon (ac_1 + b\varkappa_1 \cos\lambda \\
&+ \sigma N_2 \sin\lambda)\,\xi_2 = 0.
\end{aligned} \tag{2.59}$$

To these equations, we add the equations of the kinematic constraints (1.30), which, in this case, have the form

$$\begin{aligned}
&u + \dot\xi_1 + \dot\varkappa_1\xi_1 - \varkappa_2\chi = 0, \\
&-cw + c_1\dot\psi + (\cos\lambda + \varkappa_2 \sin\lambda)\,\psi - \xi_1 + \xi_2 - \varkappa_1\xi_1 + \varkappa_1\xi_2 = 0.
\end{aligned} \tag{2.60}$$

Equations (2.59) and (2.60) are the equations of motion of a bicycle on wheels with pneumatic tires in the case when the kinematic parameters α, β, and γ of the tires are fairly large. These equations describe the

motion of a representative point in an eight-dimensional phase space Φ in which the coordinates are u, w, χ, $\dot{\chi}$, ψ, $\dot{\psi}$, ξ_1, ξ_2.

It may be mentioned that if κ_2 is assumed to vanish in the equations of the kinematic constraints (2.60), the latter are then identical with equations (2.22) and (2.23), which were derived with allowance for only the transverse deformation of the tires. In order to obtain the dynamical equations in this case, it is sufficient to set $\sigma = \rho = b = 0$ in equations (2.59), since these coefficients are related to the deformation of the tire when the wheel is inclined and are also related to the torsional deformation of the tire, and both of these effects are ignored in Rocard's theory (see §1). Consequently, the equations of motion of a bicycle on wheels with pneumatic tires for the case when allowance is made for only transverse deformation of the tires can be written in the form

$$
\begin{aligned}
u + \dot{\xi}_1 + \varkappa_1 \xi_1 &= 0,\\
-cw + c_1\psi + \psi\cos\lambda - \dot{\xi}_1 + \dot{\xi}_2 - \varkappa_1\xi_1 + \varkappa_1\xi_2 &= 0,\\
m\dot{u} - l\dot{w} - mw + h\chi - m_2 d\dot{\psi} - \varepsilon a\xi_1 - \varepsilon a\xi_2 &= 0,\\
-l\dot{u} + J_2\dot{w} + lw - J_{12}\ddot{\chi} + (C_1^v + C_2^v)\,\dot{\chi} + J_4\dot{\psi}&\\
- \psi C_2^0\cos\lambda + \varepsilon\psi g m_2 d &= 0,\\
h\dot{u} - J_{12}\dot{w} - (h + C_1^0 + C_2^0)\,w + J_2\ddot{\chi} - \varepsilon g h\chi - J_3\dot{\psi} - \psi C_2^0\cos\lambda&\\
+ \varepsilon g m_2 d\psi &= 0,\\
-m_2 d\dot{u} + J_4\dot{w} + (m_2 d + C_2^0\sin\lambda)\,w - J_3\ddot{\chi} + \dot{\chi}C_2^0\cos\lambda&\\
+ \varepsilon g m_2 d\chi + J\dot{\psi} + \delta_1\psi - \varepsilon\dot{\psi} g m_2 d\sin\lambda - \varepsilon a c_1\xi_2 &= 0.
\end{aligned}
\tag{2.61}
$$

Although the system of equations (2.61) is simpler than (2.59) and (2.60), the number of dimensions of the phase space Φ is still equal to eight.

Finally, let us derive the equations of motion of a bicycle using Rocard's hypothesis of creep. According to this hypothesis, wheels with pneumatic tires are assumed to be quasi-rigid ($\xi_1 = \xi_2 = 0$) but subject to the phenomenon of creep. Under such a hypothesis, the wheel is subjected to a transverse force proportional to the value of the angle of creep. The angle of creep δ of the rear wheel is given by

$$\delta \approx \tan\delta = (\dot{x} + V\theta)/V,$$

and the transverse force F acting on the rear wheel is

$$F = -\frac{a}{V}\dot{x} - a\theta, \tag{2.62}$$

where a ($a > 0$) is a proportionality coefficient, which we shall assume

has the same value for both wheels of the bicycle. The analogous force acting on the front wheel of the bicycle is

$$F' = -\frac{a}{V}\dot{x}' - a\theta'.$$

Comparing the expressions (2.62) and (2.53), one can readily see that (2.62) can be obtained from (2.53) for $a_{11} = a_{31} = b = b_{11} = b_{21} = b_{31} = 0$, and $a_{21} = a$. If allowance is made for only the transverse force F, the moments M_θ and M_χ vanish. In order to obtain the corresponding expressions for F', $M_{\theta'}$, and $M_{\chi'}$ one must set $a_{12} = a_{32} = b = b_{12} = b_{22} = b_{32} = 0$ and $a_{22} = a$ is (2.54).

Hence, the equations of motion of a bicycle on wheels with pneumatic tires can be obtained on the basis of Rocard's hypothesis of creep from equations (2.56) if one assumes $a_{11} = a_{31} = a_{12} = a_{32} = b = b_{11} = b_{21} = b_{31} = b_{12} = b_{22} = b_{32} = 0$ and $a_{21} = a_{22} = a$ in the latter. As a result, we obtain the equations

$$\begin{aligned}
&m\dot{u} + 2\varepsilon a u - l\dot{w} - (m + \varepsilon a c)\, w + h\ddot{\chi} - m_2 d\ddot{\psi} + \varepsilon a c_1 \psi \\
&\qquad + \varepsilon a \psi \cos\lambda = 0, \\
&-l\dot{u} - \varepsilon a c u + (I + \varepsilon a c^2)\, w + J_2 \dot{w} - J_{12}\ddot{\chi} \\
&+ (C_1^0 + C_2^0)\,\dot{\chi} + J_4 \ddot{\psi} - (C_2^0 \sin\lambda + \varepsilon a c c_1)\,\psi \\
&\qquad - \varepsilon a c \psi \cos\lambda = 0, \\
&h\dot{u} - J_{12}\dot{w} - (h + C_1^0 + C_2^0)\, w + J_1 \ddot{\chi} - \varepsilon g h \chi - J_3 \ddot{\psi} \\
&\qquad - \psi C_2^0 \cos\lambda + \varepsilon g m_2 d\psi = 0, \\
&-m_2 d\dot{u} + \varepsilon a c_1 u + J_4 \dot{w} + (m_2 d + C_2^0 \sin\lambda - \varepsilon a c c_1)\, w \\
&- J_3 \ddot{\chi} + \dot{\chi} C_2^0 \cos\lambda + \varepsilon g m_2 d\chi + J\ddot{\psi} + (\delta_1 + \varepsilon a c_1^2)\,\psi \\
&\qquad - \varepsilon (g m_2 d - a c_1 \cos\lambda)\,\psi = 0.
\end{aligned} \tag{2.63}$$

Equations (2.63) describe the motion of the representative point in a six-dimensional phase space $\Phi(u, w, \chi, \dot{\chi}, \psi, \dot{\psi})$.

Thus, in the study of the dynamics of an uncontrolled bicycle, we obtain a differential equation whose order n depends on the specific mathematical model. If the models discussed above are arranged in order of increasing n, we obtain the following series, in which the relevant equations are given in the brackets:

$n = 4$ (2.36), (2.38) for a bicycle with disc-like rigid wheels,
$n = 4$ (2.42), (2.44) for a bicycle with toroidal rigid wheels,
$n = 6$ (2.63) for Rocard's hypothesis of creep,
$n = 6$ (2.56) for a bicycle on pneumatic tires and a fairly high velocity V,

$n = 8$ (2.59), (2.60) for the case of fairly large values of the kinematic parameters α, β, and γ of the tires,

$n = 8$ (2.61) when allowance is made for only transverse deformation of the tires,

$n = 10$ (2.52) for a bicycle on pneumatic tires when Keldyš's theory is used.

3. Stability of the equations of simplified models of an uncontrolled bicycle. a) The simplest model of a bicycle is obtained when it is assumed that the axis of the steering column is vertical and that it passes through the center of the front wheel and is a principal axis of the central ellipsoid of inertia of the front part of the bicycle. If these conditions are satisfied, the parameters of the bicycle satisfy the equations

$$c_1 = \lambda = d = D_2 = 0. \tag{2.64}$$

It follows from the expressions (2.33) that, if (2.64) is satisfied,

$$J_3 = 0,\ J_4 = B_2,\quad J = B_2,\ J_{13} = m_1 h_1 l_1 + D_1 + m_2 h_2 l_2,$$

and that the remaining coefficients are unchanged. The equations (2.36) and (2.38) of motion of a bicycle on rigid disc-like wheels can be simplified and take the form

$$\begin{aligned} cJ_1\ddot{\chi} - cgh\chi - V(J_{12} + cC_2^0)\psi - (h + C_1^0 + C_2^0)V^2\psi = 0,\\ cB_2\ddot{\psi} + (B_2 + c\delta_1)V\psi + cC_2^0 V\dot{\chi} = 0. \end{aligned} \tag{2.65}$$

If we assume $J_1 = mh_0^2$, $J_{12} = mh_0 l_0$, and $h = mh_0$, where l_0 and h_0 are the coordinates of the center of mass of the complete system and we neglect the moments of inertia of the wheels C_1 and C_2 compared with the quantity $mh_0 R$, the first equation of (2.65) can be transformed into the well-known equation[6)]

[6)] It is worth pointing out that equation (2.66) enables one, for example, to explain a rule of bicycling. This rule is as follows: In order to prevent the bicycle toppling, one must turn the handlebars in the direction in which the bicycle is beginning to fall. The same result can be obtained from equation (2.66). For suppose a bicycle which is moving along a straight line begins to topple to the right, i.e., $\dot{\chi} > 0$. In order to prevent a fall, one must create a negative second derivative of the angle χ, i.e., as is clear from equations (2.66), one must create a negative value of $\dot{\psi}$. This means that the cyclist must turn the handlebars to the right, i.e., in the direction in which the bicycle is falling. In addition, the presence of a negative derivative $\dot{\psi}$ leads to a negative value of the angle ψ, $\psi < 0$, and this, in accordance with equation (2.6), gives rise to an additional increase of $\ddot{\chi}$ in absolute magnitude.

$$\ddot{\chi} = \frac{l_0 V}{h_0 c}\psi + \frac{g}{h_0}\left(\chi + \frac{V^2}{cg}\psi\right), \tag{2.66}$$

which leads to the elementary theory of a bicycle. (See, for example, the book of Loĭcjanskiĭ and Lur'e [280]). Let us investigate the equations (2.65) of motion of the simplified model of a bicycle. We introduce the new variables $\chi_{new} = RC_2^0\chi/B_2$, and we shall take the dimensionless path length $s = Vt/R$ as an independent variable. Equations (2.65) can then be written in the form

$$\begin{aligned} \frac{d^2\chi}{ds^2} - \nu\chi - A\frac{d\psi}{ds} - B\psi &= 0, \\ \frac{d^2\psi}{ds^2} + \tau\frac{d\psi}{ds} + \frac{d\chi}{ds} &= 0, \end{aligned} \tag{2.67}$$

where

$$\begin{aligned} A &= \frac{C_2^0 R^2}{cB_2 J_1}(J_{12} + cC_2^0), \quad B = \frac{C_2^0 R^3}{cB_2 J_1}(h + C_1^0 + C_2^0), \\ \tau &= \frac{R}{cB_2}\left(B_2 + \frac{c\delta}{V}\right), \quad \nu = \frac{ghR^2}{J_1 V^2} \end{aligned} \tag{2.68}$$

are dimensionless parameters.

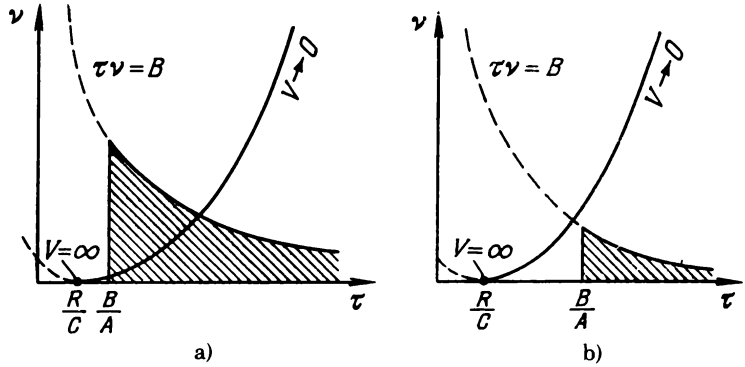


FIGURE 6.15

The parameters τ and ν depend on the velocity V of the bicycle. According to (2.67), the values of the variables χ and ψ in an equilibrium state satisfy the equation

$$\nu\chi + B\psi = 0, \tag{2.69}$$

from which it follows that the system possesses a one-dimensional manifold of equilibrium states, whose stability is determined by the roots of the characteristic equation

(2.70) $$p^3 + \tau p^2 + (A - \nu)\, p + B - \tau\nu = 0.$$

Hence, we find that the manifold is stable if the following inequalities are satisfied:

(2.71) $$\tau > \frac{B}{A}, \quad \tau\nu < B \quad (\tau > 0).$$

In the first quadrant of the (τ, ν) plane, these inequalities determine the stability regions shaded in Figure 6.15. In order to investigate the stability of the model as a function of the velocity V of the bicycle, we plot in Figure 6.15 the parabola

(2.72) $$(\tau - \rho)^2 = D\nu,$$

where

$$\rho = \frac{R}{c}, \quad D = \frac{J_1\delta^2}{ghB_2^2},$$

which is obtained by eliminating V from the expression (2.68). As V varies from zero to $+\infty$, the representative point passes along the right-hand branch of the parabola (2.72), moving from infinity to the point $(\rho, 0)$. As follows from Figure 6.15, two cases are possible: In the case of Figure 6.15a, there exists an interval of velocities $V_1 < V < V_2$ of stable motion of the bicycle, whereas in the case of Figure 6.15b the bicycle is unstable for all velocities. The relationships between the parameters that distinguish these two cases can be found from the condition of coincidence of the top left-hand point of the region of stability in Figure 6.15 with one of the points of the parabola (2.72). Hence, we obtain the inequality

(2.73) $$\delta^2 \geqslant \frac{ghB_2^3\,[c\,(h + C_1^0) - J_{12}]^2}{cC_2^0\,(J_{12} + cC_2^0)^3} = \delta_0^2,$$

which, if satisfied, ensures the existence of a stability interval $V_1 < V < V_2$. From the condition for the intersection of the parabola (2.72) with the boundaries of the stability region in Figure 6.15, we find that

(2.74) $$\nu_2 = \frac{(B - \rho A)^2}{DA^2},$$

where $\nu_2 = ghR^2/J_1 V_2^2$ and the value of $\nu_1 = ghR^2/J_1 V_1^2$ is given by the real root of the equation

(2.75) $$D\nu^3 - (\rho\nu - B)^2 = 0.$$

The condition (2.73) means that there exists an interval of values for which the motion of the bicycle is stable if the coefficient δ of viscous friction in the steering column exceeds a certain value δ_0, which depends on the remaining parameters of the bicycle. If this is not the case, the motion of this model of a bicycle is unstable. It follows from the condition (2.73) that the interval of velocities $V_1 < V < V_2$ of stable motion is also preserved for an infinitely large value of the coefficient δ of viscous friction in the steering column. This at first sight paradoxical result is explained by the fact that, as $\delta \to \infty$, the interval of velocities of stable motion of the bicycle is shifted toward infinity. For the lower boundary of this interval is determined by the real roots of equations (2.75). For a sufficiently large value of the coefficient $D = J_1\delta^2/ghB_2^2$ [the other coefficients of equation (2.75) do not depend on the parameter δ], this equation has a unique real root. For the function $f(\nu) = D\nu^3 - (\rho\nu - B)^2$, where the parameter ν varies in the interval $0 < \nu < \infty$, is negative at $\nu = 0$ and increases monotonically, so that the derivative

$$f'(\nu) = 3D\nu^2 - 2\rho(\rho\nu - B)$$

is always positive. However, it is evident from (2.75) that the real root ν_1 of this equation tends to zero as $D \to \infty$; hence, the value of the limiting velocity $V_1^2 = ghR^2/J\nu_1$, and with it the interval of stable velocities $V_1 < V < V_2$, tends to infinity.

b) Let us now consider the simplest model of a bicycle on wheels with pneumatic tires. If the conditions (2.64) are satisfied, equations (2.63), obtained on the basis of Rocard's hypothesis of creep, can be simplified and take the form

$$\begin{aligned}
&m\dot{u} + 2\varepsilon a u - l\dot{w} - (m + \varepsilon c a)w + h\ddot{\chi} + \varepsilon a\psi = 0,\\
&-l\dot{u} - \varepsilon c a u + J_2\dot{w} + (l + \varepsilon a c^2)w - J_{12}\ddot{\chi}\\
&\qquad + (C_1^0 + C_2^0)\dot{\chi} + B_2\dot{\psi} - \varepsilon c a\psi = 0,\\
&h\dot{u} - J_{12}\dot{w} - (h + C_1^0 + C_2^0)w + J_1\ddot{\chi} - \varepsilon g h\chi - C_2^0\psi = 0,\\
&B_2\dot{w} + C_2^0\dot{\chi} + B_2\dot{\psi} + \delta_1\psi = 0.
\end{aligned} \tag{2.76}$$

The equilibrium states of the system of equations (2.76) satisfy the equations

$$\begin{aligned}
&2a\varepsilon u - (m + ac\varepsilon)w + a\varepsilon\psi = 0,\\
&ac\varepsilon u - (l + a\varepsilon c^2)w + a\varepsilon c\psi = 0,\\
&(h + C_1^0 + C_2^0)w + \varepsilon g h\chi = 0,
\end{aligned} \tag{2.77}$$

from which it follows that the equilibrium values of the variables u, w, χ, and ψ are connected by the equation

$$ca\,(h + C_1^0 + C_2^0)\,\psi + gh\,(2l - mc + \varepsilon ac^2)\,\chi = 0 \tag{2.78}$$

and therefore form a one-dimensional manifold. The stability of this manifold is determined by the roots of the fifth-order characteristic equation

$$\begin{vmatrix} mp + 2a\varepsilon & -lp - (m + ac\varepsilon) & hp^2 & a\varepsilon \\ -lp - ac\varepsilon & J_2p + l + a\varepsilon c^2 & -J_{12}p^2 + (C_1^0 + C_2^0)p & B_2p^2 - a\varepsilon c \\ hp & -J_{12}p - (h + C_1^0 + C_2^0) & J_1p^2 - gh\varepsilon & -C_2^0 \\ 0 & B_2 & C_2^0 & B_2p + \delta_1 \end{vmatrix} = 0. \tag{2.79}$$

Introducing $\chi_{\text{new}} = RC_2^0\chi/B_2$ and the dimensionless path length $s = Vt/R$, and using the notation (2.68), we can write equation (2.76) in the form

$$B\psi + \nu(1 + \mu B_1)\,\chi = 0. \tag{2.78'}$$

Here, the dimensionless parameters μ and B_1 are given by the expressions

$$\mu = \frac{m}{ac\varepsilon}, \quad B_1 = \frac{2l}{mc} - 1. \tag{2.80}$$

If the transverse rigidity (hardness) of the pneumatic tire tends to infinity ($a \to \infty$), we again recover the model with rigid wheels. In this case, the parameter μ tends to zero; equation (2.78′) goes over into (2.69), as can be seen directly from (2.78′), and the equations of motion (2.76) are transformed formally into equations (2.65) or, equivalently, into equations (2.67). For as $a \to \infty$, the first two equations of (2.76) yield

$$2u - cw + \psi = 0, \quad u - cw + \psi = 0,$$

from which it follows that

$$u = 0, \quad cw = \psi. \tag{2.81}$$

Substituting these values into the remaining equations of the system (2.76), we arrive at equations (2.65).

As a result of this passage to the limit, the order of the system of differential equations is reduced by two, and it is therefore necessary to establish the conditions under which such a limiting process is in order.[7] In other words, under what conditions is the dynamics of our

[7] The various questions connected with such limiting processes were considered in Chapter IV.

model of a bicycle on wheels with pneumatic tires approximated by the dynamics of the model of a bicycle with absolutely rigid wheels?

In order to answer this question, let us consider the characteristic equation (2.79). If the parameter a has a fairly large value, this equation can be written in the form

$$\mu^2 C p^5 + \mu D p^4 + p^3 + \tau p^2 + (A - \nu)\, p + B - \tau\nu = 0, \tag{2.82}$$

where we have used the notation of (2.68) and (2.80) and introduced the dimensionless parameters

$$\begin{aligned} C &= \frac{1}{m^2 R^2 J_1}\,[m\,(J_1 J_2 - J_{12}^2) + 2lhJ_{12} + h^2 B_2 \\ &\qquad - J_1\,(mB_2 + l^2) - h^2 J_2], \\ D &= \frac{1}{mcRJ_1}\,[2\,(J_1 J_2 - J_{12}^2) + J_1\,(mc^2 - 2B_2 \\ &\qquad - 2lc) + ch\,(2J_{12} - ch)] \end{aligned} \tag{2.83}$$

neglecting small terms of higher orders in the small parameter μ. If we set $\mu = 0$ in the characteristic equation (2.82), we see immediately that it is identical with the characteristic equation (2.70). However, as $\mu \to 0$ the coefficients of the highest powers of p tend to zero, i.e., in the passage to the limit two roots of the characteristic equation (2.82) tend to infinity. Therefore, in order to answer our question, we must find the conditions under which the real parts of these two roots are negative, since only then will the system be stable with respect to the rapid motions corresponding to these two roots, i.e., only then will the simplified model of a bicycle with rigid wheels be correct. We introduce temporarily the notation

$$\sigma = \mu^2 C, \quad \rho = \mu D, \quad a = A - \nu, \quad b = B = \tau\nu \tag{2.84}$$

and write equations (2.82) in the form

$$\sigma p^5 + \rho p^4 + p^3 + \tau p^2 + ap + b = 0. \tag{2.85}$$

Suppose that the system is stable for $\rho = \sigma = 0$, i.e., the inequalities

$$\tau > 0, \quad b > 0, \quad a\tau - b > 0$$

hold. Let us now consider equation (2.85) for $\rho \neq 0$ and $\sigma \neq 0$. Making a D-decomposition [321] of the plane of the parameters (σ, ρ), we obtain the diagram shown in Figure 6.16, in which

$$\lim_{\omega\to\infty} \frac{\rho}{\sigma} = \tau > 0, \quad \lim_{\omega\to 0} \frac{\rho}{\sigma} = \frac{b}{a} > 0, \quad (\sigma)_{\omega^2 = a} = 0,$$

$$(\rho)_{\omega^2 = a} = \frac{\tau a - b}{a^2} > 0, \quad (\rho)_{\tau\omega^2 = b} = 0, \quad (\sigma)_{\tau\omega^2 = b} = \frac{\tau\,(b - a\tau)}{b^2} < 0.$$

In the case of infinite hardness of the pneumatic tire ($\mu = 0$), the representative point is situated at the origin of the coordinates on the (σ, ρ) plane. In accordance with the assumption we have made, the system is then stable. If the value of μ increases from zero, the representative point moves along the parabola $\rho^2 = D^2\sigma/C$ obtained by eliminating μ from the expressions of (2.84). A necessary condition for the representative point to move into the stability region shaded in Figure 6.16 when it is displaced from the origin is the fulfilment of the inequalities

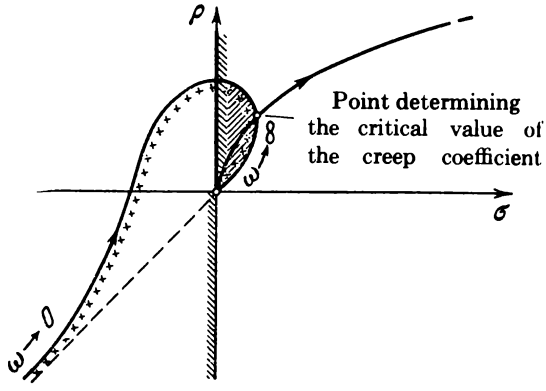


FIGURE 6.16

$$C > 0, \quad D > 0, \tag{2.86}$$

where C and D are given by the expressions (2.83). In this case, the stability of the system is preserved for $\mu \neq 0$. Thus, if the inequalities (2.68) are satisfied, the stability conditions obtained for the model of a bicycle on rigid wheels also remain valid for the model of a bicycle on wheels with pneumatic tires. However, as we see from Figure 6.16, for this to be the case the transverse hardness of the tires must not be less than a critical value determined by the condition for the intersection of the parabola $\rho^2 = D^2\sigma/C$ and the boundary of the stability region. The inequalities (2.86) are satisfied for bicycles of the usual constructions.

c) Finally, let us consider the question of the influence on the stability of a bicycle of parameters such as the stagger c_1 of the front wheel, the angle λ of inclination of the steering column, and the distance d from the center of mass of the front part of the bicycle to the steering column. In order to simplify the calculations, we shall assume that the centrifugal moment of inertia of the front part of the bicycle vanishes ($D_2 = 0$) and that the wheels of the bicycle can be regarded as absolutely rigid discs. In this case, the motion of the bicycle is described by equations (2.36) and (2.38). These equations possess a unique equilibrium state at the origin ($\chi = \psi = 0$), which corresponds to rectilinear motion of the bicycle. Thus, allowance for the stagger c_1 of the front wheel, the inclination λ of

the front part of the bicycle from the steering column reduces the manifold of equilibrium states in the phase space $(\chi, \dot{\chi}, \psi, \dot{\psi})$ into the unique equilibrium state $\chi = \dot{\chi} = \psi = \dot{\psi} = 0$. The stability of this equilibrium state is determined by the roots of the fourth-order characteristic equation

$$\alpha_0 p^4 + \alpha_1 p^3 + (\alpha_2 - \varepsilon\alpha_3)\, p^2 + (\alpha_4 - \varepsilon\alpha_5)\, p + \varepsilon\,(\varepsilon\alpha_6 - \alpha_7) = 0, \tag{2.87}$$

where

$$\begin{aligned} \alpha_0 &= a_0 b_0 - a_2 b_4, & \alpha_4 &= a_5 b_5, \\ \alpha_1 &= a_0 b_1 + a_2 b_5 - a_3 b_4, & \alpha_5 &= a_1 b_1 + a_4 b_5 - a_3 b_6, \\ \alpha_2 &= a_0 b_2 + a_3 b_5 - a_5 b_4, & \alpha_6 &= a_1 b_3 - a_4 b_6, \\ \alpha_3 &= a_1 b_0 + a_0 b_3 - a_2 b_6 - a_4 b_4, & \alpha_7 &= a_1 b_2 - a_5 b_6, \end{aligned} \tag{2.88}$$

the quantities a_i and b_i $(i = 0, 1, 2, \cdots, 6)$ are given by the expressions (2.37) and (2.39) and the quantity ϵ is, in accordance with (2.51), inversely proportional to the square of the velocity of the bicycle.

Equation (2.87) is well known in the literature (see, for example, Dohring's paper [144]) and has been used for numerical calculations for specific constructions of motorcycles.

For the normal constructions of a bicycle, the coefficients α_i $(i = 1, 2, \cdots, 7)$ in equation (2.87) are positive. It follows immediately from the form of equation (2.7) that there exists only an interval of velocities of stable motion of the bicycle for the model under consideration.

In order to be able to compare this model with the simplest model of a bicycle $(\lambda = d = c_1 = 0)$, we shall assume that the values of λ, d, and c_1 are fairly small and that the terms containing products of these quantities can be neglected in the coefficients α_i of the characteristic equation (2.87). Then, introducing dimensionless quantities and using the notation (2.68), we can write the characteristic equation (2.87) in the form

$$\begin{aligned} &(1 + 2\sigma)\, p^4 + (\tau + \beta_1\rho - \beta_2\lambda + \beta_3\sigma)\, p^3 + [A - \nu \\ &\quad + \beta_4\rho + \beta_5\lambda + (\beta_6 - 2\nu)\,\sigma]\, p^2 + [B - \tau\nu - \beta_7\nu\rho \\ &\quad + (\beta_8 - \beta_9\nu)\,\sigma]\, p + \nu\,(\beta_{10}\rho - \beta_5\lambda - \beta_{11}\sigma) = 0. \end{aligned} \tag{2.89}$$

Here, $\sigma = c_1/c$ is the dimensionless stagger of the front wheel; $\rho = d/c$ is the distance d divided by the base c of the bicycle; and the positive parameters β_i are given by the expressions

$$
\begin{aligned}
\beta_1 &= \frac{m_2 R}{J_1 B_2}(J_1 l_2 - J_{12} h_2), & \beta_6 &= \frac{1}{cJ_1 B_2}[lJ_1 R^2 + chC_2 R \\
& & & \quad + (C_1 + C_2)(J_{12} R + 2cC_2], \\
\beta_2 &= \frac{A_2 J_{12} R}{cJ_1 B_2}, & \beta_7 &= \frac{m_2 R (lh_2 - J_{12})}{hB_2}, \\
\beta_3 &= \frac{R\left(J_1 J_2 - J_{12}^2\right)}{cJ_1 B_2}, & \beta_8 &= \frac{h(C_1 + C_2)}{cJ_1 B_2}(hR + C_1 + C_2), \\
\beta_4 &= \frac{m_2 R^2}{B_2}, & \beta_9 &= \frac{J_2 R}{cB_2}, \\
\beta_5 &= \frac{C_2 R}{cB_2}, & \beta_{10} &= \frac{m_2 R (C_1 + C_2)}{hB_2}; \quad \beta_{11} = \frac{lR^2}{cB_2}.
\end{aligned} \tag{2.90}
$$

The region of stability of the equilibrium state $\psi = \chi = 0$ is determined by the inequalities

$$
\begin{aligned}
\beta_{10}\rho - \beta_5\lambda - \beta_{11}\sigma &> 0, \\
(A\tau - B)(B - \tau\nu) + C\rho + D\lambda + E\sigma &> 0, \\
\tau + \beta_1\rho - \beta_2\lambda - \beta_3\sigma &> 0, \\
A\tau - B + A_1\rho + A_2\lambda + A_3\sigma &> 0,
\end{aligned} \tag{2.91}
$$

where

$$
\begin{aligned}
C &= AB\beta_1 + \beta_4 B\tau + B(2\beta_7 - \beta_1)\nu \\
&\quad - (\beta_1 + \beta_7) A\tau\nu - (\beta_4 + \beta_{10})\tau^2\nu + (\beta_1 - \beta_7)\tau\nu^2, \\
D &= -AB\beta_2 + \beta_5 B\tau + \beta_2 B\nu + A\tau\nu - \beta_2\tau\nu^2, \\
E &= -B(2B + \beta_3 A + 2\beta_8) + (\beta_8 A + \beta_6 B)\tau \\
&\quad + B(2\beta_9 + \beta_3)\nu + (2B + \beta_8 - \beta_9 A)\tau\nu \\
&\quad - (\beta_3 + \beta_9)\tau\nu^2 - (\beta_6 - \beta_{11})\tau^2\nu, \\
A_1 &= \beta_1(A - \nu) + \tau\beta_4 + \beta_7\nu, \\
A_2 &= -\beta_2(A - \nu) + \tau\beta_5, \\
A_3 &= \beta_3 A - 2B - \beta_8 + \tau\beta_6 + (\beta_9 - \beta_3)\nu.
\end{aligned} \tag{2.92}
$$

For $\rho = \lambda = \sigma = 0$, the stability conditions (2.91) go over into the conditions (2.71). If the parameters ρ, λ, and σ all vanish, the unique equilibrium state corresponding to rectilinear motion of the bicycle is transformed into a one-dimensional manifold corresponding to circular motions. This corresponds to the fact that the characteristic equation (2.89) acquires a vanishing root. If, however, the parameters do not vanish, this root becomes negative if the condition

$$
\frac{\beta_{10}\rho - \beta_5\lambda - \beta_{11}\sigma}{B - \tau\nu} > 0 \tag{2.93}
$$

is satisfied. Since $b - \tau\nu > 0$ in the stability region of the simplified model, the condition (2.93) goes over into the first condition of (2.91). The remaining conditions of (2.91) go over into the stability conditions (2.71).

It may be noted that the stability conditions (2.91) show that if λ and σ are positive (which corresponds to the usual construction of a bicycle), the parameter ρ must be positive. Physically, this condition means that the front part of the bicycle must have static instability with respect to transverse tilts.

4. Stability of a bicycle for riding with "no hands". Hitherto, we have considered a model of a bicycle in which the rider is rigidly fixed to the frame. In reality, the rider on a bicycle frequently makes small movements to the right or left, often without realizing he is doing so. Such movements of the rider are more apparant in the case of riding with "no hands", i.e., when the rider sits straight upward and takes his hands from the handlebars. Let us consider the effects of sideward movement of the rider on the stability of the simplest model of a bicycle. To this end, we add a mass m_0 to the construction of our earlier model; this mass can slide along a weightless rod fixed at right angles to the frame at the center of mass of the rear part of the bicycle (Figure 6.17). This mass m_0 will give rise to additional terms in the expression L for the Lagrange function (2.31). In order to find these expressions, we use the formulas which express the coordinates of the mass m_0 in terms of the generalized coordinates:

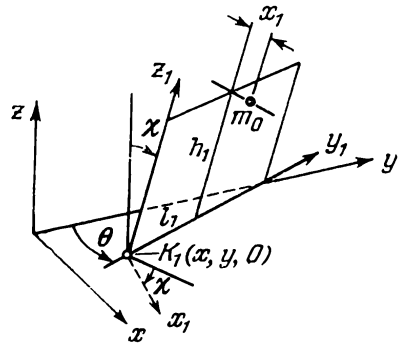


FIGURE 6.17

$$
\begin{aligned}
x_0 &= x - l_1 \sin\theta + h_1 \cos\theta \sin\chi + x_1 \cos\theta \cos\chi, \\
y_0 &= y + l_1 \cos\theta + h_1 \sin\theta \sin\chi + x_1 \sin\theta \cos\chi, \\
z_0 &= h_1 \cos\chi - x_1 \sin\chi.
\end{aligned} \tag{2.94}
$$

Differentiating these relations with respect to the time and squaring and adding them, we obtain the square of the velocity of the mass m_0. We then use the condition $\dot{y} = V = \text{const}$ and omit the constant terms and the small terms of third order and higher. As a result, we obtain, retaining the notation m_1 for the total mass $m_1 + m_0$ of the rear part of the bicycle, the following expression for the additional kinetic energy $\tilde{T}$ due to the presence of the mass m_0 on the rod:

$$\widetilde{T} = \frac{1}{2} m_0 \dot{x}_1^2 + m_0 \dot{x}_1 (\dot{x} - l_1 \dot{\theta} + h_1 \dot{\chi}) + m_0 V (\theta \dot{x}_1 + x_1 \dot{\theta}).$$

In accordance with (2.94), the additional part of the potential energy $\widetilde{U}$ can be expressed in the form

$$\widetilde{U} = - m_0 g x_1 \chi.$$

Hence, the additional part of the Lagrange function $\widetilde{L}$ has the form

$$\widetilde{L} = \frac{1}{2} m_0 \dot{x}_1^2 + m_0 \dot{x}_1 (\dot{x} - l_1 \dot{\theta} + h_1 \dot{\chi}) + m_0 V (\theta \dot{x}_1 + x_1 \dot{\theta}) + m_0 g x_1 \chi. \tag{2.95}$$

Thus, we obtain the additional terms in (2.32):

$$\widetilde{\mathcal{L}}_x = m_0 \ddot{x}_1, \quad \widetilde{\mathcal{L}}_\theta = - m_0 l_1 \ddot{x}_1, \quad \widetilde{\mathcal{L}}_\chi = m_0 h_1 \ddot{x}_1 - m_0 g x_1, \quad \widetilde{\mathcal{L}}_\psi \equiv 0. \tag{2.96}$$

In accordance with (2.65), the equation of motion of the simplified model of a bicycle on rigid disc-like wheels has the form

$$\mathcal{L}_\chi + \widetilde{\mathcal{L}}_\chi = 0, \quad \mathcal{L}_\psi + \frac{\partial F}{\partial \dot{\psi}} = 0.$$

Substituting the expressions for the operators (2.32) and (2.96) into the equation of motion and using the equations of the nonholonomic constraints, we obtain

$$\begin{aligned} c J_1 \ddot{\chi} - cgh\chi - V (J_{12} + cC_2^0) \dot{\psi} - (h + C_1^0 + C_2^0) V^2 \psi \\ + c m_0 h_1 \ddot{x}_1 - c m_0 g x_1 = 0, \\ c B_2 \ddot{\psi} + (B_2 + c\delta_1) V \dot{\psi} + c C_2^0 V \dot{\chi} = 0. \end{aligned} \tag{2.97}$$

The influence of the movements of the rider on the stability depends on his reaction to the variations of the various parameters. The rider may react to the tilt of the bicycle or to the rate of change of this tilt, to the angle through which the handlebar is turned or to the rate of change of this angle. In the limit of small variations of all these quantities, this dependence can be represented linearly as follows

$$T_1 \dot{x}_1 + x_1 = - \beta_1^0 \psi - \beta_2^0 \dot{\psi} - \gamma_1^0 \chi + \gamma_2^0 \dot{\chi}, \tag{2.98}$$

where β_1^0, β_2^0, γ_1^0, and γ_2^0 are constant coefficients and T_1 is the time constant of the rider. The equations (2.97) and (2.98) of motion of the bicycle can be used to study its directional stability in the case under consideration. The fact that these are the equations of motion of the system can be seen from the following considerations. The relation (2.98) is the equation of a nonholonomic servo constraint. This servo constraint is realized by forces whose virtual work does not, in general, vanish. In the present case, these forces are the forces of reaction of

the mass m_0 on the remaining part of the system. Since x_1 is the relative displacement of the mass m_0, the virtual work of the reaction forces for $\delta x_1 = 0$ vanishes. Consequently, the existence of forces that realize the servo constraint in no way affects the dynamical equations derived for all the generalized coordinates of the system except x_1. As regards the coordinate x_1, an additional generalized force Q_{x_1} appears on the right-hand side of the equation corresponding to this coordinate. This additional force causes the mass m_0 to move in accordance with the law (2.98). Thus, the dynamical equation for the coordinate x_1 is only needed to determine the generalized force Q_{x_1}. Since we are not concerned with the magnitude of the force Q_{x_1} that realizes the servo constraint, we can ignore the corresponding dynamical equation. The motion itself, i.e., the variation of the generalized coordinates of the system, including the coordinate x_1, is completely determined by the system of differential equations (2.97) and (2.98).

It is easily seen that the above arguments concerning the derivation of equations of motion of a system with servo constraints can be generalized to the following case. Suppose that the system S, which has the Lagrange function

$$L = L\,(q_\alpha, \dot{q}_\alpha, x_\beta, \dot{x}_\beta, t) \quad (\alpha = 1, 2, \ldots, n;\ \beta = 1, 2, \ldots, m)$$

can be represented as two interacting subsystems S_q and S_x such that the virtual work of the reaction forces of the subsystems S_q and S_x vanish for $\delta x_1 = \delta x_2 = \cdots = \delta x_m = 0$. In addition, let us assume that the servo constraints existing in the system and determined by the equations

(A) $f_\beta\,(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n; x_1, \ldots, x_m, \dot{x}_1, \ldots, \dot{x}_m) = 0\ (\beta = 1, 2, \ldots, m)$,

are realized by means of certain internal forces in the system S_x and reaction forces between S_x and S_q satisfying the conditions formulated above. The equations of motion of the system S can then be written in the form

(B) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \qquad (\alpha = 1, 2, \ldots, n),$$

(C) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\beta} - \frac{\partial L}{\partial x_\beta} = X_\beta \qquad (\beta = 1, 2, \ldots, m),$$

where X_β are the generalized forces acting on the subsystem S_x. It follows from the form of these equations that, if we wish to determine the motion of the system, it is sufficient to take the equations (A)

and (B), which are a complete system of $n + m$ equations for the $n + m$ unknowns q_α and x_β ($\alpha = 1, 2, \cdots, n$; $\beta = 1, 2, \cdots, m$). The equations (C) only serve to determine the generalized forces X_β ($\beta = 1, 2, \cdots, m$) that ensure the relations (A) are satisfied.

In order to be able to compare the results obtained with the preceding results, we shall retain the previous notation (2.68) and introduce the dimensionless quantities

$$\chi_{\text{new}} = \frac{C_2}{B_2}\chi, \qquad s = \frac{Vt}{R}, \qquad x_{1\,\text{diml}} = \frac{m_0 C_2}{hB_2}x_1$$

writing the system of equations (2.97) and (2.98) in the form

$$\begin{gathered}\frac{d^2\chi}{ds^2} - \nu\chi - A\frac{d\psi}{ds} - B\psi + K\frac{d^2x_1}{ds^2} - \nu x_1 = 0,\\ \frac{d}{ds}\left(\frac{d\psi}{ds} + \tau\psi + \chi\right) = 0,\\ T\frac{dx_1}{ds} + x_1 = -\beta_1\psi - \beta_2\frac{d\psi}{ds} - \gamma_1\chi + \gamma_2\frac{d\chi}{ds},\end{gathered} \tag{2.99}$$

where

$$\begin{gathered}T = T_1 V R^{-1}, \qquad \beta_1 = \frac{\beta_1^0 m_0 C_2}{hB_2}, \qquad \beta_2 = \frac{\beta_2^0 m_0 V C_2}{hB_2 R},\\ \gamma_1 = \frac{\gamma_1^0 m_0}{h}, \qquad \gamma_2 = \frac{\gamma_2^0 m_0 V}{hR}, \qquad K = \frac{hh_1}{J_1} < 1.\end{gathered} \tag{2.100}$$

The equilibrium states of the system (2.99) form a one-dimensional manifold whose stability is determined by the roots of the characteristic equation

$$\begin{aligned}&(T + K\gamma_2)\,p^4 + [1 + K(\beta_2 - \gamma_1) + (T + K\gamma_2)\,\tau]\,p^3\\ &\quad + [K\beta_1 + TA + (1 - K\gamma_1)\,\tau - (T + \gamma_2)\,\nu]p^2\\ &\quad + [A + TB - (1 - \gamma_1 + \beta_2)\,\nu - (T + \gamma_2)\,\tau\nu]\,p\\ &\quad + B - \beta_1\nu - (1 - \gamma_1)\,\tau\nu = 0.\end{aligned} \tag{2.101}$$

Let us consider the case $\alpha = \gamma_2 = 0$, when the equation (2.101) is of third order. In this case, (2.101) is replaced by

$$\begin{aligned}&[1 + K(\beta_2 - \gamma_1)]\,p^3 + [\beta_1 K + (1 - K\gamma_1)\,\tau]\,p^2\\ &\quad + [A - (1 - \gamma_1 + \beta_2)\,\nu]\,p + B - \beta_1\nu - (1 - \gamma_1)\,\tau\nu = 0.\end{aligned} \tag{2.102}$$

The motion of the bicycle is stable if

$$\begin{gathered}\nu[\beta_1 + \tau(1 - \gamma_1)] < B, \qquad \nu(1 - \gamma_1 + \beta_2) < A,\\ (1 - K\gamma_1)\,\tau + \beta_1 K > 0, \qquad 1 + K(\beta_2 - \gamma_1) > 0,\\ (1 - K\gamma_1)(A\tau - B) + \beta_1[AK + (1 - K)\,\nu]\\ - \beta_2[BK + (1 - K)\,\tau\nu] > 0.\end{gathered} \tag{2.103}$$

Let us consider the influence of the parameters β_1 and γ_1 in the inequalities (2.103) on the behavior of the boundaries of the stability region on the (τ, ν) plane.

a) $\gamma_1 = \beta_2 = 0$, $\beta_1 \neq 0$. In this case, the inequalities (2.103) take the form

$$\nu(\tau + \beta_1) < B, \quad \nu < A, \quad \tau + K\beta_1 > 0,$$
$$A\tau - B + \beta_1[(1 - K)\nu + AK] > 0.$$

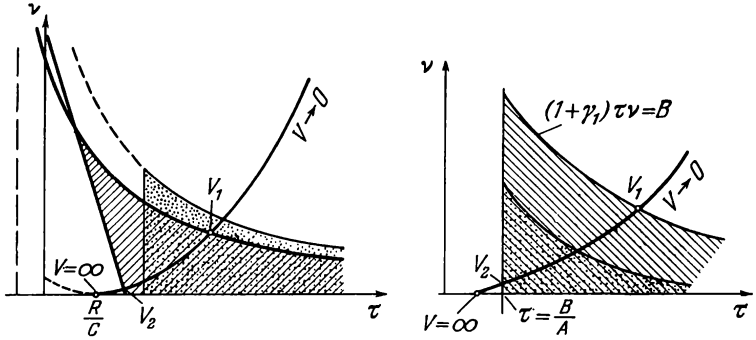


FIGURE 6.18

These inequalities determine the stability region on the (τ, ν) plane; in Figure 6.18a, this region is shaded with the oblique lines. In the same figure, the region shaded with dots is the stability region that would be obtained for the same values of the parameters in the case $\beta_1 = 0$. Comparing these regions, we see that the stability of the bicycle is greatly improved by making β_1 a positive quantity.[8] For although the lower value V_1 of the critical velocity is slightly increased the upper value V_2 may be increased to infinity. From the condition that the straight line

$$A\tau + \beta_1(1 - K)\nu + AK\beta_1 - B = 0$$

passes through the point $(R/c, 0)$, one can easily find a value of

$$\beta_1^* = \frac{cB - AR}{cAK} = \frac{J_1 R\,[c(h + C_2^0) - J_{12}]}{chh_1(J_{12} + cC_2^0)}$$

such that, for $\beta_1 \geqq \beta_1^*$, the bicycle is stable for all velocities $V > V_{cr}$, where V_{cr} is given by the real root of the equation

$$(\tau - \rho)^2(\tau - \beta_1) = BD,$$

[8] It is easily seen that the stability region on the parabola (2.72) is decreased for $\beta_1 < 0$ and vanishes for some value satisfying $|\beta_1| > |\beta_1^0|$.

which can be obtained from the condition of intersection of the parabola (2.72) and the hyperbola $\nu(\tau + \beta_1) = B$.

b) $\beta_1 = \beta_2 = 0$, $\gamma_1 \neq 0$. In this case, the inequalities (2.103) have the form

$$\begin{aligned} &\nu(1 - \gamma_1) < A, \quad (1 - K\gamma_1)\tau > 0, \quad 1 - K\gamma_1 > 0, \\ &(1 - \gamma_1)\tau\nu < B, \qquad\quad (1 - K\gamma_1)(A\tau - B) > 0, \end{aligned} \tag{2.104}$$

from which it follows that the stability region on the plane is smaller for $\gamma_1 < 0$ and, for $\gamma_1 > 0$, is altered in the manner shown in Figure 6.18b; namely, as γ_1 increases from zero to unity, the critical velocity V_1 decreases to zero and the critical velocity V_2 remains constant. If γ_1 varies in the interval $1 < \gamma_1 < 1/K$, the bicycle remains stable for all velocities in the interval $0 < V < V_2$ but it becomes unstable for $\gamma_1 > K^{-1}$ since the three inequalities (2.104) are all violated.

Thus, we have seen that a variation of the value of $\beta_1 \neq 0$ affects the position of the upper boundary V_2 of the interval of velocities of stable motion of the bicycle, whereas a variation of $\gamma_1 \neq 0$ affects the position of the lower boundary V_1 of the interval. Now let us consider the case when both of these coefficients do not vanish.

c) $\beta_2 = 0$, $\gamma_1 \neq 0$, $\beta_1 \neq 0$. In this case, the stability conditions (2.103) have the form

$$\begin{aligned} &\nu[\beta_1 + \tau(1 - \gamma_1)] < B, \quad \nu(1 - \gamma_1) < A, \\ &(1 - K\gamma_1)\tau + \beta_1 K > 0, \quad 1 - K\gamma_1 > 0, \\ &(1 - K\gamma_1)A\tau + \beta_1(1 - K)\nu + \beta_1 AK - B(1 - K\gamma_1) > 0. \end{aligned} \tag{2.105}$$

Let us find the values of the coefficients β_1 and γ_1 for which the bicycle is stable for any velocity $0 < V < +\infty$.

In accordance with our foregoing investigation, we shall consider only positive values of the parameters β_1 and γ_1, assuming $\gamma_1 < K^{-1}$ as follows from the fourth inequality of (2.105). The condition $V_2 = \infty$ can be satisfied if the straight line

$$(1 - K\gamma_1)A\tau + \beta_1(1 - K)\nu + \beta_1 AK - B(1 - K\gamma_1) = 0,$$

which can be obtained from the last inequality (2.105) if it is transformed into an equality, passes through the point $(R/c, 0)$ on the (τ, ν) plane. Hence, we obtain the relation

$$\beta_1 = \frac{BC - AR}{cAK}(1 - K\gamma_1), \tag{2.106}$$

which must be satisfied by the desired values of β_1 and γ_1. The remaining inequalities of (2.105) are certainly satisfied if we take γ_1 in the interval

(2.107) $$1 < \gamma_1 < K^{-1} \qquad (K < 1).$$

Thus, if the coefficients β_1 and γ_1 satisfy the conditions (2.106) and (2.107), the bicycle is stable for any velocity V.

We conclude that the movement of the rider when he is riding with "no hands" has an extremely important effect on the stability of the motion of the bicycle.

5. Directional stability of the simpliest model of a controlled bicycle. In order to study the stability of the simplest model of a controlled bicycle, it is sufficient to consider only the first equation of the system (2.65). Instead of the second equation, which was derived for the angle ψ through which the front part of the bicycle is turned, we take a relationship between ψ and χ, which expresses the manner in which the handlebars are turned. In order to simplify the form of the final results, we replace the first equation of (2.65) by equation (2.66), which we write in the form

(2.108) $$\ddot{\chi} - c_1\chi - c_2\psi - c_3\dot{\psi} = 0,$$

where

(2.109) $$c_1 = \frac{g}{h_0}, \qquad c_2 = \frac{l_0 V}{h_0 c}, \qquad c_3 = \frac{V^2}{h_0 c},$$

and introduce the following control for the bicycle:

(2.110) $$\psi = \alpha\chi + \beta\dot{\chi} + \gamma\int_0^t \chi\, dt,$$

where α, β, and γ are constant coefficients. The problem consists of finding the region of values of α, β, and γ for which rectilinear motion of the bicycle is stable. In doing so, we shall also find how the quantities χ, $\dot{\chi}$, and $\int_0^t \chi\, dt$ affect the angle ψ through which the bicycle is turned.

Eliminating the variable ψ from equations (2.108) and (2.110), we obtain the differential equation

(2.111) $$(1 + c_2\beta)\dddot{\chi} + (c_2\alpha + c_3\beta)\ddot{\chi} + (c_3\alpha + c_2\gamma - c_1)\dot{\chi} + c_3\gamma\chi = 0.$$

For positive values of V, $V > 0$, all the coefficients c_i $(i = 1, 2, 3)$ are positive. The characteristic equation for (2.111) is

(2.112) $$(1 + c_2\beta)p^3 + (c_2\alpha + c_3\beta)p^2 + (c_3\alpha + c_2\gamma - c_1)p + c_3\gamma = 0.$$

In order to find the stability region for the parameters α, β, and γ we shall use the D-decomposition method. First, let us assume that the parameter γ is fixed and determine the boundaries of the stability region on the (α, β) parameter plane. We shall then study the way in which a variation of γ affects the boundaries of the stability region. We

shall also assume that the velocity V of the bicycle is constant. Substituting $p = j\omega$ $(j = \sqrt{-1})$ into equation (2.112), we obtain the boundary of the D-decomposition in the form

$$-\omega^2 (c_2\alpha - c_3\beta) + c_3\gamma = 0,$$
$$-\omega^3 (1 + c_2\beta) + \omega (c_3\alpha + c_2\gamma - c_1) = 0.$$

Solving these equations for α and β, we obtain the equation of a unicursal curve in a parametric representation ($\Omega = \omega_2 > 0$ is the running parameter)

$$\alpha = \frac{c(h\Omega + g)}{l^2\Omega + V^2}, \qquad \beta = -\frac{l\Omega(hc\Omega - lV\gamma + cg) - \gamma V^3}{V\Omega(l^2\Omega + V^2)}, \tag{2.113}$$

where c_1, c_2, and c_3 are replaced by the expressions (2.109), in which the subscript 0 is omitted. The determinant of the system is

$$\Delta = \omega^3 V (l^2\Omega + V^2).$$

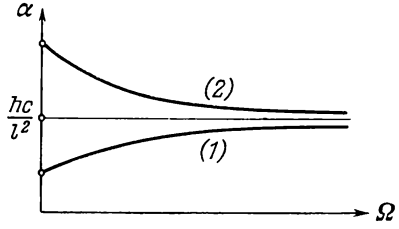


FIGURE 6.19

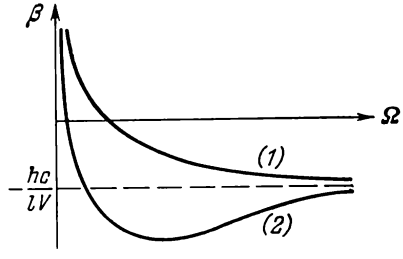


FIGURE 6.20

For the value $\Omega = \infty$, we obtain a singular straight line of the D-decomposition $\beta = -hc/l^2V$ and, for the value $\Omega = 0$, the limiting value $\gamma = 0$. Hence, and by (2.112), it follows that a stability region exists only for positive values of γ and we shall therefore assume $\gamma > 0$ for the remainder of the investigation. Before we construct the curve of the D-decomposition on the (α, β) plane, let us establish the qualitative behavior of the curves $\alpha = \alpha(\Omega)$ and $\beta = \beta(\Omega)$ defined by the expressions (2.113). The curve $\alpha = \alpha(\Omega)$ is shown in Figure 6.19, curve (1) referring to the case $V^2 > gl^2/h$ and curve (2) to the case $V^2 < gl^2/h$. In the case $V^2 = gl^2/h$, the curve $\alpha = \alpha(\Omega)$ degenerates into the straight line $\alpha = hc/l^2$. In order to construct the curve $\beta = \beta(\Omega)$, we calculate the first derivative

$$\frac{d\beta}{d\Omega} = -\frac{l(hcV^2 + l^3\gamma V - l^2cg)\Omega^2 + 2l^2V^3\gamma\,\Omega + \gamma V^5}{V\Omega^2(l^2\Omega + V^2)^2}.$$

Since we are considering only positive values $V > 0$, the derivative $d\beta/d\Omega$ is certainly negative for $V^2 \geqq dl^2/h$, as can be verified by direct substitution of the value $V^2 = gl^2/h$ into the coefficient of Ω^2 in the

numerator. On the other hand, it is obvious that this coefficient is negative for very small positive values $V > 0$, and, hence, for all values $\Omega > \Omega^*$, where Ω^* is the value that makes the numerator vanish, the derivative $d\beta/d\Omega$ is positive. Hence and from the asymptotic behavior of the curve $\beta = \beta(\Omega)$ it follows that the curve has the form shown in Figure 6.20. Curve (1) refers to the case $V > V_1$ and curve (2) to the case $0 < V < V_1$, where V is the positive root of the equation

$$hcV^2 + \gamma l^3 V - cgl^2 = 0.$$

It follows from the foregoing analysis that $V_1^2 < gl^2/h$. Comparing the graphs in Figures 6.19 and 6.20, we obtain the three cases of the behavior

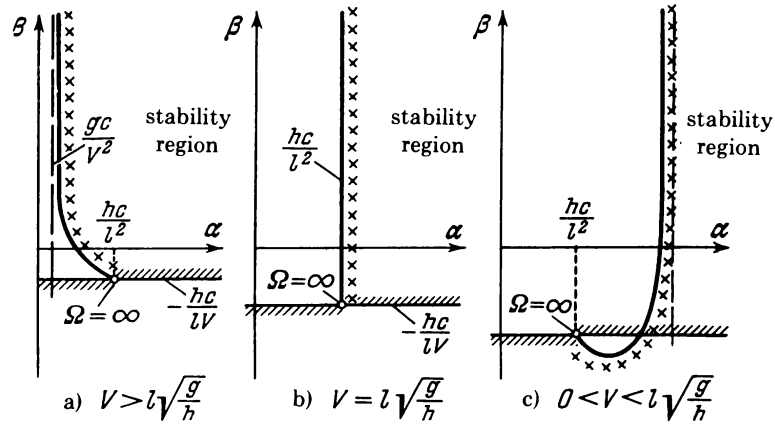


FIGURE 6.21

of the boundary of the stability region shown in Figure 6.21. In constructing the curves of Figure 6.21, we took into account the expression for the derivative

$$\frac{d\beta}{d\alpha} = -\frac{l(hcV^2 + \gamma l^3 V - cgl^2)\Omega^2 + 2l^2\gamma V\Omega + \gamma V^5}{cV(hV^2 - gl^2)\Omega^2}.$$

It can be seen immediately from Figure 6.21 that the stability region occupies practically the whole of the first quadrant of the (α, β) plane if the bicycle has a high velocity. As the velocity decreases, the boundary is shifted to the right and downward, i.e., the stability region is reduced with respect to the parameter α but is increased with respect to the parameter β in the direction of negative values of β. Let us now consider how a change in the value of $\gamma > 0$ affects the boundaries of the stability region. First, we note that the behavior of the curve $\alpha = \alpha(\Omega)$ does not depend on γ. The coefficient γ also has no effect on the asymptotic behavior of the curve $\beta = \beta(\alpha)$ (see Figure 6.21). Hence, the qualitative

behavior of the boundaries of the stability region is preserved. However, as the parameter γ increases, the curve $\beta = \beta(\Omega)$ has a more shallow form (Figure 6.20) and the stability region is accordingly slightly extended (Figure 6.21a) or reduced (Figures 6.21b and 6.21c).

6. Stability of simplest model of a bicycle with given controlling law with allowance for a time lag. Let us now consider the equation of motion (2.108) of a simplified bicycle whose steering is governed by the law

$$\psi(t) = \alpha\chi(t-\tau) + \beta\dot{\chi}(t-\tau), \tag{2.114}$$

where α and β are constant coefficients of a feedback with the time lag $\gamma > 0$. Eliminating the variable ψ from equations (2.108) and (2.114), we obtain the differential equation

$$\ddot{\chi}(t) + c_2\beta\ddot{\chi}(t-\tau) + (c_2\alpha + c_3\beta)\dot{\chi}(t-\tau) + c_3\alpha\chi(t-\tau) - c_1\chi(t) = 0. \tag{2.115}$$

We shall also solve this problem of finding the stability region with respect to the parameters α, β, and τ by the D-decomposition method.

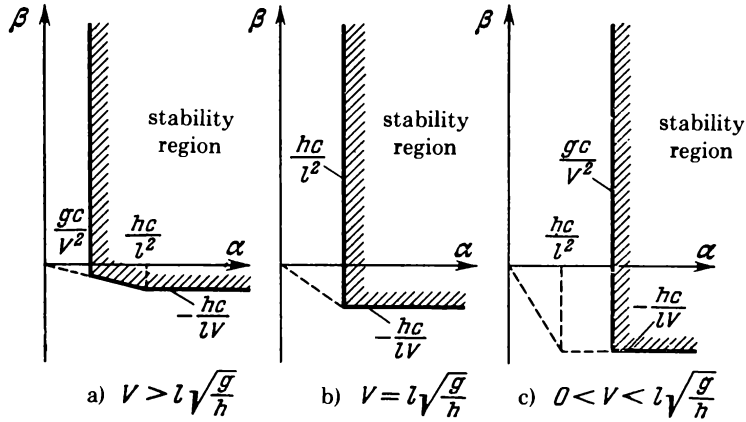


FIGURE 6.22

First, let us consider the special case $\tau = 0$, when the steering is effected without a time lag. In this case, the characteristic equation has the form

$$(1 + \beta c_2)p^2 + (c_2\alpha + c_3\beta)p + c_3\alpha - c_1 = 0.$$

It follows that the stability region is determined by the inequalities

$$\alpha > \frac{gc}{V^2}, \quad \beta > -\frac{hc}{lV}, \quad \beta > -\frac{l}{V}\alpha.$$

In accordance with these inequalities, we obtain the boundary of the stability region shown in Figure 6.22. We should mention that the diagrams of Figure 6.22 can also be obtained from Figure 6.21 by means of the passage to the limit $\gamma \to 0$.

Let us now consider the general case $\tau \neq 0$, when the feedback circuit has a time lag. Substituting a solution of the form $\chi = \chi_0 e^{pt}$ into (2.115), we arrive at the characteristic equation

$$(1 + c_2\beta e^{-p\tau})\, p^2 + (c_2\alpha + c_3\beta)\, e^{-p\tau} p + c_3\alpha e^{-p\tau} - c_1 = 0$$

or, equivalently,

$$(c_2 p + c_3)\,\alpha + (c_2 p^2 + c_3 p)\,\beta + (p^2 - c_1)\, e^{p\tau} = 0. \tag{2.116}$$

Into this equation we substitute $p = j\omega$ $(j = \sqrt{-1})$ and separate the real and imaginary parts

$$\begin{aligned} c_3\alpha - c_2\omega^2\beta &= (\omega^2 + c_1)\cos\omega\tau, \\ c_2\omega\alpha + c_3\omega\beta &= (\omega^2 + c_1)\sin\omega\tau. \end{aligned} \tag{2.117}$$

Solving this system of equations for α and β, we obtain

$$\begin{aligned} \alpha &= \rho(\omega)\,[c_3\cos\omega\tau + c_2\omega\sin\omega\tau], \\ \beta &= \rho(\omega)\left[c_3\frac{\sin\omega\tau}{\omega} - c_2\cos\omega\tau\right], \end{aligned} \tag{2.118}$$

where $\rho(\omega) = (\omega^2 + c_1)/(c_2^2\omega^2 + c_3^2)$ is the function whose graph is shown in Figure 6.23. The Jacobian of the system of equations (2.117) is

$$\omega(c_2^2\omega^2 + c_3^2)$$

and is positive for $\omega > 0$. As ω varies from $-\infty$ to $+\infty$, we obtain a unicursal curve, which, together with the singular straight line

$$\alpha = c_1/c_3$$

determines the boundary of the D-decomposition on the (α, β) plane.

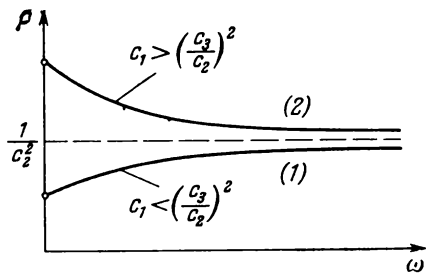


FIGURE 6.23

In order to find the stability region, it is sufficient to note that the point $\alpha = \beta = 0$ corresponds to a quasi-polynomial having one root with a positive real part.

We shall now find the form of the D-decomposition of the (α, β) plane for a very short time lag τ and then establish the manner in which the stability region changes as the value of τ increases. Since the expression (2.118) is unaffected by the substitution of $-\omega$ for ω, it is

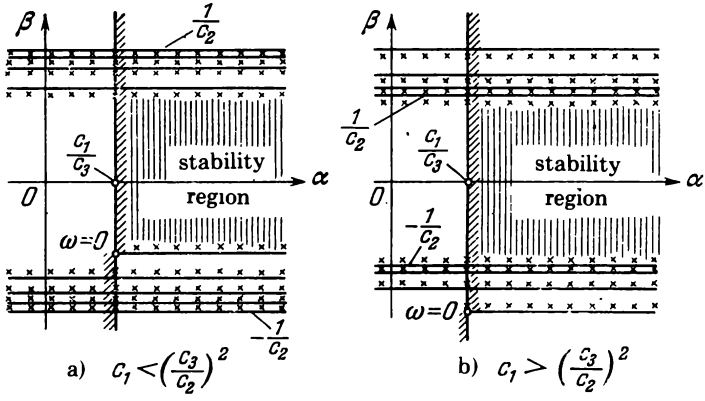


FIGURE 6.24

sufficient to construct the curve (2.118) only for the values $0 \leqq \omega < \infty$. For a very small value of τ, the curve (2.118) begins at the point $\alpha = c_1/c_3$, $\beta = -c_1c_2/c_3^2$ and, as ω increases to infinity, travels repeatedly along horizontal straight lines that tend monotonically to the straight lines $\beta = \pm 1/c_2$. After the shaded region has been plotted, we obtain the diagram of the D-decomposition shown in Figure 6.24, in which the continuous shading indicates the stability region. Figure 6.24a corresponds to curve (1) in Figure 6.23a, and Figure 6.24b to curve (2) in Figure 6.23.

Now suppose that the value of τ increases. The boundary of the D-decomposition will vary and, for some value of τ, will take the form shown in Figure 6.25. A further increase in the value of τ will lead to an even greater decrease and, possibly, to the disappearance of the stability region.

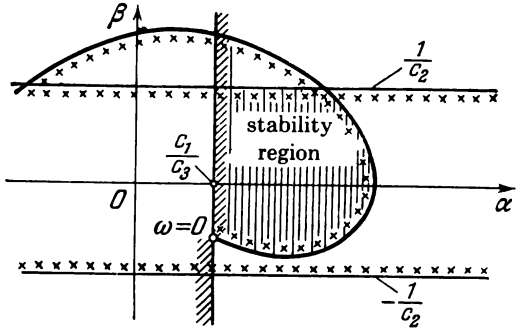


FIGURE 6.25

§3. Shimmy of the nose wheel of a three-wheel aircraft landing gear

The word shimmy is used for the phenomenon of selfexcitation of vibrations of the nose wheel of a three-wheeled aircraft landing gear (and also of the front wheels of an automobile) during the motion of the aircraft on the runway. These vibrations consist of rotations of the wheel about the vertical axis and transverse displacements. The phenomenon of shimmy is connected with the elasticity of the pneumatic

tire and the front strut of the landing gear and also with the fact that the nose wheel can rotate about the vertical axis. On an aircraft with a three-wheeled landing gear, the nose wheel is self-orientating and there it therefore always has the possibility of rotating about the vertical axis. Shimmy is eliminated principally by hydraulic or friction dampers joined to the strut. Shimmy has been studied theoretically by many investigators. Different simplifying assumptions have been made: Some authors have taken into account the elasticity of the strut holding the wheel but have neglected the elasticity of the tire, whereas others have taken into account the elasticity of the latter but assumed that the strut is rigid. In all cases it is assumed that the wheel rolls without slipping. In the absence of slipping, the problem of shimmy of a rigid wheel becomes a problem of the analytical mechanics of nonholonomic systems and can be completely solved without additional assumptions. An additional investigation into the behavior of a pneumatic tire during rolling is necessary to solve the problem of the shimmy of a wheel with an elastic tire.

In the present section, this problem will be treated first from the point of view of Keldyš's theory [228] and then by means of the generalized hypothesis of creep (see §1 of the present chapter).

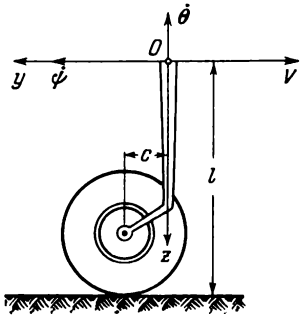


FIGURE 2.26

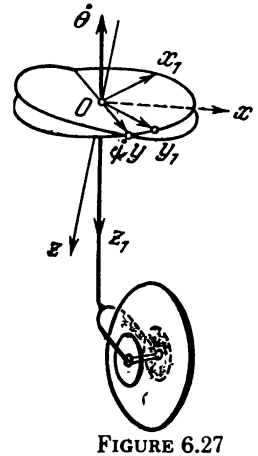


FIGURE 6.27

1. Equations of kinematic constraints for the rolling of the nose wheel of a three-wheel aircraft landing gear. Suppose the wheel with the pneumatic tire is fixed in an inclined fork attached to the lower end of the vertical strut of the landing gear (Figure 6.26). We shall assume that the strut possesses elasticity in the transverse direction and can turn together with the fork about its axis. For the sake of simplicity,

we shall consider a rigid strut with an elastic joint instead of an elastic strut. The calculation for an elastic strut can always be reduced to this case.

In accordance with Figure 6.26, the position of the system (without allowance for the rotation of the wheel in its own plane) is determined by four generalized coordinates: the angle ψ of rotation about the Oy axis, the angle θ of rotation about the axis of the strut, and also the deformation parameters ξ and φ of the pneumatic tire. The deformation parameter χ can be expressed in terms of ψ and θ. To see this we introduce a moving coordinate system $Ox_1y_1z_1$, the strut and the central plane of the wheel always lying in the y_1Oz_1 plane (Figure 6.27). The cosines of the angles between the axes of the coordinate systems $Oxyz$ and $Ox_1y_1z_1$ are given in the table

(3.1)

	x_1	y_1	z_1
x	$\cos\psi\cos\theta$	$\cos\psi\sin\theta$	$\sin\psi$
y	$-\sin\theta$	$\cos\theta$	0
z	$-\sin\psi\cos\theta$	$-\sin\psi\sin\theta$	$\cos\psi$

In accordance with this table,

$$\sin\chi = -\sin\psi\cos\theta$$

or, for small angles, $\chi = -\psi$. Suppose the aircraft moves along the runway with constant velocity $V = \text{const}$ in the negative direction of the Oy axis. A steady motion regime for the strut with the pneumatic tire is realized for null values of the generalized coordinates $\psi = \theta = \xi = \varphi = 0$. In investigating the small deviations from this steady motion, we shall assume that the quantities ψ, θ, ξ, and φ are small. Using the notation introduced for the deviation of the center of contact of the tire from the steady position, we obtain the following expression:

$$x_1 = l\psi + c\theta + \xi, \tag{3.2}$$

where l is the distance from the point of the joint of the strut with the wheel to the projection onto the axis of the strut of the center of contact, and c is the distance from the center of contact to the axis of the strut, the so-called stagger of the wheel, which we shall take to be positive if the wheel is staggered backward and negative in the opposite case. The relation (3.2) gives the equation of the rolling line. In accordance with the table (3.1), the line of intersection of the diametral plane of the wheel forms an angle θ with the direction of motion of the aircraft. Denoting by $s = Vt$ the distance traversed by the aircraft, we can write

the condition that the tangent to the rolling line and the axis of the surface of contact coincide in the form $dx_1/ds = -(\theta + \varphi)$ or, in accordance with (3.2),

$$\frac{d}{ds}(l\psi + c\theta + \xi) + \theta + \varphi = 0. \tag{3.3}$$

Bearing in mind that the curvature of the rolling line is

$$\frac{1}{R} = -\frac{d^2x_1}{ds^2} = \frac{d}{ds}(\theta + \varphi),$$

we write the second condition of rolling of the pneumatic tire in the form

$$\frac{d}{ds}(\theta + \varphi) = \alpha\xi - \beta\varphi + \gamma\psi. \tag{3.4}$$

Here, we have used the relation $\chi = -\psi$ for small angles. The relations (3.3) and (3.4) are the equations of the nonholonomic constraints imposed on the system for the case when the pneumatic tire rolls in the idealized manner. In accordance with the equations of the nonholonomic constraints (3.3) and (3.4), the variations of the generalized coordinates ψ, θ, ξ, and φ must satisfy the relations

$$l\delta\psi + c\delta\theta + \delta\xi = 0, \quad \delta\theta + \delta\varphi = 0. \tag{3.5}$$

It follows that only two coordinates, say $\delta\psi$ and $\delta\theta$, can have independent variations. The variations of the remaining coordinates are then determined by equations (3.5).

Thus, the system has two degrees of freedom.

2. Equations of motion of the nose strut of an aircraft's landing gear. Let us find the linearized equations of motion of the system for small deviations from the steady state. Let A be the moment of inertia of the strut and wheel about the Oy_1 axis; B, the moment of inertia about the Oz_1 axis; D, the centrifugal moment of inertia about these axes; and C, the moment of inertia of the wheel about the axis of rotation in the plane of the wheel. In accordance with Figure 6.27 and the table (3.1), the projections onto the $Ox_1y_1z_1$ axes of the instantaneous angular velocity of the system is (to within small terms of second order) $\Omega\{-(\omega + \theta\psi), \psi, -\dot{\theta}\}$, where ω is the angular velocity of rotation of the wheel in its own plane, which, if the compression of the tire is neglected, is $\omega = V/r$. In the same approximation, the kinetic energy of the system is

$$T = \frac{1}{2}(A\psi^2 + B\dot{\theta}^2 + 2D\psi\dot{\theta} + 2\omega C\theta\psi).$$

We now find an expression for the potential energy of the system. If we denote by k the stiffness coefficient of the elastic joint of the strut, the potential energy is $U_1 = k\psi^2/2$ plus the previously found expression for the deformation potential energy of a pneumatic tire. Thus, the potential energy of the system is

$$U = \frac{1}{2} k\psi^2 + \frac{1}{2}(a\xi^2 + b\varphi^2 + \rho N\psi^2 - 2\sigma N\xi\psi).$$

Thus, the system is characterized by the Lagrange function

$$L = T - U_1 - U = \frac{1}{2}(A\dot{\psi}^2 + B\dot{\theta}^2 + 2D\dot{\psi}\dot{\theta} + 2C\omega\theta\psi) - \frac{1}{2}(k + \rho N)\psi^2 - \frac{1}{2}(a\xi^2 + b\varphi^2) + \sigma N\xi\psi. \tag{3.6}$$

Apart from the forces taken into account in the expression for the potential energy U_1, the strut carrying the wheel is subjected to a normal reaction N, which we shall assume to be constant. In the frame $Oxyz$, the point of application of the force $\boldsymbol{N}(0, 0, -N)$ has the coordinates $\boldsymbol{r}_N(l\psi + c\theta, c, l)$. In order to calculate the generalized forces corresponding to the normal reaction $\boldsymbol{N}$, we write down an expression for the virtual work:

$$\delta A = \boldsymbol{N}\delta\boldsymbol{r}_N = \boldsymbol{N} \cdot (\delta\boldsymbol{\vartheta} \times \boldsymbol{r}_N) = (\boldsymbol{r}_N \times \boldsymbol{N})\,\delta\boldsymbol{\vartheta},$$

where $\delta\vartheta$ is the virtual rotation corresponding to the instantaneous angular velocity $\boldsymbol{\Omega} = \psi\boldsymbol{j} - \dot{\theta}\boldsymbol{k}_1 = -\psi\dot{\theta}\boldsymbol{i} + \psi\boldsymbol{j} - \dot{\theta}\boldsymbol{k}$.

Hence,

$$\delta A = [-cN\boldsymbol{i} + N(l\psi + c\theta)\boldsymbol{j}] \cdot (-\psi\delta\theta\boldsymbol{i} + \delta\psi\boldsymbol{j} - \delta\theta\,\boldsymbol{k}) = cN\psi\delta\theta + N(l\psi + c\theta)\delta\psi.$$

Consequently,

$$Q_\psi = N(l\psi + c\theta), \quad Q_\theta = cN\psi.$$

Further, we assume that a hydraulic damper with a linear characteristic is fitted onto the strut. The damping moment corresponding to an angular velocity $\dot{\theta}$ about the axis of the strut is equal to $-h\dot{\theta}$. Thus, we obtain the final expression for the generalized forces:

$$Q_\psi = N(l\psi + c\theta), \quad Q_\theta = cN\psi - h\dot{\theta}. \tag{3.7}$$

In accordance with (3.6) $L = L(\psi, \theta, \xi, \varphi, \dot{\psi}, \dot{\theta})$; therefore equations (1.18) take the form

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} + l\frac{\partial L}{\partial \xi} - Q_\psi = 0, \\ &\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + c\frac{\partial L}{\partial \xi} + \frac{\partial L}{\partial \varphi} - Q_\theta = 0. \end{aligned} \tag{3.8}$$

Using the expressions (3.6) and (3.7) in (3.8), we obtain the equations of motion of the system:

$$\begin{aligned} &A\ddot{\psi} + [k + \rho N + lN(\sigma - 1)]\psi + D\ddot{\theta} + C\omega\dot{\theta} - cN\theta \\ &\qquad - (la + \sigma N)\xi = 0, \\ &D\ddot{\psi} - C\omega\psi - cN(\sigma - 1)\psi + B\ddot{\theta} + h\dot{\theta} - ca\xi - \beta\varphi = 0. \end{aligned} \tag{3.9}$$

Equations (3.9) must be augmented by the equations (3.3) and (3.4) of the nonholonomic constraints, which, using the relation $s = Vt$, we rewrite in the form

$$\begin{aligned} l\psi + c\dot{\theta} + V\theta + \dot{\xi} + V\varphi = 0, \\ \dot{\theta} + \dot{\varphi} + \beta V\varphi - \alpha V\xi - \gamma V\psi = 0. \end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) are a complete system for the determination of ψ, θ, ξ, and φ as functions of the time. Equations (3.9) and (3.10) are the linearized equations of motion in the neighborhood of the steady motion. They enable one to investigate the stability of the steady motion of the aircraft.

We shall now find the conditions for the occurrence of shimmy of the nose wheel of a three-wheeled landing gear of an aircraft for different cases.

3. Shimmy of the wheel in the case of a rigid vertical strut ($k = \infty$). In the case of a rigid joint of the strut of the nose wheel, the displacement of the wheel is determined solely by rotation of the wheel about the vertical strut and the deformation of the tire. In this case ($k = \infty$), the first equation of (3.9) yields $\psi = 0$ and the remaining equations take the form

$$\begin{aligned} B\ddot{\theta} + h\dot{\theta} - ca\xi - b\varphi = 0, \\ c\dot{\theta} + \dot{\xi} + V\theta + V\varphi = 0, \\ \dot{\theta} + \dot{\varphi} + \beta V\varphi - \alpha V\xi = 0. \end{aligned} \tag{3.11}$$

Setting $\theta = \theta_0 e^{pt}$, $\xi = \xi_0 e^{pt}$, and $\varphi = \varphi_0 e^{pt}$, we rewrite the characteristic equation of the system (3.11) as follows:

$$\begin{aligned} &Bp^4 + (h + \beta BV)p^3 + (c^2a + b + h\beta V + \alpha BV^2)p^2 \\ &\qquad + V(c^2a\beta + cab + \alpha hV)p + V^2(ca\beta + \alpha b) = 0. \end{aligned} \tag{3.12}$$

Let us begin the investigation of the roots of equation (3.12) with the special case $h = 0$.

a) *Conditions of stability in the absence of a damper* ($h = 0$). If there is no damping ($h = 0$), the characteristic equation (3.12) has the form

$$\begin{aligned} &Bp^4 + \beta BVp^3 + (ac^2 + b + \alpha BV^2)p^2 \\ &\qquad + cV(ca\beta + \alpha b)p + V^2(ca\beta + \alpha b) = 0. \end{aligned} \tag{3.13}$$

Since the parameters B, V, α, β, a, and b are positive, the stability conditions reduce by virtue of the Routh-Hurwitz criterion to the inequalities

$$c > 0, \quad (\alpha c - \beta)(\beta B V^2 - bc) > 0. \tag{3.14}$$

It follows that, in the absence of a damper, the rolling of the wheel with elastic tire is stable in each of the two cases:

$$\text{I.} \quad 0 < c < \frac{\beta}{\alpha}, \quad V^2 < \frac{bc}{\beta B}. \qquad \text{II.} \quad c > \frac{\beta}{\alpha}, \quad V^2 > \frac{bc}{\beta B}.$$

The conditions (3.14) enable one to distinguish the stability regions on the (c, V^2) half-plane shown in Figure 6.28. On the transition through the boundary of the stability region, the system departs from the state of steady motion, either aperiodically or by executing vibrations. An investigation shows that we have an aperiodic instability in the region $c < 0$ if $c < -\alpha b/\beta a$ and vibrational instability, generally speaking, if $c > -\alpha b/\beta a$. For small velocities, these unstable motions approximate to the motions of a weightless wheel. In the case of vibrational instability, this motion has been given the name "kinematic shimmy", since it does not depend on the inertial properties of the wheel. In the region $c > \beta/\alpha$, $V^2 < bc/\beta B$, the instability generally has a vibrational nature and is determined by two roots which give a motion that degenerates into the normal vibrations of a fixed wheel in the limit $V \to 0$. Thus, the instability is of a dynamical nature in this region.

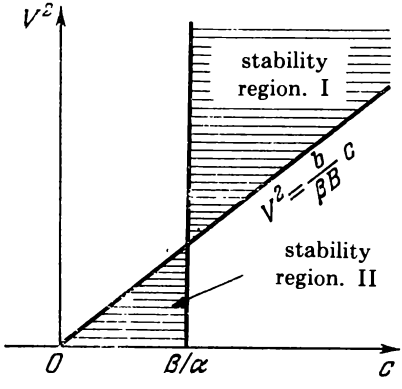


FIGURE 6.28

b) *Influence of damping on the stability in the case* $k = \infty$. In order to study the influence of damping, we shall use the D-decomposition method. First of all, we replace the variable p by the new variable $q = pV^{-1}$ $(V \neq 0)$ in the characteristic equation (3.12):

$$BV^2q^4 + (\beta BV^2 + hV)\, q^3 + (\alpha BV^2 + \beta hV + ac^2 + b)\, q^2 + (\alpha hV + \alpha bc + a\beta c^2)\, q + ab + a\beta c = 0. \tag{3.15}$$

Taking $\tau = BV^2$ and $\nu = hV$ as the parameters of the D-decomposition, we write the characteristic equation (3.15) in the form

$$(\tau q^2 + \nu q + ac^2 + b)(q^2 + \beta q + \alpha) + (bq + ac)(\alpha c - \beta) = 0. \tag{3.16}$$

Substituting $q = j\omega$ $(j = \sqrt{-1})$ into this equation, we arrive at the parametric representation of the unicursal curve of the D-decomposition

$$\tau = \frac{A_1\Omega^2 + A_2\Omega + A_3}{\Omega\,[\Omega^2 + (\beta^2 - 2\alpha)\,\Omega + \alpha^2]}, \qquad \nu = \frac{(\alpha c - \beta)\,[b\Omega - (\alpha b + \beta ac)]}{\Omega^2 + (\beta^2 - 2\alpha)\,\Omega + \alpha^2}. \tag{3.17}$$

Here,

$$\Delta = -\omega^3\,[\Omega^2 + (\beta^2 - 2\alpha)\,\Omega + \alpha^2].$$

The numerator in the expression for τ has no positive roots for all $c > 0$. The determinant necessary for determining the shading of the unicursal curve [322] is

$$A_1 = ac^2 + b > 0, \qquad A_3 = \alpha\,(ab + \beta ac), \tag{3.18}$$
$$A_2 = (\beta^2 - \alpha)\,ac^2 + (ab - a)\beta c - 2ab, \qquad \Omega = \omega^2 > 0.$$

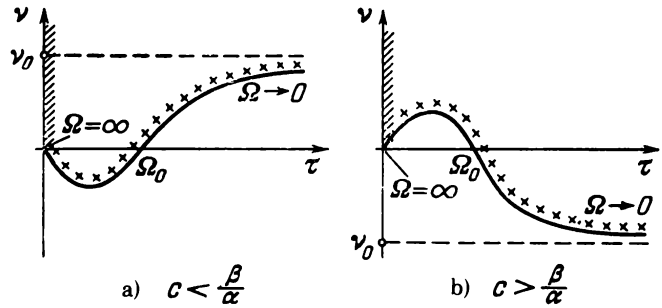


FIGURE 6.29

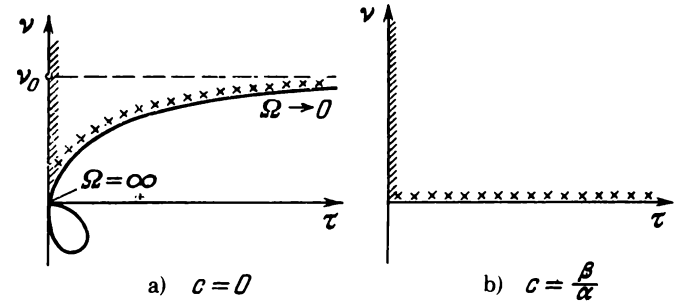


FIGURE 6.30

Qualitatively, the form of the curve (3.17) is shown in Figure 6.29, in which $\Omega^* = (\alpha b + \beta ac)/b$, and the asymptote $\nu = \nu_0$ is determined by the

expression $\nu_0 = -(\alpha c - \beta)(\alpha b + \beta ac)/\alpha^2$. The curve (3.17) in Figure 6.29 is shaded and marked with crosses on the side of the stability region. Apart from the curve (3.17), we have the singular straight line $\tau = 0$.

In the cases $c = 0$ and $c = \beta/\alpha$, the diagram of the D-decomposition on the (τ, ν) plane has the form shown in Figure 6.30.

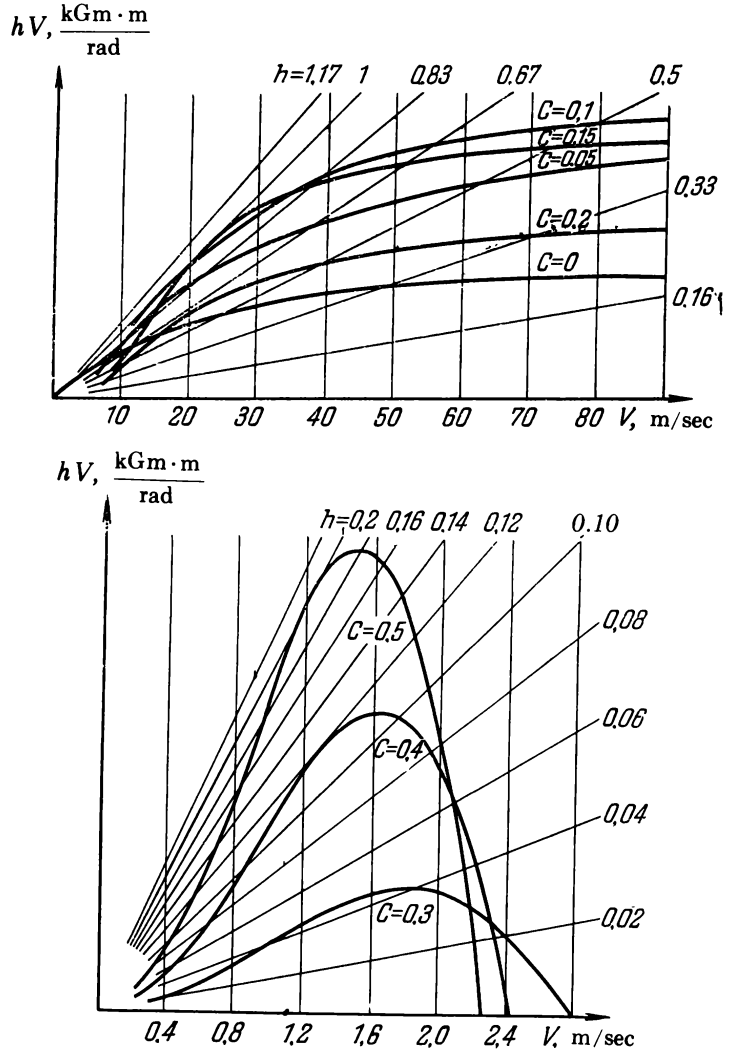


FIGURE 6.31

After substitution of the numerical values of the parameters of an

aircraft wheel [228] with the ratio $\beta/\alpha = 0.25$, the curves of Figures 6.29 and 6.30 take the form shown in Figure 6.31, in which the values of $\sqrt{\tau}$ are plotted along the abscissa in order to facilitate the representation of the coordinate mesh $h = \text{const}$ and $V = \text{const}$.

Using the curves of Figure 6.31, one can easily construct the graph of the dependence of the minimum necessary damping $h_{\min}$ on the stagger c. Such a graph is shown in Figure 6.32.

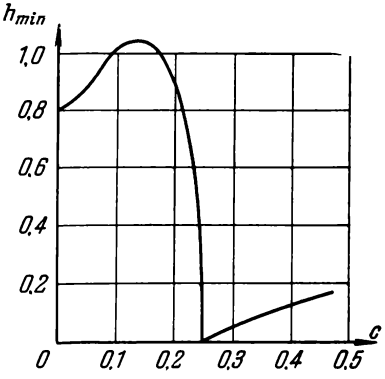


FIGURE 6.32

The following conclusions can be drawn from the study of the case of an absolutely rigid strut. With the usual construction of the strut and a backward stagger c of the wheel to some limit c_0 of the order of half the diameter of the wheel, stability only obtains at very small velocities. However, if the stagger exceeds c_0, instability only occurs at low velocities and can be completely eliminated by slight damping. The friction in the construction of the strut is probably sufficient to provide this amount of damping. If there is no backward stagger or the stagger satisfies $0 < c < c_0$, the onset of shimmy can always be prevented by a damper which gives rise to a damping moment when the strut rotates about its axis. Using the formulas given above, one can easily estimate the required amount of damping and represent it in the form of the graph in Figure 6.32.

4. Influence of transverse elasticity of the strut on the shimmy of the nose wheel of a three-wheeled aircraft landing gear. A study of the effect of transverse elasticity of the strut on the stability of the steady motion must be based on the complete system of equations (3.9) and (3.10). Substituting a solution of the form

$$\theta = \theta_0 e^{Vqt}, \quad \psi = \psi_0 e^{Vqt}, \quad \xi = \xi_0 e^{Vqt}, \quad \varphi = \varphi_0 e^{Vqt}$$

into these equations, we obtain the equation for the characteristic roots q:

$$\begin{vmatrix} AV^2q^2 + k + pN + lN(\sigma - 1), & DV^2q^2 + \dfrac{CV^2}{r}q - cN, & -(la + \sigma N), & 0 \\ DV^2q^2 - \dfrac{CV^2}{r}q - cN(\sigma - 1), & BV^2q^2 + hVq, & -ac, & -b \\ lq, & cq + 1, & q, & 1 \\ -\gamma, & q, & -\alpha, & q + \beta \end{vmatrix} = 0. \tag{3.19}$$

For what follows, we shall find it convenient to consider the inverse of the stiffness k, namely, $\kappa = 1/k$. We can then write the characteristic equation (3.19) in the form

$$P_4 + (P_2P_4 - P_6)\kappa = 0, \tag{3.20}$$

where we have just defined κ and P_2, P_4, and P_6 are polynomials of second, fourth, and sixth degree, respectively:

$$\begin{aligned}
P_2 &= AV^2q^2 + N\,[l\,(2\sigma - 1) + \rho] + al^2,\\
P_4 &= BV^2q^4 + (\beta BV^2 + hV)\,q^3 + (\alpha BV^2 + ac^2 + b + \beta hV)\,q^2\\
&\qquad + (abc + \beta ac^2 + \alpha hV)\,p + \alpha b + \beta ac,\\
P_6 &= D^2V^4q^6 + D^2V^4\beta q^5 + V^2\left[2Dd + V^2\left(D^2\alpha - \frac{C^2}{r^2}\right)\right]q^4\\
&\quad + V^2\left[D\,(2\beta d + \alpha bl - b\gamma) - V^2\,\frac{C^2\beta}{r^2}\right]q^3\\
&\quad + \left\{d^2 + V^4\left[2D\alpha d - Dac\gamma - D\sigma N\alpha c - V^2\,\frac{C^2\alpha}{r^2} - \frac{Cb}{r}\,(\alpha l - \gamma)\right.\right.\\
&\qquad\qquad \left.\left. + al\,[D\beta + B\,(\alpha l - \gamma + \sigma N)] + hV\,(\alpha l - \gamma)\right]\right\}q\\
&\quad + \alpha d^2 + (\alpha l - \gamma)\,[(ac^2 + b)\,(al + \sigma N) - acd] - d\,(al + \sigma N)(\alpha c - \beta),
\end{aligned}$$

where $d = c[N(\sigma - 1) + hl]$. Assuming that the system is stable for $\kappa = 0$, i.e., assuming that P_4 is a Hurwitz polynomial, we first find the conditions for stability to be preserved for very small, nonvanishing values of κ. To this end, we shall construct a D-decomposition on the plane of the parameters κ_1 and κ_2, which we define as follows:

$$\begin{aligned}
\varkappa_1 &= (AB - D^2)\,V^4\varkappa > 0,\\
\varkappa_2 &= [V\beta\,(AB - D^2) + hA]\,V^3\varkappa > 0.
\end{aligned}$$

In this notation, the characteristic equation (3.20) takes the form

$$\varkappa_1q^6 + \varkappa_2q^5 + P_4 + \varkappa Q_4 = 0, \tag{3.21}$$

where Q_4 is a polynomial of fourth degree, being the remainder of the polynomial $(P_2P_4 - P_6)$ after the higher terms have been removed. Replacing q by $j\omega$ $(j = \sqrt{-1})$ in equation (3.21) and equating the real and imaginary parts to zero, we obtain expressions for κ_1 and κ_2:

$$\begin{aligned}
\varkappa_1 &= \frac{1}{\omega^6}\,[(a_0 + \varkappa b_0)\,\omega^4 - (a_2 + \varkappa b_2)\,\omega^2 + a_4 + \varkappa b_4],\\
\varkappa_2 &= \frac{1}{\omega^4}\,[(a_1 + \varkappa b_1)\,\omega^2 - (a_3 + \varkappa b_3)],
\end{aligned} \tag{3.22}$$

which are a parametric representation of a unicursal curve on the (κ_1, κ_2) plane. In the limiting case $\kappa \to 0$, the expression (3.22) tends to

a limit represented by the curve shown in Figure 6.33. Apart from this curve, the straight line $\kappa_1 = 0$ is a solution of equation (3.21). It corresponds to the exceptional value $\omega = 0$ which makes the determinant

$$\Delta = -\omega^{11}$$

vanish. In accordance with the shading rule, we obtain a stability region G near the origin on the (κ_1, κ_2) plane. The origin corresponds to a stable system with an absolutely rigid strut. It can be seen from Figure 6.33 that the stability is lost if the representative point moves away from the origin in a direction contained within the angle δ but that it is preserved if the point moves into the region G. The condition for the preservation of stability for very small nonvanishing κ can be expressed by the inequality[9]

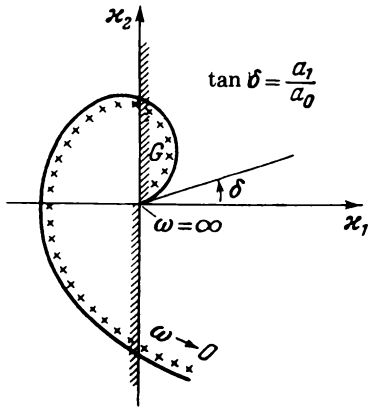


FIGURE 6.33

$$\kappa_2/\kappa_1 > a_1/a_0.$$

Substituting the expressions for the various quantities, we obtain

$$hD^2 > 0. \tag{3.23}$$

Since the centrifugal moment of inertia D is proportional to the stagger c, the condition (3.23) shows that if the damping and stagger are both nonvanishing and the original system ($\kappa = 0$) is stable, there always exists an interval of nonvanishing values of κ for which stability is not lost.

Pursuing this investigation further, we find that if the original system is stable in the cases a) $c = 0$, $h \neq 0$ or b) $c > 0$, $h = 0$, the representative point on the (κ_1, κ_2) plane arrives at the boundary of the stability region for $\kappa \neq 0$, so that we must study the stability problem more thoroughly. To this end, we take into account the manner in which the angular coefficient

$$\tan\tilde{\delta} = \frac{\tilde{a}_1}{\tilde{a}_0} \approx \frac{a_1 + \varkappa b_1}{a_0 + \varkappa b_0}$$

varies for a small, nonvanishing value of κ as the relevant parameters take different values ($\tilde{\delta}$ is the value of the angle δ in the case $0 < \kappa \ll 1$). If the condition

[9] This stability condition also follows from Meerov's theorems [298], as is easily verified. Here, we have obtained the results by a different method.

(3.24) $$\tilde{a}_1/\tilde{a}_0 > a_1/a_0$$

is satisfied, the boundary is stable in the cases a) and b). Substituting the values of the coefficients, we obtain

$$BV\left[(\beta^2-\alpha)\,BhV+(al-\gamma)\,Db+(AB-D^2)\,V^2\alpha\beta-Ab\,(\alpha c-\beta)\right]+h\left[A\,(\beta hV+\alpha BV^2+ac^2+b)-V^2\left(D^2\alpha-\frac{C^2}{r^2}\right)-2Dd\right]<0.$$

In the case a), i.e., $c=0$, $h\neq 0$, we have

$$A\beta Vh^2+\left[V^2\left(AB\beta^2+\frac{C^2}{r^2}\right)+AB\right]h+B\beta V\,(B\alpha V^2+Ab)<0.$$

Since the coefficients of the quadratic polynomial h are positive, this inequality cannot be satisfied for any $h>0$. Hence, the boundary is stable in this case.

In the case b), i.e., $c>0$, $h=0$, we have

(3.25) $$\alpha Abc > (AB-D^2)\,\alpha\beta V^2+b\,[A\beta+D\,(al-\gamma)].$$

Assuming a dependence of the moment of inertia B and the centrifugal moment D on the stagger of the form $B=B_0c^2$ and $D=D_0c$, where $B_0=\text{const}$ and $D_0=\text{const}$, we obtain an inequality which can be satisfied at small velocities and a very large stagger. However, substitution of the numerical values of the parameters of the aircraft wheel considered above shows that this inequality can only be satisfied for staggers much greater than those usually used in construction. In the case b), the boundary is therefore stable in practice.

The next stage of the investigation is to determine the intervals of values of the stiffness k (or, equivalently, of κ) of the strut for which the stability of the original system is not lost; to study the manner in which the finite stiffness alters the stability curve; and to find how this effect depends on the parameters V, c, and h. The most natural way to solve this problem would be to construct a D-decomposition with respect to the complex parameter κ. On the complex κ plane, the positive segment of the real axis lying in the stability region would give the interval of desired values of κ. Constructing the curves giving the dependence of the length of this segment on the relevant parameters V, h, and c, we obtain the complete answer to the question of the influence of the stiffness of the strut on the shimmy. If we keep all the parameters constant except κ, this method readily yields an accurate result, but the dependence on V, h, and c remains obscure. The solution can be

found much more readily by using the results of §3.3, in which a D-decomposition with respect to the parameters v^2 and hV was made for the case $\kappa = 0$, and then studying the variation of the resulting curves for different positive values of κ.

The investigation of the form of the stability regions for the parameters V and hV for different values of κ reduces to a study of the space of parameters (V, hV, κ), in which it is necessary to determine the boundary surface and the stability region. The intersection of this surface with the plane $\kappa = \text{const}$ gives us the boundary curve for the given stiffness. In the space (τ, ν, κ) where $\tau = V$, $\nu = hV$, and $\kappa = 1/k$, we construct the boundary surface and the stability region for the case $c = \beta/\alpha$. As is shown in §3.3, the boundary curve is the abscissa in this case for $\kappa = 0$. Keeping all the parameters constant apart from τ, ν, and κ, we obtain a parametric equation of the surface $hV = f(\kappa, V)$, which has the form shown in Figure 6.34. From the form of the surface, one can easily gauge the required damping for a given value of the transverse stiffness of the strut, and also the way in which the required damping varies with the stiffness. For example, suppose we must ensure the absence of shimmy for velocities from zero to 100 m/sec and that the boundary surface is constructed only for these values. The damping required to eliminate the shimmy must then be greater than the tangent of the angle of slope of the straight line drawn on the plane $\kappa = \text{const}$ in such a way that it passes through the $O\kappa$ axis and rests on the boundary surface.

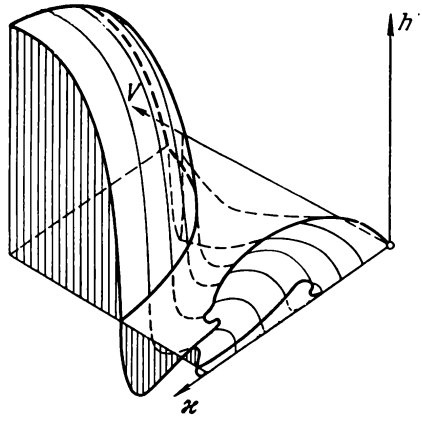


FIGURE 6.34

If the velocity of the aircraft on the runway does not exceed 100 m/sec, a study of the surface shown in Figure 6.34 shows that appreciable damping (of the order of 10) is required to eliminate shimmy if the stiffness of the strut is of the order of 10^5 or less. If the stiffness is further decreased, the amount of damping required begins to decrease. This is explained by the fact that, as $k \to 0$, the system approximates to the model of a wheel which can be displaced freely along the axis of rotation in its own plane. In an investigation of this limiting case [228],

it was found that such a model is not subject to shimmy. In conclusion, we should like to point out that the methods used here to determine the dependence of the required damping on the stiffness of the strut can be applied for any value of the stagger.

5. Shimmy at high velocities. Let us consider the problem of the shimmy of the nose wheel of a three-wheel aircraft landing gear in the case of fairly high velocities when the generalized hypothesis of creep (§1) can be applied.

In accordance with §3.1, the generalized coordinates of the system are $q_1 = \psi$ and $q_2 = \theta$. The kinetic energy T is

$$T = \frac{1}{2}\left(A\psi^2 + B\dot{\theta}^2 + 2D\psi\dot{\theta} + 2\frac{CV}{r}\theta\psi\right), \tag{3.26}$$

where A is the moment of inertia about the longitudinal axis passing through the joint of the strut; B is the moment of inertia about the axis of the strut; D is the centrifugal moment of inertia about these axes; C is the moment of inertia of the wheel about its axis of rotation in its own plane; and r is the radius of the wheel. The generalized forces are

$$Q_\psi = (Nl - k)\psi + cN\theta, \quad Q_\theta = cN\psi - h\dot{\theta}, \tag{3.27}$$

where N is the value of the normal load on the nose strut of the landing gear; l is the distance from the point of joint of the strut to the runway; k is the stiffness coefficient of the elastic joint of the strut; c is the stagger of the wheel ($c > 0$ for backward stagger); and h is the coefficient of viscous friction in the damper. The generalized forces R_j occurring in the equations of motion (1.18) are given by

$$R_j = F\frac{\partial x}{\partial q_j} + M_\theta\frac{\partial\theta}{\partial q_j} + M_\chi\frac{\partial\chi}{\partial q_j}, \tag{3.28}$$

in which the quantities x and χ are related to the generalized coordinates $q_1 = \psi$ and $q_2 = \theta$ by the equations

$$x = l\psi + c\theta, \quad \chi = -\psi \tag{3.29}$$

and, in accordance with (1.25), the quantities F, M_θ, and M_χ are

$$\begin{aligned} F &= \frac{a_1}{V}\dot{\theta} - a_2\theta - \frac{a_2}{V}\dot{x} + a_3\chi, \quad M_\theta = -\frac{b}{V}\dot{x} - b\theta, \\ M_\chi &= -\frac{b_1}{V}\theta + b_2\theta + \frac{b_2}{V}\dot{x} - b_3\chi. \end{aligned} \tag{3.30}$$

Here, the coefficients a_i and b_i are the deformation parameters of the pneumatic tire and can be expressed in terms of Keldyš's coefficients by means of the relations (1.26). With allowance for (3.29) and (3.30), equation (3.28) yields the following expressions for the generalized forces:

$$
\begin{aligned}
R_\psi = lF - M_\chi = &- \frac{c(a_2 l + b_2) - (a_1 l + b_1)}{V}\dot\theta \\
&- \frac{l(a_2 l + b_2)}{V}\psi - (a_2 l + b_2)\theta - (a_3 l + b_3)\psi, \\
R_\theta = cF + M_\theta = &- \frac{c(a_2 c + b - a_1)}{V}\dot\theta - \frac{l(a_2 c + b)}{V}\psi \\
&- (a_2 c + b)\theta - a_3 c\psi.
\end{aligned}
\tag{3.31}
$$

Writing down the equations (1.18), where the kinetic energy T is given by (3.26) and the generalized forces by (3.27) and (3.31), we obtain

$$
\begin{aligned}
&A\ddot\psi + \frac{l(a_2 l + b_2)}{V}\psi + (a_3 l + b_3 + k - Nl)\psi + D\ddot\vartheta \\
&\quad + \left[\frac{CV}{r} + \frac{c(a_2 l + b_2) - (a_1 l + b_1)}{V}\right]\dot\theta + (a_2 + b_2 - cN)\theta = 0, \\
&D\ddot\psi - \left[\frac{CV}{r} - \frac{l(a_2 c + b)}{V}\right]\psi + (a_3 c - cN)\psi + B\ddot\theta \\
&\quad + \left[h + \frac{c(a_2 c + b - a_1)}{V}\right]\dot\theta + (a_2 c + b)\theta = 0.
\end{aligned}
\tag{3.32}
$$

Equations (3.32) enable one to find the conditions for the occurrence of shimmy in the case of fairly high velocities V of the aircraft on the runway. In accordance with (3.32), the characteristic equation has the form

$$
\begin{aligned}
&\tau^2 p^4 + (A_1\nu + \alpha_1 - c\alpha_2)\tau p^3 + [c\alpha_3 + \alpha_4\nu + (c\alpha_5 + \alpha_6)\tau + C_1^2\tau^2]p^2 \\
&\quad + [c\alpha_7 + c^2\alpha_8 + \alpha_9\nu + (\alpha_{10} - c\alpha_{11})\tau]p + \alpha_{12} + c\alpha_{13} + c^2\alpha_{14} = 0,
\end{aligned}
\tag{3.33}
$$

in which we have separated the parameters $\tau = V^2(AB - D^2)^{1/2}$, $\nu = hD$, and c (the stagger of the wheel) and the remaining quantities are as follows:

$$
\begin{aligned}
&D_1 = D\,(AB - D^2)^{-1/2},\ A_1 = A\,(AB - D^2)^{-1/2}, \\
&B_1 = B\,(AB - D^2)^{-1/2},\ C_1 = \frac{C}{r}(AB - D^2)^{-1/2}, \\
&\alpha_1 = B_1 l\,(a_2 l + b_2) + D_1\,(a_1 l + b_1 - lb), \\
&\alpha_2 = D_1\,(2a_1 l + b_2), \\
&\alpha_3 = l[a_2\,(lb + b_1) + b_2\,(b - a_1)], \\
&\alpha_4 = l\,(a_2 l + b_2), \\
&\alpha_5 = A_1 a_2 + C_1 b_2 + D_1\,(a_3 - 2N), \\
&\alpha_6 = B_1 k + A_1 b + B_1 l\,(a_3 - N) - D_1\,(a_2 l + b_2),
\end{aligned}
$$

$$\alpha_7 = (b - a_1)(k + b_3) + a_3 b l + b_1 (a_3 - N),$$
$$\alpha_8 = a_2 (b_3 + k) - a_3 b_2 + N (a_2 l + b_2),$$
$$\alpha_9 = a_3 l + b_3 + k - Nl,$$
$$\alpha_{10} = C_1 (a_2 l + b_2),$$
$$\alpha_{11} = C_1 a_3,$$
$$\alpha_{12} = b(a_3 l + b_3 + k - Nl),$$
$$\alpha_{13} = a_2 (b_3 + k) - b_2 (a_3 - N),$$
$$\alpha_{14} = N (a_3 - N).$$

In the case of a very high stiffness of the strut ($k \to \infty$), the characteristic equation can be simplified and its order reduced by two:

$$B_1 \tau p^2 + [c^2 a_2 - c (a_1 - b) + \nu] p + c a_2 + b = 0. \tag{3.34}$$

It follows that for $c > -b/a_2$ and an arbitrarily small damping h there always exists a velocity $V = V^*$ such that for all $V > V^*$ the coefficient of p in equation (3.34) is positive and the motion is therefore stable ($\nu = hV$). This accords with the results of Keldyš, who found that, in this case, the required damping tends to zero as the velocity of rolling increases. The physical explanation of this fact is that if the wheel rolls very rapidly, the rate of damping of the inertia-free pneumatic tire is determined by the rolling velocity, whereas the rate of rotation of the inert strut remains finite since the finite deformations of the pneumatic tire give rise to finite forces. Therefore, at high velocities, there is always sufficient time for the pneumatic tire to take up the deformed configuration, which is identical with that corresponding to rolling of the wheel with a constant angle θ. By the same token, the conditions are satisfied under which the generalized hypothesis of creep can be applied, and we obtain the characteristic equation (3.34).

Let us now find the conditions under which a system, stable for $k = \infty$, preserves its stability for finite, fairly large values of k. To this end, we introduce the small parameter $\kappa = 1/k$ and write the characteristic equation (3.33) in the form

$$B_1 \tau p^2 + [c^2 a_2 - c (a_1 - b) + \nu] p + c a_2 + b + \varkappa \{\tau^2 p^4 + (A_1 \nu + \alpha_1 - c\alpha_2) \tau p^3 + \ldots\} = 0,$$

where the dots stand for the terms with lower powers of p. Suppose that the system is stable for $\kappa = 0$. As is well known [298], the system also preserves its stability for fairly small values of κ if

$$a_1/a_0 < b_1/b_0, \tag{3.35}$$

where a_0 and a_1 are the coefficients of the highest terms of the first polynomial and b_1 and b_2 are the coefficients of the highest terms of the polynomial with the factor κ. In our case,

$$a_0 = B_1\tau,\ a_1 = c^2a_2 - c\,(a_1 - b) + \nu,\ b_0 = \tau^2,\ b_1 = (A_1\nu + \alpha_1 - c\alpha_2)\tau.$$

Substituting these expressions into the inequality (3.35), we obtain

$$D^2\nu + \alpha_1 B_1 + c\,(a_1 - b - B_1a_2) - c^2a_2 > 0. \tag{3.36}$$

This inequality can always be satisfied for all $V > V^{**}$ or any arbitrarily small damping $h \neq 0$, where the critical velocity V^{**} depends on the relationship between the remaining parameters of the system and can be determined from the condition of vanishing of the left-hand side of the inequality (3.36).

6. Shimmy for large values of the kinematic parameters of a pneumatic tire. In accordance with the generalized hypothesis of creep, the generalized coordinates of a strut with a pneumatically tired wheel in the case of fairly large values of the kinematic parameters of the tire (see §1) are θ, ψ, and ξ, where ξ is the transverse deformation of the tire. In accordance with (1.30), the equation of the kinematic constraint in this case can be written in the form

$$l\psi + c\dot{\theta} + \dot{\xi} + V\theta + \varkappa_1 V\xi + \varkappa_2 V\psi = 0, \tag{3.37}$$

where κ_1 and κ_2 are the creep coefficients which are related to the kinematic parameters of the tire by the equations $\kappa_1 = \alpha/\beta$ and $\kappa_2 = \gamma/\beta$. The equations of motion can be written in the form (1.18), where the kinetic energy T of the system is given by (3.26). In order to find the generalized forces Q_ψ and Q_θ, we write down an expression for the virtual work of the force of the normal reaction N of the runway, the forces of the elastic joint of the strut, and the forces of viscosity in the damper:

$$\delta A = N\delta r_N - k\psi\delta\psi - h\dot{\theta}\delta\theta = [(Nl - k)\,\psi + cN\theta]\,\delta\psi + (cN\psi - h\dot{\theta})\delta\theta.$$

Hence,

$$Q_\psi = (Nl - k)\,\psi + cN\theta,\ \ Q_\theta = cN\psi - h\dot{\theta}.$$

The generalized forces R_j are given by equations (3.28) and (3.29), in which F, M_θ, and M_χ are given by the expressions (1.31):

$$F = a\xi - \sigma N\psi,\ \ M_\chi = -\sigma N\xi + \rho N\psi,\ \ M_\theta = b\varkappa_1,\ \xi + b\varkappa_2\psi.$$

Hence,

$$R_\psi = lF - M_\chi = (al + \sigma N)\ \xi - (\sigma l + \rho)\,N\psi,$$
$$R_\theta = cF + M_\theta = (ac + b,\varkappa_1)\ \xi + (b\varkappa_2 - c\sigma N)\psi.$$

Substituting these expressions into (1.18), we obtain the following equations

$$\begin{aligned} &A\ddot{\psi} + \alpha_1\psi + D\ddot{\theta} - \frac{CV}{r}\dot{\theta} - cN\theta - \alpha_2\xi = 0, \\ &D\ddot{\psi} + \frac{CV}{r}\psi + \alpha_3\psi + B\ddot{\theta} + h\dot{\theta} - \alpha_4\xi = 0, \end{aligned} \tag{3.38}$$

where

$$\alpha_1 = k + N[(\sigma - 1)l + \rho], \quad \alpha_3 = (\sigma - 1)lN - b\varkappa_2,$$
$$\alpha_2 = al + \sigma N, \qquad \alpha_4 = ac + b\varkappa_1.$$

The system of equations (3.37) and (3.38) describes the small vibrations of the nose strut of the landing gear in the present case. The system has the unique steady motion $\psi = \theta = \xi = 0$, whose stability is determined by the roots of the characteristic equation

$$\begin{vmatrix} Ap^2 + \alpha_1 & , & Dp^2 - \frac{CV}{r}p - cN, & -\alpha_2, \\ Dp^2 + \frac{CV}{r}p + \alpha_3, & & Bp^2 + hp & , & -\alpha_4 \\ lp + \varkappa_2 V & , & cp + V & , & p + \varkappa_1 V \end{vmatrix} = 0. \tag{3.39}$$

In order to be able to compare the results obtained on the basis of the generalized hypothesis of creep with the results following from Keldyš's theory, let us consider the case of an absolutely rigid strut ($k = \infty$). In this case, the characteristic equation (3.39) can be simplified and takes the form

$$Bp^3 + (\varkappa_1 BV + h)p^2 + (\varkappa_1 hV + c\alpha_4)p + \alpha_4 V = 0. \tag{3.40}$$

Since $V > 0$ by hypothesis, a necessary condition for stability is $\alpha_4 > 0$. Therefore, if we introduce the positive parameters τ and ν:

$$\tau = BV^2/\alpha_4, \qquad \nu = hV/\alpha_4,$$

and replace p by pV, we can write the characteristic equation (3.40) most simply in the form

$$\tau p^3 + (\varkappa_1\tau + \nu)p^2 + (c + \varkappa_1\nu)p + 1 = 0. \tag{3.41}$$

Apart from the parameters τ and ν, the left-hand side of equation (3.41) contains the positive parameter $\varkappa_1$, the first creep coefficient, and the parameter c, the stagger of the wheel, which, in accordance with the inequality $\alpha_4 > 0$, may take any values in the interval $-\varkappa_1 b/a < c < +\infty$.

It should be noted that equation (3.41) contains only the first creep coefficient $\varkappa_1$. Therefore, the plane of the wheel always remains in the vertical position in this case ($k = \infty$). According to the foregoing analysis,

the second creep coefficient plays no role at all in this case in the dynamics of a wheel with a pneumatic tire.

Let us make a D-decomposition of the (τ, ν) parameter plane and investigate the behavior of the boundary of the stability region for different values of the stagger of the wheel c, assuming that κ_1 is fixed. Setting $p = j\omega$ ($j = \sqrt{-1}$) in (3.41), we obtain a parametric expression for the unicursal curve

$$\tau = \frac{c\Omega + \kappa_1}{\Omega(\Omega + \kappa_1^2)}, \quad \nu = \frac{1 - c\kappa_1}{\Omega + \kappa_1^2}, \tag{3.42}$$

where $\Omega = \omega^2$ and the determinant needed to shade this curve is $\Delta = -\omega^3(\Omega + \kappa_1^2)$. As $\Omega \to 0$, the curve $\nu = \nu(\tau)$ has the asymptote $\nu = (1 - c\kappa_1)/\kappa_1^2$, and, as $\Omega \to \infty$,

$$\lim_{\Omega \to \infty} \tau(\Omega) = \frac{c}{\Omega}, \quad \lim_{\Omega \to \infty} \nu(\Omega) = \frac{1 - c\kappa_1}{\Omega}.$$

Hence

$$\left(\frac{d\nu}{d\tau}\right)_{\Omega=\infty} = \frac{1 - c\kappa_1}{c}.$$

In addition to the curve (3.42), a solution of equation (3.41) as $p \to \infty$ is the straight line $\tau = 0$, the singular straight line of the D-decomposition. For different values of the parameter c, the curve (3.42) of the D-decomposition has the form shown in Figure 6.35. For the value $c = 1/\kappa_1$, the curve (3.42) degenerates into the straight line $\nu = 0$. In this case, the stability region is the whole of the first quadrant of the (τ, ν) plane, i.e., the system does not require damping for any velocity V. Comparison of the diagram of Figure 6.35 with the corresponding curves of Figure 6.30 reveals that the only deviation from Keldyš's theory in the result of the investigation of the shimmy of the nose wheel occurs in the region of self excitation of vibrations at high frequencies.

§4. Shimmy of the front suspension of an automobile

Among the perfections of the chassis of an automobile, the invention of pneumatic tires was of particular importance. It made great improvements possible in the speed attainable and also ensured particularly soft suspension of the automobile. However, the transition to wheels with pneumatic tires gave rise in the case of various constructions of the front suspension of the automobile to very unpleasant vibrations of the automobile at certain, usually high velocities. These vibrations were caused by the stable vibrations of the front suspension. These vibrations of the front suspension, which came to be known as shimmy,

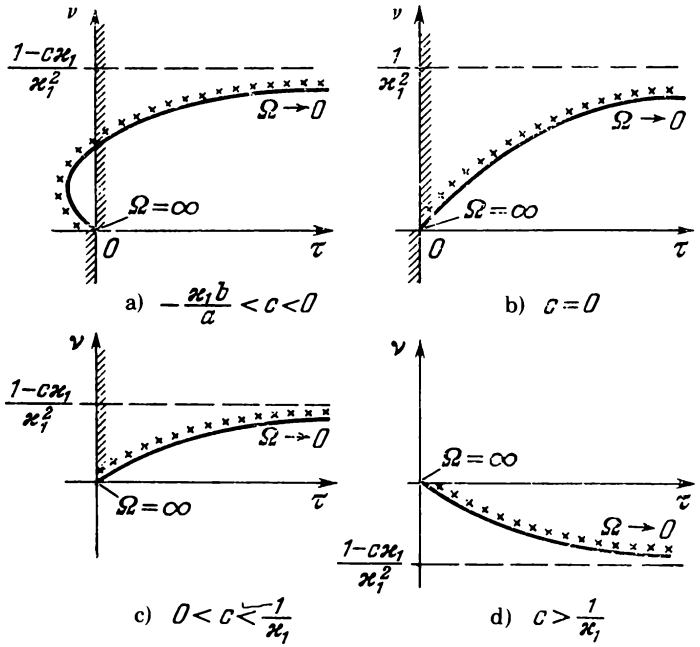


FIGURE 6.35

usually consisted of vibrations of the wheels about the vertical axis and rotations of the front axle with the wheels about the longitudinal axis of the automobile. Shimmy gives rise to large dynamical stresses on the parts of the suspension and the steering and may mean that the automobile cannot be controlled.

In this connection, a theoretical study of shimmy acquires particular importance.

Shimmy has been studied theoretically by many authors. The first attempts were unsuccessful, since it was originally assumed that shimmy is due to forced vibrations of the suspension arising as a result of the influence of an external force (roughness of the road, unbalance of the wheels, etc.) and the gyroscopic effect of the wheels amplified at resonance. That this explanation was unsatisfactory could already be deduced from the fact that shimmy arises when an automobile is driven on a smooth road and the wheels are well balanced.

The correct theory of shimmy only began when it was realized that it is a self-oscillation phenomenon arising as a result of random excitation due to the existence of an instability of the system and maintained by means of a constant source of energy, the automobile motor.

This suggestion was first put forward by B. A. Gluh [172]. A complete treatment of shimmy, including a calculation of the amplitude, is a complicated nonlinear problem. If it is only desired to find the conditions of excitation of self-oscillations i.e., the condition for the occurrence of shimmy (a problem of great practical importance), it is sufficient to investigate only the linearized problem and consider small perturbations of the original motion. In the present section, we shall consider the problem of finding the conditions of selfexcitation of shimmy, first on the basis of Keldyš's theory and then using the generalized hypothesis of creep. The investigation will allow us to make a D-decomposition with respect to two real parameters and, on the basis of the decomposition, to find formulas for the critical velocities. These will enable us to make an appropriate choice of the relevant parameter if the remaining parameters of the system are specified.

1. Statement of the problem and the equations of motion. We shall consider the simplest arrangement of a dependent front suspension of an automobile (Figure 6.36), when the center of gravity of the suspension is at an equal distance from both wheels; the suspension is dynamically symmetric with respect to its center of gravity; the wheels are perfectly balanced, their central planes are always parallel, and the axis of the kingpin is vertical.

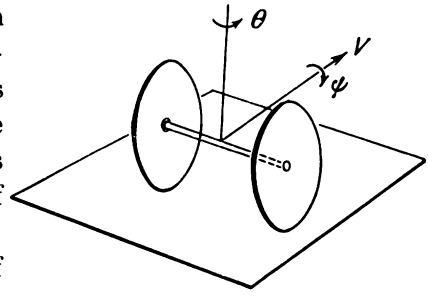


FIGURE 6.36

We shall assume that the body of the automobile is so massive that it does not vibrate at all and moves along a horizontal plane with a constant rectilinear velocity $V = \text{const}$. The problem is to determine the velocity V of the automobile for which the front suspension becomes unstable. In accordance with the assumptions made, we shall assume that the load N on the front suspension is constant for all conditions of motion.

Leaving aside the motion of the center of gravity of the suspension, we shall derive the equations of small vibrations of the suspension when it moves relative to its center of mass. The position of the suspension relative to the center of mass can be specified by two angles: the angle ψ of rotation of the front axle together with the wheels about the longitudinal axis of the automobile and the angle θ of rotation of the wheels about the kingpin. The positive values of ψ will be measured from the horizontal position of the front axle in the clockwise direction

looking along the direction of motion of the automobile. A positive value of θ corresponds to the wheels being turned to the left. We shall assume that the wheels are rigidly connected to each other and rotate simultaneously about the kingpin through the same angle θ. Apart from the angles ψ and θ, the generalized coordinates are also the deformation parameters of the pneumatic tires: ξ_1, θ_1, ξ_2, and φ_2, where the subscript 1 refers to the left-hand wheel and the subscript 2 to the right-hand wheel. In accordance with the theory of rolling of an elastic pneumatic tire, the motion of the system is subject to kinematic constraints expressed by equations (1.14). In this case, the latter can be written in the form

$$\begin{aligned} \dot{\xi}_1 - r\dot{\psi} + V\theta + V\varphi_1 &= 0, \\ \dot{\xi}_2 - r\dot{\psi} + V\theta + V\varphi_2 &= 0, \\ \dot{\theta} + \dot{\varphi}_1 - \alpha V\xi_1 + \beta V\varphi_1 + \gamma V\dot{\psi} &= 0, \\ \dot{\theta} + \dot{\varphi}_2 - \alpha V\xi_2 + \beta V\varphi_2 + \gamma V\dot{\psi} &= 0 \end{aligned} \tag{4.1}$$

and they represent the conditions of rolling without slipping of wheels with pneumatic tires. In writing down equations (4.1), we have taken into account the relations

$$x_1 = -l - r\psi, \quad x_2 = l - r\psi, \quad \chi = \psi, \tag{4.2}$$

where l is half the length of the front axle and r is the radius of the wheels. The relations (4.2) connect the generalized coordinates ψ and θ to the quantities occurring in equations (1.14). The dynamical equations of the front suspension can be written in the form (1.18). The kinetic energy T of the system is

$$T = \frac{1}{2}\,(J_1\dot{\psi}^2 + J_2\dot{\theta}^2) - \frac{JV}{r}\theta\dot{\psi}, \tag{4.3}$$

where J_1 is the moment of inertia of the suspension about the longitudinal axis of the automobile; J_2 is the moment of inertia of both wheels about the axis of the kingpin; and J is twice the moment of inertia of a wheel about the axis of rotation in its own plane. The generalized forces acting on the front suspension are

$$Q_\theta = -k_1\theta - h_1\dot{\theta}, \quad Q_\psi = -k_2\psi - h_2\dot{\psi}. \tag{4.4}$$

Here, k_1 is the stiffness of the steering; k_2 is the stiffness of the springs and pneumatic tires in the vertical direction; and h_1 and h_2 are the coefficients of viscous friction in the shock absorbers and the steering. We shall assume that the damping in the suspension has linear characteristics within the limits of variations of the variables ψ and θ.

In our case, the potential energy of the deformed pneumatic tires can be written in the form

$$U = \frac{1}{2}\left[a(\xi_1^2 + \xi_2^2) + 2\sigma N\chi(\xi_1 + \xi_2) + 2\rho N\chi^2 + b(\varphi_1^2 + \varphi_2^2)\right], \tag{4.5}$$

and therefore, taking into account (4.2), we obtain

$$\begin{aligned} R_\psi &= -(ar + \sigma N)(\xi_1 + \xi_2) - 2N(\sigma r + \rho)\psi, \\ R_\theta &= b(\varphi_1 + \varphi_2). \end{aligned} \tag{4.6}$$

Substituting the expressions for the kinetic energy T and the generalized forces into the equations of motion (1.18), we obtain the equations

$$\begin{aligned} J_1\ddot{\psi} + h_2\dot{\psi} + [k_2 + 2N(\sigma r + \rho)]\psi - \frac{JV}{r}\dot{\theta} \\ + (ar + \sigma N)(\xi_1 + \xi_2) = 0, \\ J_2\ddot{\theta} + h_1\dot{\theta} + k_1\theta + \frac{JV}{r}\psi - b(\varphi_1 + \varphi_2) = 0, \end{aligned} \tag{4.7}$$

which, together with the kinematic equations (4.1), form a complete system of equations for the small vibrations of the suspension.

2. Characteristic equation. Setting $\psi = \psi_0 e^{pt}$, $\theta = \theta_0 e^{pt}$, $\xi_{1,2} = \xi_{1,2}^0 e^{pt}$, and $\varphi_{1,2} = \varphi_{1,2}^0 e^{pt}$, we obtain the characteristic equation for the determination of p. It can be written in the form

$$P_2(p) \cdot P_6(p) = 0,$$

where $P_2(p)$ and $P_6(p)$ are, respectively, polynomials of second and sixth degree in p. Since both roots of the polynomial $P_2(p) \equiv p^2 + \beta Vp + \alpha V^2$ have a negative real part, the stability of the system is governed by the roots of the equation

$$P_6(p) = 0. \tag{4.8}$$

Introducing the notation

$$\begin{gathered} \frac{J}{rJ_1} = A_1, \quad \frac{J}{rJ_2} = A_2, \quad A_1A_2 = A, \quad \frac{2b}{J_2} = B, \quad \frac{k_1}{J_2} = K, \\ \frac{2(ar + \sigma N)}{J_1} = C, \quad \frac{k_2 + 2N(\sigma r + \rho)}{J_1} = M, \quad \frac{h_1}{J_2V} = \delta_1, \quad \frac{h_2}{J_1V} = \delta_2 \end{gathered}$$

defining the positive parameters

$$\tau = 1/V^2, \qquad \nu = M/V^2, \tag{4.9}$$

and replacing p by pV and cancelling V^6 $(V \neq 0)$, we can write the characteristic equation (4.8) in the form

$$p^6 + c_1 p^5 + c_2 p^4 + c_3 p^3 + c_4 p^2 + c_5 p + c_6 = 0, \tag{4.10}$$

where

$$\begin{aligned}
c_1 &= \beta + \delta_1 + \delta_2,\\
c_2 &= A + \alpha + (K + B + rC)\,\tau + \nu + \beta\delta_1 + \beta\delta_2 + \delta_1\delta_2,\\
c_3 &= \beta A + \beta\nu + \beta\,(K + rC)\,\tau + (a + \nu + rC\tau)\,\delta_1\\
&\qquad + [\alpha + (K + B)\,\tau]\,\delta_2,\\
c_4 &= [\alpha\,(K + B - A_1 Br) + \gamma\,(C + A_1 B)]\,\tau + \alpha\nu\\
&\quad + rC\,(K + B)\,\tau^2 + \alpha A + (K + B)\,\tau\nu + \beta\,(\nu + rC\tau)\,\delta_1\\
&\qquad + \beta K\tau\delta_2 + \alpha\delta_1\delta_2,\\
c_5 &= \beta A_2 C\tau + \beta rKC\tau^2 + \beta K\nu\tau + (\gamma C\tau + \alpha\nu)\,\delta_1,\\
c_6 &= (K + B)\,(C\gamma\tau^2 + \alpha\nu\tau).
\end{aligned}$$

Equation (4.10) is very difficult to investigate in the general case and we shall therefore restrict ourselves to a special case when the damping of the suspension is negligibly small and the parameters of the wheels and tires satisfy the relation $\alpha r - \gamma = 0$.

3. Stability of the front suspension in the absence of damping. In the case when the parameters of the wheels and tires satisfy the relation $\alpha r - \gamma = 0$ and the damping of the suspension is negligibly small ($\delta_1 = \delta_2 = 0$), the coefficients of the characteristic equation (4.10) simplify somewhat. Eliminating the parameter γ ($\gamma = \alpha r$), we can write the characteristic equation (4.10) in the form

$$\begin{aligned}
&p^6 + \beta p^5 + [A + \alpha + \nu + (K + B + rC)\,\tau]\,p^4\\
&+ \beta\,[A + \nu + (K + rC)\,\tau]p^3 + [\alpha A + \alpha\,(K + rC + B)\tau\\
&+ \alpha\nu + (K + B)\,\nu\tau + rC\,(K + B)\,\tau^2]\,p^2 + \beta\,[rKC\tau^2 + A_2 C\tau\\
&\qquad + K\nu\tau]\,p + \alpha\,(K + B)\,(\nu\tau + rC\tau^2) = 0.
\end{aligned} \tag{4.11}$$

Using the D-decomposition method, we can find the stability region of the system on the (τ, ν) parameter plane. Substituting the value $p = j\omega$ ($j = \sqrt{-1}$) into equation (4.11), we separate out and then equate to zero the real and imaginary parts, obtaining ($\Omega = \omega^2$)

$$\begin{aligned}
&-(\Omega - \alpha)\,\{rC\,(K + B)\,\tau^2 - (K + B + rC)\,\Omega\tau\\
&\qquad + (\Omega - A)\,\Omega - \nu\,[\Omega - (K + B)\,\tau]\} = 0,\\
&\omega\,\{rCK\tau^2 - [(rC + K)\,\Omega - A_2 C]\,\tau + \Omega^2\\
&\qquad - A\Omega - \nu(\Omega - K\tau)\}\,\beta = 0.
\end{aligned} \tag{4.12}$$

We now express τ and ν in terms of Ω. From the second equation of (4.12), we obtain

$$\nu = \frac{rCK\tau^2 - [(rC + K)\,\Omega - A_2 C]\,\tau + \Omega\,(\Omega - A)}{\Omega - K\tau}. \tag{4.13}$$

Substituting this expression into the first equation of (4.12) and performing simple transformations, we obtain

$$\tau\,[-A_2C\,(K+B)\,\tau+(AB+A_2C)\,\Omega]=0. \tag{4.14}$$

From equations (4.13) and (4.14), we find the equations of two branches of the boundary of the D-decomposition. For, setting $\tau=0$ in (4.13) and (4.14), we find

$$\tau=0,\quad \nu=\Omega-A \tag{4.15}$$

which is the parametric equation of the ray $\tau=0$, $\nu\geqq -A$, traversed twice as the parameter ω varies from $-\infty$ to $+\infty$. Equating the square brackets in (4.14) to zero, we obtain, together with (4.13),

$$\tau=\lambda\Omega,\quad \nu=(1-rC\lambda)\,\Omega+\lambda_1, \tag{4.16}$$

where

$$\lambda=\frac{A_2C+AB}{A_2C\,(K+B)},\qquad \lambda_1=\frac{A_2C}{B}>0.$$

Equations (4.16) are the parametric equation of the ray $\nu=(1/\lambda-rC)\tau+\lambda_1$, traversed twice as the parameter ω varies from $-\infty$ to $+\infty$.

In order to find the singular straight lines and the rules for shading, we write down the determinant

$$\Delta=\begin{vmatrix}\dfrac{\partial f_1}{\partial\tau}, & \dfrac{\partial f_1}{\partial\nu}\\[2ex] \dfrac{\partial f_2}{\partial\tau}, & \dfrac{\partial f_2}{\partial\nu}\end{vmatrix},$$

where f_1 and f_2, the real and imaginary parts of the polynomial (4.11), are given by the left-hand sides of equations (4.12). Performing the necessary differentiation and expanding the determinant, we obtain after simplifications

$$\Delta=\omega\,(\Omega-\alpha)\,\{[rCB\Omega+A_2C\,(K+B)]\,\tau+\Omega\,(B\nu-A_2C-B\Omega)\}. \tag{4.17}$$

It follows from the expression (4.17) that Δ vanishes for $\omega=0$ and $\Omega=\alpha$ and also when the curly brackets vanish. In accordance with (4.12), the singular straight lines correspond to $\omega=0$ and $\Omega=\alpha$. Substituting $\omega=0$ into (4.12) or, equivalently, into (4.11), we obtain

$$\tau(\nu+rC\tau)=0.$$

It follows that the value $\omega=0$ corresponds to two singular straight lines:

$$\tau=0,\quad \nu=-rC\tau, \tag{4.18}$$

the straight line $\tau = 0$ corresponding to a twofold vanishing root and the other straight line in (4.18) to a simple vanishing root.

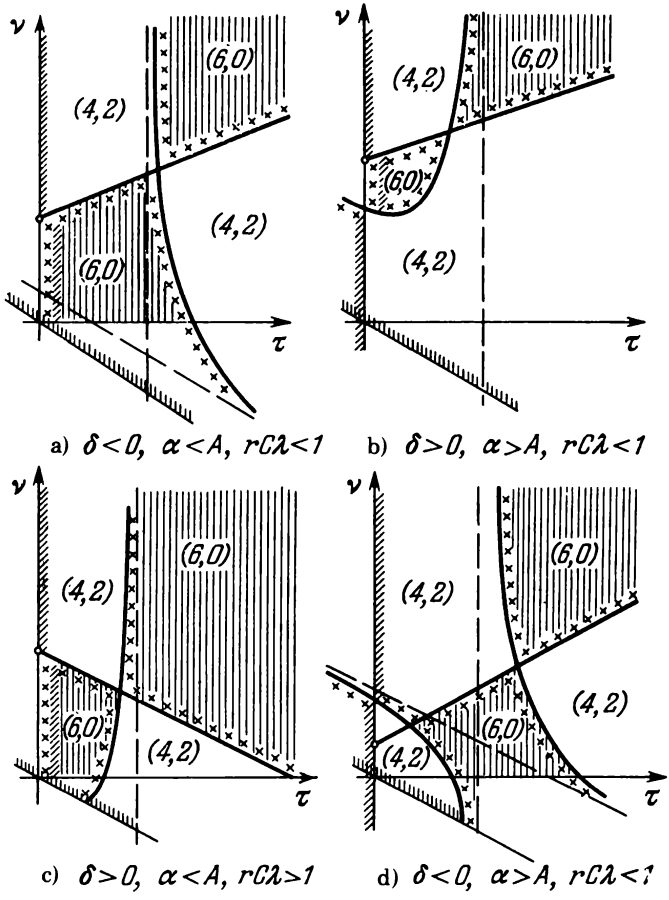


FIGURE 6.37

For $\Omega = \alpha$, the first of the equations of (4.12) is satisfied identically for τ and ν and the second equation yields

$$rCK\tau^2 + [A_2C - \alpha\,(rC + K)]\,\tau + \alpha\,(\alpha - A) - \nu\,(\alpha - K\tau) = 0,$$

which can be solved for the parameter ν:

$$\nu = \frac{rCK\tau^2 + [A_2C - (rC + K)\,\alpha]\,\tau + \alpha\,(\alpha - A)}{\alpha - K\tau}. \tag{4.19}$$

Equation (4.19) is a hyperbola with the asymptotes

$$\tau = \frac{\alpha}{K}, \quad \nu = -rC\tau + \alpha - \frac{A_2C}{K}. \tag{4.20}$$

Depending on the sign of

$$\delta = A_2C - AK$$

the branches of this hyperbola are as shown in parts a) and d) or b) and c) of Figure 6.37.

On the curves (4.15) and (4.16), the determinant (4.17) has the form

$$\Delta = \pm\,\omega^3(\Omega - \alpha)(A_2C + AB), \tag{4.21}$$

where the minus sign corresponds to the first curve and the plus sign to the second. The expression (4.21) shows that Δ changes sign on the curves (4.15) and (4.16) only at the values $\omega = 0$ and $\Omega = \alpha$. In accordance with the foregoing, the D-decomposition of the parameter plane has one of the forms depicted in Figure 6.37, in which the straight line (4.16) is shaded in accordance with the sign of the determinant Δ and the branch of the singular curve (4.19), which intersects the curve (4.16) in the first quadrant, is then shaded uniquely in accordance with the ordinary rules. The shading of the singular straight line $\tau = 0$ for large positive values of ν follows directly from equation (4.11), since only one root has a positive real part if τ vanishes. The shading of the second of the singular straight lines of (4.18) also follows directly from the equations. In order to determine the stability region, it is now sufficient to note that, at the point $\tau = \nu = 0$, the characteristic equation (4.11) can be written in the form

$$p^2(p^2 + A)(p^2 + \beta p + \alpha) = 0$$

and has two imaginary roots, two pure-imaginary roots, and two roots with a negative real part. If we move along the axis $\nu = 0$ by a small amount $\tau \ll 1$ from the origin, we can write the characteristic equation in the form

$$p^2(p^2 + A)(p^2 + \beta p + \alpha) + \tau\,[(K + B + rC)p^4 + \beta(K + rC)p^3 + \alpha(K + B + rC)p^2 + \beta A_2 C p + \alpha r C(K + B)\tau] = 0,$$

where small terms of higher order have been neglected. It follows from this equation that to the right of the origin on the (τ, ν) plane there is a $D(4,2)$ region if $\alpha > A$ or a $D(6,0)$ region (i.e. a stability region) if $\alpha < A$.[10] To the left of the origin, we have, accordingly, a $D(4,2)$ or

[10] The notation $D(4,2)$ indicates that the polynomial (4.11) has four roots with a negative real part and two roots with a positive real part for values of the parameters in this region of the (τ, ν) plane.

$D(2,4)$ region. Thus, we obtain a D-decomposition of the (τ,ν) plane, on which, in accordance with the physical meaning (4.9) of the parameters τ and ν, only the first quadrant is of interest. In Figure 6.37 we show the four possible cases for the D-decomposition; the stability region is shaded continuously.

In order to get a clear idea of the manner in which the stability of the front suspension of the automobile depends on its velocity and the parameters of the suspension, we have plotted the straight line $\nu = M\tau$ in Figure 6.37. The origin, through which this straight line passes, corresponds to an infinitely high velocity of the automobile. A displacement along this straight line to infinity corresponds to a decrease in the velocity of the automobile to zero. From the diagrams of Figure 6.37, we see that if $\alpha > A$ the front suspension is always unstable for high velocities V of the automobile but that it is always stable at these velocities if $\alpha < A$. Let us consider in more detail these two cases for the relationships between the parameters of the suspension corresponding to the D-decomposition of Figures 6.37a and 6.37d. Figure 6.38a corresponds to the first case; in this figure, the straight lines drawn through the origin correspond to three values of the parameter M $(M_1 > M_2 > M_3)$, the straight line $\nu = M_1\tau$ passing through the

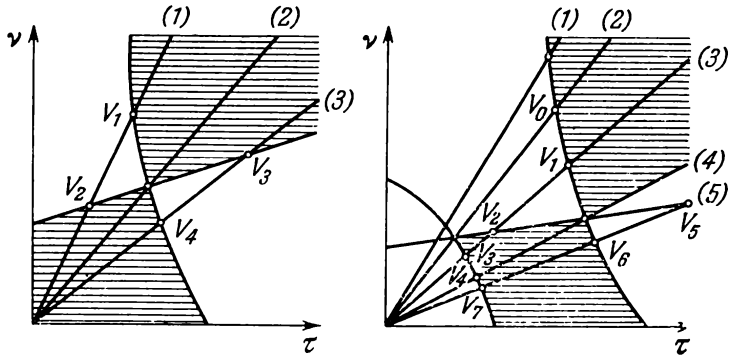


FIGURE 6.38

point of intersection of the straight line (4.16) and the singular curve (4.19). In accordance with Figure 6.38a, the front suspension of the automobile is, in this case, stable at all velocities of the automobile only for the unique value $M = M_2$. For the remaining values of M, for example, $M = M_1$ [the straight line (1) in Figure 6.38a], there exists an interval of velocities $V_1 > V > V_2$ for which shimmy arises. In order to calculate the critical velocities V_{cr} at the limits of this interval, it is

sufficient to write down the conditions for the intersection of the straight line $\nu = M\tau$ with the corresponding curves. Thus, from the condition for the intersection of the straight line $\nu = M\tau$ and the straight line (4.16), we obtain

$$\tau_{cr} = \frac{1}{V_{cr}^2} = \frac{\lambda_1}{M + rC - \frac{1}{\lambda}}.$$

Hence,

$$V_{cr}^2 = \frac{B}{A_2C}\left[M + rC - \frac{A_2C(K+B)}{A_2C + AB}\right]. \tag{4.22}$$

Similarly, from the condition for the intersection of the straight line $\nu = M\tau$ and the singular curve (4.19), we obtain

$$\tau = \frac{\alpha(rC + K + M) - A_2C \pm \sqrt{[(rC + K + M)\alpha - A_2C]^2 - 4\alpha(\alpha - A)(M + rC)K}}{2K(rC + M)}.$$

Hence,

$$V_{cr}^2 = \frac{2K(rC + M)}{(rC + K + M)\alpha - A_2C \pm \sqrt{[(rC + K + M)\alpha - A_2C]^2 - 4\alpha K(\alpha - A)(M + rC)}}. \tag{4.23}$$

Since $\alpha < A$ in this case, the plus sign in front of the root in the expression (4.23) should be taken. For the value $M = M_3$ [curve (3) in Figure 6.38a], the suspension is unstable for velocities in the interval $V_3 < V < V_4$, where V_3 is given by equation (4.22) and V_4 by equation (4.23).

In Figure 6.38b, we show the case $\alpha > A$, when the front suspension is always unstable for velocities exceeding a certain value V_{cr}. For values $M > M_2$, where the value M_2 in Figure 6.38b corresponds to the straight line (2), the suspension is stable up to a certain velocity V given by equation (4.23) with the plus sign in front of the root and is unstable at high velocities. For values $M < M_2$, we again find a stability region as the velocity V increases, but the suspension is thereafter unstable for all $V \to \infty$. For example, for $M = M_3$ [the straight line (3) in Figure 6.38b] the stability intervals are $0 < V < V_1$ and $V_2 < V < V_3$, where the values of V and V_3 are given by equations (4.23) and the value of V_2 by equation (4.22). For $M = M_5$ [the straight line (5) in Figure 6.38b] the stability intervals are $(0, V_5)$ and (V_6, V_7). For $M = M_4$ [the straight line (4) in Figure 6.38b], the suspension is stable for all velocities in the interval $(0, V_4)$, where V_4 is given by equation (4.23) with the minus sign in front of the root. The value $M = M_4$ is the optimal value for the given relationship between the parameters.

Our equations enable one to choose the optimal value for any parameter if the values of the remaining parameters are given, and also to establish the way in which a variation of any particular parameter affects the stability of the system. Thus, the D-decomposition of the (τ, ν) plane, which we have used as the basis for the theoretical derivation of the formulas for calculating the critical velocities, shows how shimmy affects the front suspension of the automobile in the case considered.

4. Shimmy of the front suspension of an automobile for large values of the kinematic parameters. Let us assume that the kinematic parameters of the pneumatic tires α, β, and γ are sufficiently large that we can use the generalized hypothesis of creep of §1.4b. In this case, the dynamical equations of the front suspension of the automobile can be written in the form (1.18), in which the generalized forces R_j are to be calculated by means of formulas (1.31). Taking into account the relation (4.2), we obtain

$$R_\psi = -(ar + \sigma N)(\xi_1 + \xi_2) - 2\rho N\psi,$$
$$R_\theta = b\varkappa_1(\xi_1 + \xi_2) - 2b\varkappa_2\psi.$$

Suppose, as before, that the generalized forces Q_φ and Q_θ are given by the expressions (4.3). The dynamical equations (1.18) then have the form

$$\begin{aligned} J_2\ddot{\theta} + h_1\dot{\theta} + k_1\theta + \frac{JV}{r}\dot{\psi} + 2b\varkappa_2\psi - b\varkappa_1(\xi_1 + \xi_2) &= 0, \\ J_1\ddot{\psi} + h_2\dot{\psi} + (k_2 + 2\rho N)\psi - \frac{JV}{r}\dot{\theta} + (ar + \sigma N)(\xi_1 + \xi_2) &= 0. \end{aligned} \tag{4.24}$$

These equations must be augmented by the equations of the kinematic constraints (1.31), which, in our case, have the form

$$\begin{aligned} \dot{\xi}_1 - r\psi + V\theta + \varkappa_1 V\xi_1 - \varkappa_2 V\psi &= 0, \\ \dot{\xi}_2 - r\psi + V\theta + \varkappa_1 V\xi_2 - \varkappa_2 V\psi &= 0. \end{aligned} \tag{4.25}$$

The system of equations (4.24) and (4.25) gives us the equations of motion of the front suspension of the automobile in the case of fairly large values of the kinematic parameters and describes the motion of the representative point in the six-dimensional phase space $(\theta, \dot{\theta}, \psi, \dot{\psi}, \xi_1, \xi_2)$. Steady motion of the suspension is represented in this space by an equilibrium state at the origin. The stability of this state is governed by the roots of the characteristic equation

$$(p + \varkappa_1 V)P_5(p) = 0, \tag{4.26}$$

where $P_5(p)$ is a polynomial of fifth degree in p. Since one root of equation (4.26) is certainly negative, we consider the equation

$$P_5(p) = p^5 + c_1p^4 + c_2p^3 + c_3p^2 + c_4p + c_5. \tag{4.27}$$

Here,

$$\begin{aligned}
c_1 &= \varkappa_1 + \delta_1 + \delta_2,\\
c_2 &= A + (K + rC)\,\tau + \nu + \varkappa_1\delta_1 + \varkappa_1\delta_2 + \delta_1\delta_2,\\
c_3 &= \varkappa_1 A + \varkappa_1\nu + [\varkappa_1(B + K) - A_1B(r\varkappa_1 - \varkappa_2) + C\varkappa_2]\,\tau\\
&\quad + (\nu + rC\tau)\,\delta_1 + K\tau\delta_2 + \varkappa_1\delta_1\delta_2,\\
c_4 &= K\nu\tau + A_2C\tau + rCK\tau^2 + (\varkappa_1\nu + \varkappa_2C\tau)\,\delta_1 + \varkappa_1\tau(B + K)\,\delta_2,\\
c_5 &= (B + K)(\varkappa_1\tau\nu + \varkappa_2C\tau^2)
\end{aligned}$$

and we have introduced the notation

$$\frac{J}{rJ_1} = A_1,\quad \frac{J}{rJ_2} = A_2,\quad A = A_1A_2,\quad \frac{2b}{J_2} = B,\quad \frac{k_1}{J_2} = K,$$
$$\frac{k_2 + 2\rho N}{J_1} = M,\quad \frac{2(ar + \sigma N)}{J_1} = C,\quad \frac{h_1}{J_2V} = \delta_1,\quad \frac{h_2}{J_1V} = \delta_2.$$

In order to be able to compare the results of the present subsection with the results of subsection 3, let us consider the case when the relations $\delta_1 = \delta_2 = 0$ and $r\varkappa_1 = \varkappa_2$ hold. In this case, the characteristic equation (4.27) can be written in the form

$$\begin{aligned}
&p^5 + \varkappa_1p^4 + [A + (K + rC)\,\tau + \nu]\,p^3\\
&+ \varkappa_1[A + \nu + (B + K + rC)\,\tau]\,p^2 + (K\nu\tau + A_2C\tau + rCK\tau^2)\,p\\
&+ \varkappa_1(B + K)(\tau\nu + rC\tau^2) = 0.
\end{aligned} \tag{4.28}$$

In order to make a D-decomposition of the (τ, ν) plane, we substitute $p = j\omega$ into equation (4.28) and equate the real and imaginary parts of the resulting expressions to zero:

$$\begin{aligned}
&\Omega^2 - [A + \nu + (B + K + rC)\,\tau]\,\Omega + (B + K)(\tau\nu + rC\tau^2) = 0,\\
&\omega\{\Omega^2 - [A + (K + rC)\,\tau + \nu]\,\Omega + K\tau\nu + A_2C\tau + rCK\tau^2\} = 0,
\end{aligned} \tag{4.29}$$

where $\Omega = \omega^2$. We use these relations to express τ and ν in terms of the parameter Ω. From the second equation, we find

$$\nu = \frac{\Omega^2 - [A + (K + rC)\,\tau]\,\Omega - A_2C\tau + rCK\tau^2}{\Omega - K\tau}. \tag{4.30}$$

Substituting this expression into the first equation of (4.29), we obtain the equation

$$\tau[\Omega(A_2C + AB) - A_2C(B + K)\,\tau] = 0. \tag{4.31}$$

From equations (4.30) and (4.31), we obtain two equations of the unicursal curve: the first,

$$\tau = 0,\quad \nu = \Omega - A, \tag{4.32}$$

is identical with equation (4.15), and the second,

$$\tau = \lambda\Omega, \quad \nu = (1 - rC\lambda)\,\Omega + \lambda_1, \tag{4.33}$$

where

$$\lambda = \frac{A_2C + AB}{A_2C\,(K + B)}, \quad \lambda_1 = \frac{A_2C}{B},$$

is identical with equation (4.16). We now write down the expression for the functional determinant necessary for separating the singular straight lines and shading the boundaries of the D-decomposition:

$$\Delta = \omega\,[B\Omega^2 - (rCB\tau + B\nu - A_2C)\,\Omega - \tau\,A_2C\,(B + K)]. \tag{4.34}$$

It follows from the expression (4.34) that $\Delta = 0$ for $\omega = 0$. Substituting $\omega = 0$ into equation (4.31), we obtain two singular straight lines: 1) $\tau = 0$; 2) $\nu = -rC\tau$. On the unicursal curve (4.32), the determinant (4.34) is equal to

$$\Delta = \omega^3(A_2C + AB),$$

and on the curve (4.33) it is given by the expression

$$\Delta = -\omega^3(A_2C + AB).$$

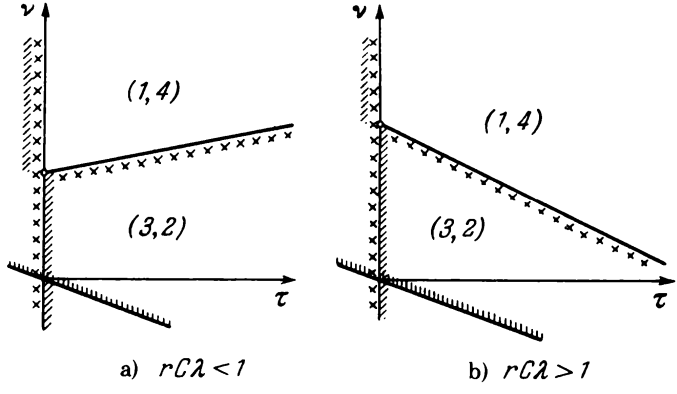


FIGURE 6.39

These results enable us to construct the boundaries of the D-decomposition on the (τ, ν) plane (Figure 6.39). In order to determine the number of roots with positive and negative real parts in the different regions of the D-decomposition on the (τ, ν) plane, we first find the form of the polynomial (4.26) when $\tau = \nu = 0$:

$$p^2(p + \kappa_1)(p^2 + A) = 0.$$

This polynomial has one negative root, a twofold vanishing root, and two complex imaginary roots. Moving from the origin to the right ($\nu = 0$, $\tau > 0$) we have, for small values of τ,

$$p^5 + \varkappa_1 p^4 + Ap^3 + \varkappa_1 Ap^2 + A_2 C\tau p + \varkappa_1 rC\,(K + B)\,\tau^2 = 0.$$

Setting $p = q\tau$, we obtain an equation for q as $\tau \to 0$:

$$\varkappa_1 Aq^2 + A_2 Cq + \varkappa_1 r\,C\,(B + K) = 0,$$

from which it follows that for $\tau > 0$ ($\tau \ll 1$) the twofold vanishing root is transformed into two complex conjugate roots with negative real part and, for $\tau < 0$, into two complex conjugate roots with positive real part. We now write down equation (4.28) for $\nu = 0$ in the form

$$p^2\,(p + \varkappa_1)\,(p^2 + A) + \tau\{(K + rC)\,p^3 + \varkappa_1\,(B + K + rC)\,p^2 + A_2 Cp\} + O\,(\tau^2) = 0,$$

where $O(\tau^2)$ stands for small terms of second order in τ. Substituting $p = \delta\tau + j(\sqrt{A} + \mu\tau)$ into this expression, we obtain equations for δ and μ as $\tau \to 0$:

$$2\mu\varkappa_1\sqrt{A} + 2\delta A = \varkappa_1\,(B + K + rC),$$

$$2\mu A\,\sqrt{A} - 2\delta\varkappa_1 A = A\,(K + rC) - A_2 C.$$

Hence,

$$\delta = \frac{\varkappa_1\,(A_2 C + AB)}{2A\,(\varkappa_1^2 + A)}$$

and, consequently, the imaginary roots are transformed for $\tau > 0$ ($\tau \ll 1$) into complex conjugate roots with positive real parts. For $\tau < 0$, we obtain the opposite result.

Thus, to the right of the origin of the (τ, ν) plane, as also to the left, we have a $D(3, 2)$ region. As a result, we obtain a complete picture of the D-decomposition (Figure 6.39), from which it follows that, in the case under consideration, there is no stability region on the (τ, ν) plane for any relationship between the parameters. Mathematically, this result is explained by the fact that at very large values of α, β, and γ we have a special case when the characteristic equation (4.11) is degenerate and its order is reduced by one. At the same time, the singular curve (4.19) disappears to infinity together with the stability region as $\alpha \to \infty$. From the point of view of the diagram of the D-decomposition in Figures 6.37b and 6.37d, the case under consideration is described only the neighborhood of the origin. Outside this neighborhood, the

conditions for the applicability of the generalized hypothesis of creep are not satisfied for large values of the kinematic parameters.

From all this one can conclude that a front suspension on pneumatic tires with large values of the kinematic parameters is unstable if there is no damping.

§5. Directional stability of an automobile

The problem of shimmy of the front suspension of an automobile considered in the previous section is a special case of the more general problem of the directional stability of an automobile. In solving this latter problem, one must take into account not only the motion of the suspension but also the perturbed motion of the body of the automobile. Together with the forces already considered, we must now add the forces of the road on the rear wheels of the automobile.

In analogy with the shimmy of the front suspension, the vibrations of the body of an automobile that arise during driving along a smooth road came to be called shimmy of the automobile. Different reasons for the occurrence of shimmy have been given in the literature. Becker et al. [52] associate shimmy of an automobile with the effect of the so-called transverse creep force arising because of the elasticity of the pneumatic tire. Rocard [387, 390] showed that shimmy of an automobile can also arise when there is no creep (i.e., in the case of absolute transverse hardness of the tires) because of the effect of a centrifugal force when there are small deviations of the trajectory of the automobile from rectilinear motion. According to Rocard, transverse creep forces are not necessary for the explanation of shimmy, but they do play a certain role. Aronovič [26] considered the question of the directional stability of an automobile for absolute transverse rigidity of the tires in the case when the pneumatic effect of the latter is taken into account by the hypothesis of creep.

1. Statement of the problem and the equations of motion. We shall consider a model of an automobile moving on a horizontal plane Oxy along the Oy axis. The model consists of a body rigidly fixed to the axle of the back wheels and a dependent front suspension (Figure 6.40). We shall assume that the body of the automobile can execute only plane-parallel motions, i.e., it can move translationally on the Oxy plane and rotate about the vertical axis. The axle of the front suspension of the automobile, which is elastically connected to the body, can also rotate about the longitudinal axis of the automobile. Let (x, y) be the co-

ordinates of the center of mass of the automobile; θ, the angle of rotation of the automobile about the vertical axis; ϑ, the angle through which the front wheels are turned, measured from the direction of the longitudinal axis of the automobile; ψ, the angle of rotation of the axle of the front suspension about the longitudinal axis of the automobile; and ω, the angular velocity of rotation of the wheels of the automobile in their own planes. We shall assume that a positive value of the angles ϑ and θ corresponds to a turning of the wheels to the left and that a positive value of the angle ψ corresponds to a raising of the left-hand wheel.

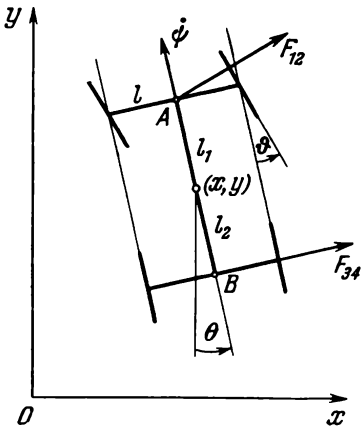


FIGURE 6.40

Let us consider the motion of the automobile if it makes small deviations from a motion along the Oy axis with a constant velocity V. In this case, $\omega = V/r$, where r is the radius of the wheels, which we shall assume identical, and the angles θ, ϑ, and ψ are small.

Let m be the mass of the automobile; l_1 and l_2, the distances from the center of mass of the automobile to its front and rear axles, respectively, so that $L = l_1 + l_2$ is the base length of the automobile; J_1, the moment of inertia of the automobile without the front wheels about the vertical axis passing through the center of mass of the automobile; J_2, the moment of inertia of the front wheels about the vertical diameter; J_3, the moment of inertia of the front suspension about the longitudinal axis of the automobile; J_4, the axial moment of inertia of the front wheels; and finally, $J = J_1 + J_2$ and $J_5 = 2J_4$.

The kinetic energy of the automobile can then be written in the form

$$T = \frac{1}{2}[m\dot{x}^2 + J_1\dot{\theta}^2 + J_2(\dot{\theta} + \dot{\vartheta})^2 + J_3\dot{\psi}^2 + 2J_4\omega\psi(\dot{\theta} + \dot{\vartheta}) - 2J_4\omega\vartheta\psi], \tag{5.1}$$

where the constant additive part has been omitted.

Let us now write down the equations of motion of the automobile. In accordance with §1, the dynamical equations of the automobile can be written in the form (1.18), in which all forces except those associated with the deformation of the tires must be taken into account in the calculation of Q_j. In our case, the generalized forces Q_j are due to viscous

friction in the shock absorbers, the elastic connection between the front suspension and the body of the automobile, and also the elasticity of the steering. Let k_1 and k_2 be the stiffness coefficients of the steering and the springs of the front suspension, respectively, and h_1 and h_2 the coefficients of viscous friction. We then have

$$Q_\vartheta = -k_1\vartheta - h_1\dot{\vartheta}, \quad Q_\psi = -k_2\psi - h_2\dot{\psi}. \tag{5.2}$$

Denoting by R_j the generalized reaction forces of the road on the automobile, we obtain, in accordance with (5.1) and (5.2), the following equation of motion

$$\begin{aligned} m\ddot{x} &= R_x, \\ J\ddot{\theta} + J_2\ddot{\vartheta} + J_4\omega\dot{\psi} &= R_\theta, \\ J_2\ddot{\vartheta} + h_1\dot{\vartheta} + k_1\vartheta + J_2\ddot{\theta} + J_5\omega\dot{\psi} &= R_\vartheta, \\ J_3\ddot{\psi} + h_2\dot{\psi} + k_2\psi - J_5\omega\left(\dot{\vartheta} + \frac{1}{2}\dot{\theta}\right) &= R_\psi. \end{aligned} \tag{5.3}$$

The right-hand sides of equations of (5.3) can only be made more precise if we make assumptions about the way in which the tires are deformed. The simplest assumption (Rocard's hypothesis of creep) is to take into account only the transverse deformation of the tire. In this case, the reaction forces of the road on the automobile reduce to transverse forces F_{12} and F_{34} perpendicular to the planes of the front and back wheels, respectively. In order to simplify the calculations, we shall assume that the force F_{12} passes through the point A (Figure 6.40) and the force F_{34} through the point B. If the tires do not slip, the transverse reaction forces F_{12} and F_{34} are proportional to the angles of creep δ_A and δ_B, which can be readily determined by the following arguments.

Consider the rolling of a fictitious wheel whose center is at the point B and whose plane is parallel to the plane of the back wheels. If there were no creep, the velocity of the transverse displacement of the point B would be $-V\theta$. In reality, this velocity is $\dot{x} + l_2\dot{\theta}$ when there is creep. Consequently, the creep velocity is $\dot{x} + l_2\dot{\theta} + V\theta$. Hence, we can find the creep angle for the back wheels:

$$\tan\delta_B \approx \delta_B \approx \frac{1}{V}\dot{x} + \frac{l_2}{V}\dot{\theta} + \theta. \tag{5.4}$$

A similar argument for the front wheels leads to the relation

$$\delta_A \approx \frac{1}{V}\dot{x} - \frac{l_1}{V}\dot{\theta} + \theta + \vartheta. \tag{5.5}$$

Denoting by a_1 and a_2 the coefficients of the resistance to creep, we obtain

$$(5.6)\quad F_{12} = -\frac{a_1}{V}\dot{x} + \frac{l_1a_1}{V}\dot{\theta} - a_1\theta - a_1\vartheta,\quad F_{34} = -\frac{a_2}{V}\dot{x} - \frac{l_2a_2}{V}\dot{\theta} - a_2\theta.$$

In order to obtain expressions for the generalized forces R_x, R_θ, R_ϑ, and R_ψ, we write down an expression for the virtual work of the forces (5.6):

$$\delta A = F_{12}(\delta x - l_1\delta\theta) + F_{34}(\delta x + l_2\delta\theta) - F_{12}h\delta\psi.$$

Hence,

$$(5.7)\quad R_x = F_{12} + F_{34},\ R_\theta = -l_1F_{12} + l_2F_{34},\ R_\vartheta = 0,\ R_\psi = -hF_{12},$$

where h is the height of the center of mass of the front axle with the wheels above the road (in the general case, $h \neq r$). Substituting (5.6) into (5.7), we find the desired expressions for the generalized forces R_j:

$$\begin{aligned} R_x &= -\frac{a_1+a_2}{V}\dot{x} + \frac{l_1a_1 - l_2a_2}{V}\dot{\theta} - (a_1+a_2)\theta - a_1\vartheta,\\ R_\theta &= \frac{l_1a_1 - l_2a_2}{V}\dot{x} - \frac{l_1^2a_1 + l_2^2a_2}{V}\dot{\theta} + (l_1a_1 - l_2a_2)\theta + l_1a_1\vartheta,\\ R_\vartheta &= 0,\ R_\psi = \frac{ha_1}{V}\dot{x} - \frac{hl_1a_1}{V}\dot{\theta} + ha_1\theta + ha_1\vartheta. \end{aligned} \qquad (5.8)$$

With allowance for the expressions (5.8), the equations of small vibrations of the automobile take the form ($u = \dot{x}$)

$$\begin{aligned} &m\dot{u} + \frac{a_1+a_2}{V}u - \frac{l_1a_1 - l_2a_2}{V}\dot{\theta} + (a_1+a_2)\theta + a_1\vartheta = 0,\\ &J\ddot{\theta} + \frac{l_1^2a_1 + l_2^2a_2}{V}\dot{\theta} - (l_1a_1 - l_2a_2)\theta + J_2\ddot{\vartheta} - l_1a_1\vartheta\\ &\qquad - \frac{l_1a_1 - l_2a_2}{V}u + J_4\frac{V}{r}\psi = 0,\\ &J_2\dddot{\vartheta} + h_1\ddot{\vartheta} + k_1\dot{\vartheta} + J_2\ddot{\theta} + J_5\frac{V}{r}\psi = 0,\\ &J_3\ddot{\psi} + h_2\dot{\psi} + k_2\psi - \left(\frac{J_5V}{2r} - \frac{hl_1a_1}{V}\right)\dot{\theta} - ha_1\theta\\ &\qquad - \frac{J_5V}{r}\dot{\vartheta} - ha_1\vartheta - \frac{ha_1}{V}u = 0. \end{aligned} \qquad (5.9)$$

Equations (5.9) describe the motion of the representative points in the seven-dimensional phase space $\Phi(u, \theta, \dot{\theta}, \vartheta, \dot{\vartheta}, \psi, \dot{\psi})$. An equilibrium state in the space Φ corresponds to a steady motion of the automobile. Assuming that all the derivatives with respect to the time in (5.9) vanish, we obtain the system of equilibrium equations

$$u/V + \theta = 0,\quad \vartheta = 0,\quad \psi = 0,$$

from which it follows that the automobile possesses a two-dimensional manifold of steady motions.[11] Physically, this manifold corresponds to the freedom in the choice of the straight line along which steady motion of the automobile is possible.

2. Analysis of the directional stability. Let us consider first the directional stability of rectilinear motion of an automobile in the case of appreciable rigidity corresponding to the angular deflections ψ and ϑ. In this case, $\psi = \vartheta = 0$, and, in accordance with (5.9), the equations of motion can be written in the form

$$\begin{gathered} m\dot{u} + \frac{a_1 + a_2}{V} u - \frac{l_1 a_1 - l_2 a_2}{V} \dot{\theta} + (a_1 + a_2)\theta = 0, \\ J\ddot{\theta} + \frac{l_1^2 a_1 + l_2^2 a_2}{V} \dot{\theta} - (l_1 a_1 - l_2 a_2)\theta - \frac{l_1 a_1 - l_2 a_2}{V} u = 0. \end{gathered} \tag{5.10}$$

In accordance with what we have said earlier, the characteristic equation of this system of equations has one vanishing root. The remaining roots, which determine the stability of the manifold of steady motions, satisfy a second-order equation

$$\begin{gathered} p^2 + \frac{1}{mV}\left[\left(1 + \frac{l_1^2}{\rho^2}\right) a_1 + \left(1 + \frac{l_2^2}{\rho^2}\right) a_2\right] p \\ + \frac{1}{m^2\rho^2}\left[\frac{(l_1 + l_2)\, a_1 a_2}{V^2} - m(l_1 a_1 - l_2 a_2)\right] = 0, \end{gathered} \tag{5.11}$$

where ρ is the radius of gyration ($J = m\rho^2$). It is immediately evident from the form of the coefficients of the characteristic equation (5.11) that if

$$l_1 a_1 < l_2 a_2 \tag{5.12}$$

we have stability for all velocities V; if the opposite inequality holds, a skidding-type instability can arise at high velocities V, since a positive root is then possible.

Let us now take into account the angular deflection of the wheels ϑ. In this case, the equations of motion (5.9) without allowance for the ϑ deformation ($h_1 = 0$) take the form

$$\begin{gathered} m\dot{u} + \frac{a}{V} u - \frac{b}{V} \dot{\theta} + a\theta + a_1\vartheta = 0, \\ J\ddot{\theta} + \frac{c}{V} \dot{\theta} - b\theta + J_2\ddot{\vartheta} - l_1 a_1 \vartheta - \frac{b}{V} u = 0, \\ J_2\ddot{\vartheta} + k_1\vartheta + J_2\ddot{\theta} = 0, \end{gathered} \tag{5.13}$$

[11] The presence of the cyclic coordinate x indicates that the abscissa x can be arbitrary in the case of steady motion of the automobile.

where $a = a_1 + a_2$, $b = l_1a_1 - l_2a_2$, $c = l_1^2a_1 - l_2^2a_2$. The characteristic equation of the system (5.13) can be reduced to the form

$$pP_4(p) = 0, \tag{5.14}$$

where $P_4(p)$ is a polynomial of fourth degree in p. The stability of the manifold is governed by the roots of the equation $P_4(p) = 0$, which can be written in the form

$$\tau\,(p^4 + Ap^3 + Bp^2) + \nu k_1 + Cp^2 + Dp + k_1B = 0. \tag{5.15}$$

Here,

$$A = \frac{aJ_1 + mc}{mJ_1},\quad B = \frac{a_1a_2L}{mJ_1},\quad C = \frac{k_1J + a_2l_2J_2}{J_1},\quad D = \frac{k_1(aJ + ml) + a_1a_2LJ_2}{mJ_1}$$

and we have distinguished the parameters

$$\tau = J_2/V^2,\quad \nu = bV^2/J_1. \tag{5.16}$$

Apart from the notation, equation (5.15) is identical with the characteristic equation obtained by Aronovič [**26**], when allowance is made for Dolgolenko's correction [**145**] to the former paper.

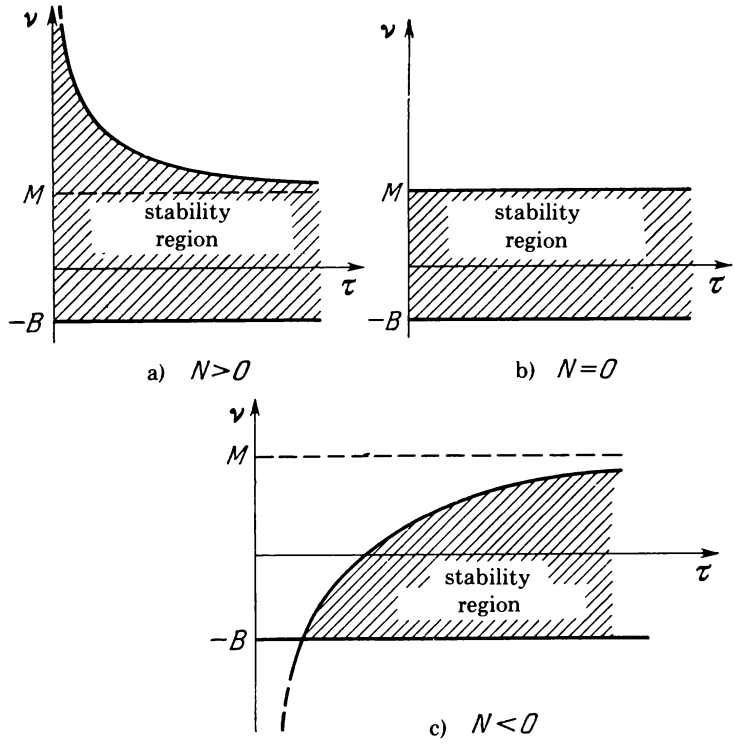


FIGURE 6.41

In accordance with the physical meaning of the quantities that occur in the coefficients k_1, A, B, C, and D and the parameters τ and ν, the quantities k_1, A, B, C, and D and the parameter τ can take only positive values, whereas ν may be positive or negative. Let us investigate equation (5.15) for different values of the parameters τ and ν. Using the Routh-Hurwitz criterion, one can easily show that the stability condition reduces to the inequalities

$$B + \nu > 0, \quad \nu < M + N/\tau, \tag{5.17}$$

where

$$N = \frac{DJ_2}{k_1 A^2}\left[mc\,(k_1 + a_2 l_2) - J_1 a_2 b\right]\frac{1}{mJ_1^2},$$
$$M = \frac{BJ_2}{mk_1 A J_1}\,(ak_1 + a_1 a_2 L) > 0. \tag{5.18}$$

It follows that, depending on the sign of N, the stability region on the half-plane $\tau > 0$ can take one of the forms shown in Figure 6.41. In order to investigate the stability of the automobile as a function of its velocity V by means of the diagrams of Figure 6.41, it is sufficient to plot on the latter the branch of the hyperbola

$$\tau\nu = J_2 b/J_1, \tag{5.19}$$

which is obtained from (5.16) by eliminating V^2. It follows from (5.19) that the hyperbola is situated in the first quadrant of the (τ, ν) plane for $b > 0$ and in the fourth quadrant for $b < 0$ (we are only interested in the half-plane $\tau > 0$). Like the coefficient N, the right-hand side of (5.19) depends on $b = l_1 a_1 - l_2 a_2$, which can, in general, take any value. We therefore obtain the different possibilities for the relative positions of the curve (5.19) and the stability region shown in Figure 6.42, in which b^* can be found from the condition $N = 0$, i.e.,

$$b^* = \frac{mc\,(k_1 + a_1 a_2)}{a_2 J_2}. \tag{5.20}$$

The critical value τ_{cr} can be found from the intersection of the curve (5.19), shown as the heavy line in Figure 6.42, with the boundary of the stability region. This, in turn, enables one to determine the critical velocity V_{cr} of the automobile at which the motion of the latter becomes unstable. An increase in the velocity V of the automobile corresponds to a displacement of the representative point along the curve (5.19) from right to left. The condition for the intersection of the curve (5.19) with the boundary of the stability region leads, in the case of Figure 6.42a, for example, to the value

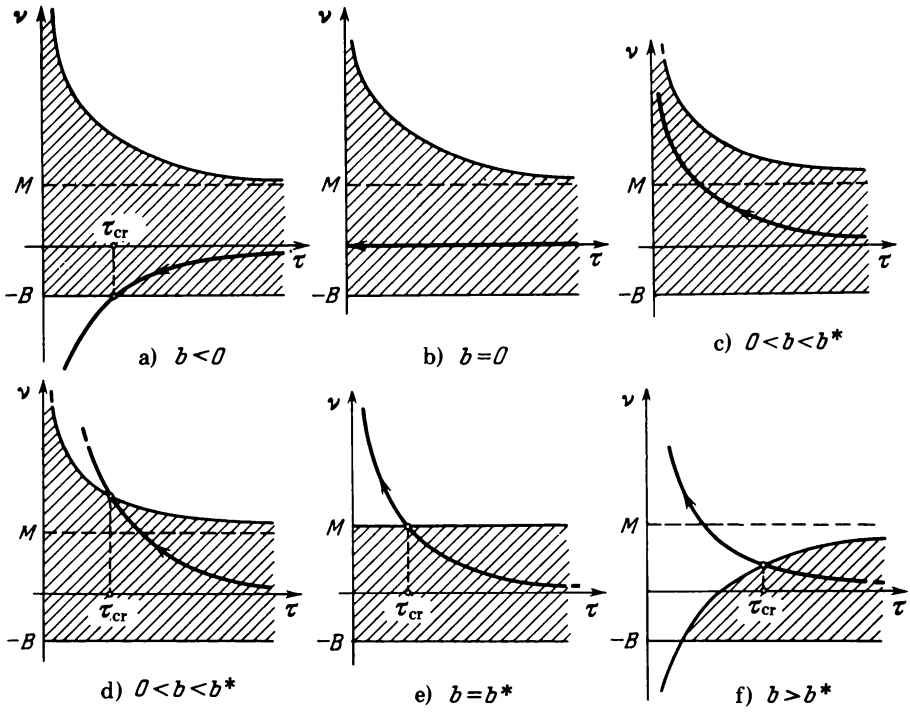


FIGURE 6.42

$$V_{cr}^2 = \frac{a_1 a_2 L}{m(l_2 a_2 - l_1 a_1)}. \tag{5.21}$$

In the case $b = 0$, the motion of the automobile is stable at all velocities, as follows from the diagram of Figure 6.42b. As the value of b increases, another value of the critical velocity V_{cr} appears from infinity. It is determined by the condition of intersection of the curve (5.19) with the curve

$$\nu = M + N/\tau.$$

This condition leads to a value of V_{cr}^2 given by

$$V_{cr}^2 = MJ_1J_2/(bJ_2 - NJ_1)$$

or, written out in full,

$$V_{cr}^2 = \frac{mk_1MJ_1^2A^2}{bJ_1(mk_1A^2 + a_2D) - mC(k_1 + a_2l_2D)}. \tag{5.22}$$

It follows immediately from formula (5.22) that, for any value of b in the interval $0 < b < b^{**}$, where b^{**} is given by

$$b^{**} = \frac{mC(k_1 + a_2l_2D)}{J_1(mk_1A^2 + a_2D)},$$

the motion of the automobile is stable at all velocities.

The phenomenon of directional stability of an automobile is also possible if classical nonholonomic constraints of rolling are satisfied. In this case, however, one must take into account the angular deflections of the suspension ψ and ϑ. If there is no transverse creep, the following nonholonomic constraints of rolling are satisfied:

$$\begin{aligned} u + l_2\dot{\theta} + V\theta &= 0, \\ u - l_1\dot{\theta} + V\theta + V\vartheta &= 0, \end{aligned} \tag{5.23}$$

which, together with the dynamical equations

$$\begin{aligned} &m\dot{u} = F_{12} + F_{34}, \\ &J\ddot{\theta} + J_2\ddot{\vartheta} + \frac{J_4V}{r}\dot{\psi} = -l_1F_{12} + l_2F_{34}, \\ &J_2\ddot{\vartheta} + h_1\dot{\vartheta} + k_1\vartheta + J_2\ddot{\theta} + \frac{J_5V}{r}\psi = 0, \\ &J_3\ddot{\psi} + h_2\psi + k_2\psi - \frac{J_4V}{r}\dot{\theta} - \frac{J_5V}{r}\dot{\vartheta} = -hF_{12} \end{aligned} \tag{5.24}$$

form a complete system of equations for the small vibrations of the automobile in the case under consideration. The characteristic equation of the system of equations (5.23) and (5.24) can be written in the form

$$\begin{aligned} &\left[J_2p^2 + \left(\frac{J_2V}{L} + h_1\right)p + k_1\right]\left[J_3p^2 + \left(h_2 - \frac{J_4Vh}{rL}\right)p + k_2\right] \\ &+ \frac{J_5V}{r}p\left\{\frac{J_2h}{L}p^2 + \left[\frac{(J + ml_2^2)Vh}{L^2} + \frac{J_5V}{R}\right]p + \frac{mV^2l_2h}{L^2} + \frac{J_4V^2}{2L}\right\} = 0. \end{aligned} \tag{5.25}$$

We distinguish the parameters

$$\mu = J_5\frac{V}{r}, \quad \nu = \left(\frac{J_5V}{r}\right)^2 \tag{5.26}$$

and investigate equation (5.25) for different values of these parameters. We represent equation (5.25) in the form

$$\mu P + \nu Q + R = 0, \tag{5.27}$$

where

$$\begin{aligned} P = p&\left[\frac{J_2h}{L}p^2 + \frac{J_0Vh}{L^2}p + \frac{mV^2l_2h}{L^2} + \frac{J_4V^2}{rL}\right] \\ &- \frac{h}{2L}p\left[J_2p^2 + \left(J_2\frac{V}{L} + h_1\right)p + k_1\right], \\ Q = p^2 \qquad &(J_0 = J + ml_2^2), \\ R = &\left[J_2p^2 + \left(\frac{J_5V}{L} + h_1\right)p + k_1\right](J_3p^2 + h_2p + k_2). \end{aligned}$$

Setting $p = j\omega$ in (5.27) and separating the real and imaginary parts, to which we add the subscripts 1 and 2, respectively, we obtain

$$\mu P_1(\omega) + \nu Q_1(\omega) + R_1(\omega) = 0, \quad \mu P_2(\omega) + R_2(\omega) = 0.$$

Hence, the parametric equation of the boundaries of the stability region has the form

$$\mu = -\frac{R_2(\omega)}{P_2(\omega)} = \frac{R_2}{\omega}\left[\frac{J_2 h}{2L}\omega^2 + \frac{k_1 h}{2L} - \frac{mV^2 l_2 h}{L^2} - \frac{J_4 V^2}{rL}\right]^{-1},$$

$$\nu = \frac{P_1(\omega) R_2(\omega)}{P_2(\omega) Q_1(\omega)} - \frac{R_1(\omega)}{Q_1(\omega)} = \frac{\left[\frac{h}{2L}\left(J_2\frac{V}{L} + h_1\right) - \frac{J_0 V h}{L^2}\right] R_2}{\omega\left[\frac{J_2 h}{2L}\omega^2 + \frac{k_1 h}{2L} - \frac{mV^2 l_2 h}{L^2} - \frac{J_4 V^2}{rL}\right]} + \frac{R_1}{\omega^2}. \tag{5.28}$$

Let us first construct the curve $R(j\omega)$. It has the form shown in Figure 6.43. Knowing the form of the curve $R(j\omega) = R_1 + jR_2$, we use formulas (5.28) to construct the curves $\mu(\omega)$ and $\nu(\omega)$. As a result, we obtain the curves shown in Figure 6.44. The value $\omega = 0$ corresponds to the straight line

$$\mu_0 = \frac{h_2 k_1 + k_2\left(\frac{J_2 V}{L} + h_1\right)}{\frac{k_1 h}{2L} - \frac{mV^2 l_2 h}{L^2} - \frac{J_4 V^2}{rL}}. \tag{5.29}$$

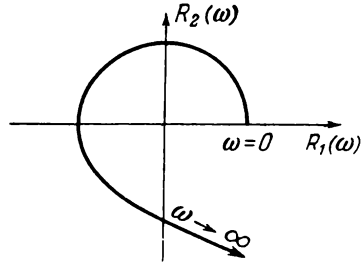


FIGURE 6.43

The value $\omega = \infty$ corresponds to the straight line

$$\mu_\infty = -\frac{2L}{J_2 h}\left[J_3\left(\frac{J_2 V}{L} + h_1\right) + J_2 h_2\right]. \tag{5.30}$$

Thus, we obtain the diagrams of the D-decomposition of the (μ, ν) plane shown in Figure 6.45. If $\omega \neq 0$ and $R_2 = 0$, we have $R_1 < 0$, and therefore some negative value of ν always corresponds to the value $\mu = 0$. For

$$d = \frac{k_1 h}{2L} - \left(\frac{m l_2 h}{L^2} + \frac{J_4}{2L}\right) V^2 > 0,$$

i.e., for a sufficiently low velocity V of the automobile, the value of the functional determinant

$$\Delta = -\omega^3\left(\frac{J_2 h}{2L}\omega^2 + d\right) \tag{5.31}$$

is always negative in the region $\omega > 0$ and, hence, the curves must be shaded on the right going along the direction of motion. However, if $d < 0$, then $\Delta > 0$ for small values of ω and the curves must be shaded

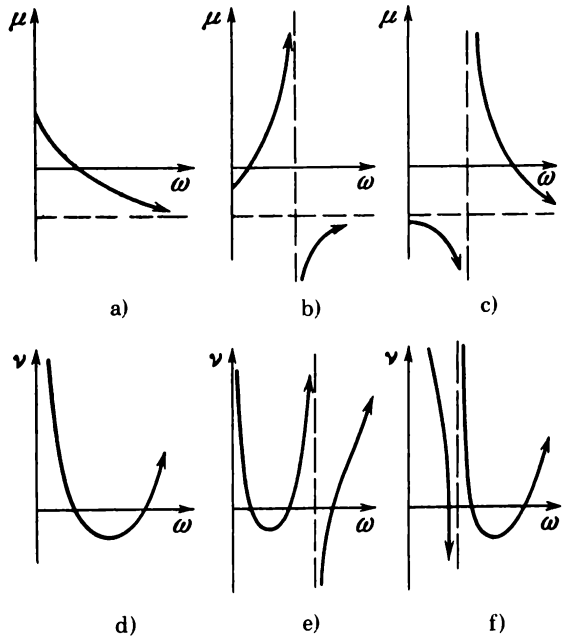


FIGURE 6.44

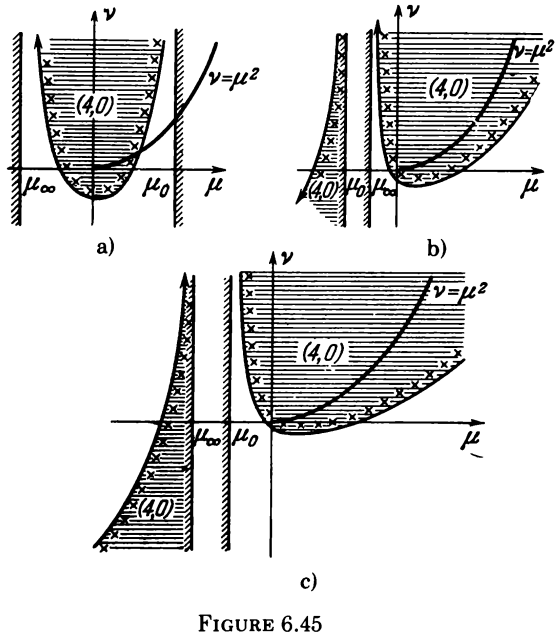


FIGURE 6.45

on the left, but for large values of ω, when $\Delta < 0$, on the right going along the direction of motion. For $\mu = \nu = 0$, we have $R = 0$, and since R is a polynomial satisfying Hurwitz's conditions, the point ($\mu = 0$, $\nu = 0$) always lies in the stability region (i.e., it corresponds to four roots with negative real parts). We can complete our qualitative investigation of the stability of the system by plotting on the diagrams of the D-decomposition the parabola $\nu = \mu^2$ obtained by eliminating V from the relations (5.26) (Figure 6.45). In accordance with the physical meaning of the parameters, we are obviously only interested in the first quadrant of the (μ, ν) plane. The intersection of the parabola $\nu = \mu^2$ with the boundary of the stability region indicates a change in the stability of the system.

§6. Traveling stability of pairs of railroad wheels and trucks

As in the case of an automobile, vibrations can also arise during the motion of railroad carriages. These vibrations may become very intense and prevent a further increase in the velocity. Many papers have been devoted to the theoretical study of the conditions of occurrence of the traveling instability of railroad carriages (steam locomotives, trucks, and electrical locomotives).

An essentially new feature in the study of the traveling stability of railroad carriages compared with the study of the directional stability of vehicles on pneumatically tired wheels is the tapered shape of the wheels and their rigid, pairwise coupling on common axles.

Whereas in the case of a bicycle or an automobile, it was possible to understand the causes of the occurrence of instability by assuming classical nonholonomic constraints of rolling without slipping, and, a fortiori, without allowance for apparent slipping, this is quite impossible in the case of a railroad carriage. The nonholonomic constraints of rolling of two tapered wheels fitted on a single axle completely determine their motion; in this case, the vibrations have a purely kinematic nature. In the case of several rigidly coupled axles, the conditions of rolling deprive the system of any dynamical freedom. The phenomenon of apparent slipping was, it seems, first observed on belt drives. It arises because elongation of part of the belt drive means that the ratio of the angular velocities of rotation of the flywheels is not equal to the ratio of their radii. This deviation increases with increasing transferred angular momentum. A similar phenomenon of apparent slipping arises during the rolling of a railroad wheel on rails.

It is due to local expansions or contractions of the rail and the rim of the wheel near their common area of contact.

Suppose a cylindrical wheel of radius r rolls on a rail with velocity v and angular velocity $\dot{\varphi}$ (Figure 6.46). Let P be the vertical load on the wheels and F the force of the rail on the wheel in the direction of the Ox axis (along the rail). Under the influence of the force F, local deformations of the rail and the rim of the wheels occur (as shown in Figure 6.46), as a result of which the path dx traversed by the wheels is not equal to $rd\varphi$. At not very large values of the force F, the relative slipping (or creepage)

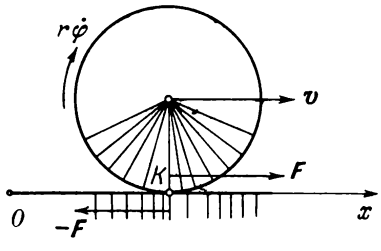


FIGURE 6.46

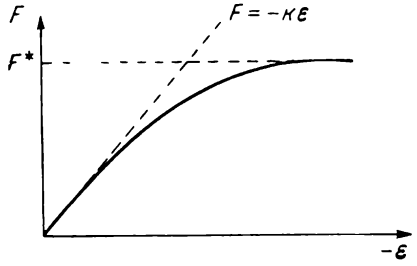


FIGURE 6.47

$$\varepsilon = \frac{dx - r\,d\varphi}{r\,d\varphi} = \frac{v - r\dot{\varphi}}{v} = 1 - \frac{r\dot{\varphi}}{v} \tag{6.1}$$

is proportional to F, so that

$$F = -k\epsilon, \tag{6.2}$$

where the coefficient k is proportional to $\sqrt{Pr}$.

The relation (6.2), which was put forward as a hypothesis by Carter [92], came to be known as the hypothesis of creep. Many experiments have shown that formula (6.2) is a good approximation only at relatively small values of the force F. If F has a large value, the dependence (6.2) breaks down and, once a certain critical value of F^* is reached, ordinary slipping occurs and the force F becomes the force of friction of slipping. The more accurate experimental dependence (shown in Figure 6.47) between the relative apparent slipping ϵ and the force F can be represented in the form

$$F = -\frac{k\varepsilon}{\sqrt{(\mu\varepsilon)^2 + 1}}, \tag{6.3}$$

where μ is a constant. It is clear that formula (6.3) goes over into (6.2) for small values of $\mu\epsilon$.

If the force F acts perpendicularly to the plane of the wheels, it is deformed and the phenomenon of creep arises, as a result of which the point of contact K is displaced through a certain angle ψ to the plane of rolling of the wheel. The angle ψ is called the angle of creep. It is assumed that at not very large forces F

$$F = -k\psi = -k\,dy/dx, \tag{6.4}$$

where dy/dx is usually called the relative transverse apparent slipping.

For a theoretically obscure reason, it has proved impossible to detect experimentally a difference between the values of the proportionality coefficients in the relations (6.2) and (6.4). The relation (6.4), which can be regarded formally as a generalization of the relation (6.2) to the case of transverse displacements, was also proposed by Carter.

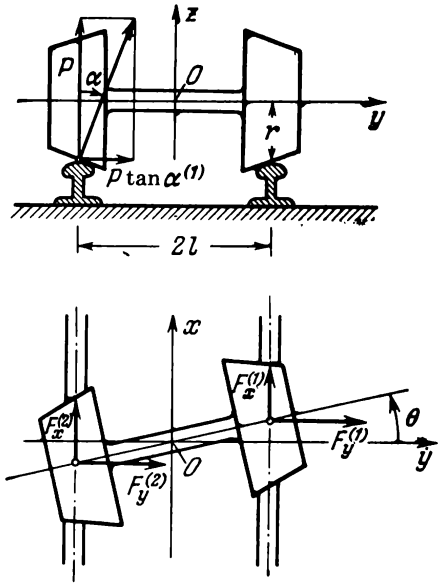


FIGURE 6.48

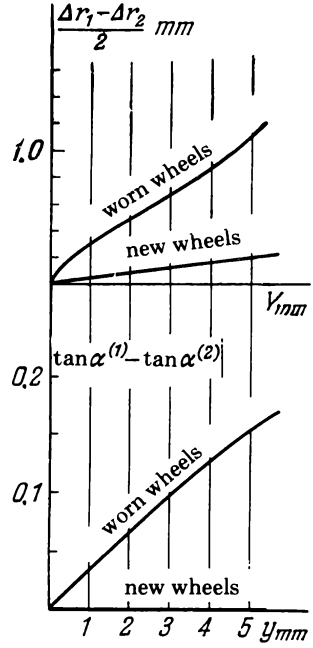


FIGURE 6.49

Let us now consider the small vibrations and stability of an individual pair of wheels (shown in Figure 6.48). Let $2l$ be the distance between the rolling circles and r the average radius of the wheels (the radius of the rolling circles in the average position of the pair). The arrangement of the coordinate system $Oxyz$ and the sense of the angle θ are shown in Figure 6.48. If P is the vertical load on the wheel, the horizontal component of the normal pressure of the rail on the wheel is $P\tan\alpha$ (Figure

6.48). When the wheel pair rolls in the direction Ox, its position is determined by the angle of rotation φ of the wheels about their axles, the angle θ of rotation about the Oz axis, and the transverse displacement y. The circles of rolling of the left- and right-hand wheels when the transverse displacement y does not vanish have different radii $r + \Delta r_1$ and $r + \Delta r_2$, respectively. Suppose that after a time dt the center of the pair of wheels has traversed a distance dx and the angles φ and θ have changed by $d\varphi$ and $d\theta$. Then the relative amounts of slipping of the right- and left-hand wheels are

$$\begin{aligned} \varepsilon_x^{(1)} &\approx \frac{dx + l\,d\theta - (r + \Delta r_1)\,d\varphi}{r\,d\varphi}, \\ \varepsilon_x^{(2)} &\approx \frac{dx - l\,d\theta - (r + \Delta r_2)\,d\varphi}{r\,d\varphi}. \end{aligned} \tag{6.5}$$

For the transverse relative apparent slipping of the right- and left-hand wheels we have similarly

$$\begin{aligned} \varepsilon_y^{(1)} &\approx \frac{dy + (r + \Delta r_1)\,\theta\,d\varphi}{r\,d\varphi}, \\ \varepsilon_y^{(2)} &\approx \frac{dy + (r + \Delta r_2)\,\theta\,d\varphi}{r\,d\varphi}. \end{aligned} \tag{6.6}$$

Assuming that the values of the relative apparent slipping do not exceed the limit of validity of formulas (6.2) and (6.4), we can find the forces that act on the pair of wheels and write down its equation of motion in the form

$$\begin{aligned} m\ddot{y} = F_y^{(1)} + F_y^{(2)} - P(\operatorname{tg}\alpha^{(1)} - \operatorname{tg}\alpha^{(2)}) &\approx -k\frac{2ly + (2r + \Delta r_1 + \Delta r_2)\,\theta\,d\varphi}{r\,d\varphi} \\ &\quad - P(\alpha^{(1)} - \alpha^{(2)}), \end{aligned} \tag{6.7}$$

$$m\rho^2\ddot{\theta} = lk(F_x^{(1)} - F_x^{(2)}) = lk\frac{2l\,d\theta + (\Delta r_1 - \Delta r_2)\,d\varphi}{r\,d\varphi},$$

where m is the mass of the pair of wheels and ρ is the radius of gyration. We assume that the wheels are only slightly tapered ($\approx 1/20$). Bearing in mind that $\Delta r_1 - \Delta r_2 \approx \gamma y$ and $\alpha^{(1)} \approx \delta y$ (graphs of the dependences of γ and δ on y for new and worn wheels are shown in Figure 6.49), we can write the equations of motion (6.7) in the form

$$\begin{aligned} m\ddot{y} + 2k\left(\frac{\dot{y}}{v} + \theta\right) + P\delta y &= 0, \\ m\rho^2\ddot{\theta} + 2k\left(l^2\frac{\dot{\theta}}{v} - \frac{l\gamma}{r}\,y\right) &= 0. \end{aligned} \tag{6.8}$$

The characteristic equation of this system of differential equations is

$$\begin{vmatrix} mp^2 + \frac{2k}{v} p + P\delta, & 2k \\ -2k \frac{l\gamma}{r}, & m\rho^2 p^2 + \frac{2kl^2}{v} p \end{vmatrix} = 0. \tag{6.9}$$

Expanding the determinant, we can write it in the form

$$m^2\rho^2 p^4 + \frac{2mk}{v}(l^2 + \rho^2) p^3 + \left[m\rho^2 P\delta + \left(\frac{2k}{v}\right)^2 l^2\right] p^2 + \frac{2k}{v} l^2 P\delta p + 4k^2 \frac{l\gamma}{r} = 0. \tag{6.9}$$

As $m \to 0$ or $v \to 0$, the equations of motion (6.8) can be simplified and take the form

$$\frac{2k}{v} \dot{y} + 2k\theta + P\delta y = 0, \qquad \frac{2kl^2}{v} \dot{\theta} - \frac{2kl\gamma}{r} y = 0. \tag{6.10}$$

Eliminating θ and replacing the derivative with respect to the time by the derivative with respect to x, we obtain the equation

$$2k \frac{d^2y}{dx^2} + P\delta \frac{dy}{dx} + \frac{\gamma}{rl} y = 0, \tag{6.11}$$

from which it follows that the center of the wheel pair describes a damped "sinusoid".

Considering the general case, we should first like to point out the interesting fact that a wheel pair with unworn tapered wheels is always unstable, since the coefficient of the first power of p in the characteristic equation (6.9) vanishes for $\delta = 0$. For $\delta > 0$, Hurwitz's inequalities for equation (6.9) reduce to the single condition

$$\frac{1}{v^2} + m\left[\frac{P\delta\rho^4}{4k^2l^2(\rho^2 + l^2)} - \frac{\gamma}{r} \frac{\rho^2 + l^2}{P\delta l^3}\right] > 0, \tag{6.12}$$

from which it follows that if the quantity in the square brackets is negative there exists a critical value of the velocity v_{cr} such that the motion is stable for $v < v_{cr}$ and unstable for $v > v_{cr}$. If the quantity in the square brackets is positive the motion of the wheel pair is stable for any velocity of the carriage ($v_{cr} = \infty$).

Let us now consider a truck with several axles. We shall assume that it is symmetric about its center of mass; that the load is distributed uniformly over all the wheels; and that the axles of the wheel pairs are rigidly fixed to the frame of the truck. We shall denote by F_x^{1i} and F_x^{2i} the components of the force of apparent slipping in the direction of

motion for the right- and left-hand wheels of the ith wheel pair, respectively, and by F_y^{1i} and F_y^{2i} the corresponding components in the transverse direction for the same wheel pair. We shall also denote by a_i the distance from the center of mass of the truck to the ith pair, taking a_i positive for axles in front of the center of mass and negative for those behind the center of mass. The equations of motion have the form

$$\begin{aligned} m\ddot{y} &= \sum_{i=1}^{n} (F_y^{1i} + F_y^{2i}) - \sum_{i=1}^{n} P(\alpha^{1i} - \alpha^{2i}), \\ m\rho^2\ddot{\theta} &= l\sum_{i=1}^{n} (F_x^{1i} - F_x^{2i}) - \sum_{i=1}^{n} a_i (F_y^{1i} + F_y^{2i}) + \sum_{i=1}^{n} a_i P(\alpha^{1i} - \alpha^{2i}). \end{aligned} \tag{6.13}$$

Expressing, as in the case of a single wheel pair, the forces F_x^{1i}, F_x^{2i}, F_y^{1i}, and F_y^{2i} in terms of the magnitudes of the relative apparent slipping and noting that

$$\Delta r_1^i - \Delta r_2^i = \gamma(y - a_i\theta), \quad \alpha^{1i} - \alpha^{2i} = \delta(y - a_i\theta),$$

we can write the equation of motion (6.13) in the following form (it should be borne in mind that $\Sigma a_i = 0$ by virtue of the assumed symmetry):

$$\begin{aligned} m\ddot{y} + 2nk\left(\frac{\dot{y}}{v} + \theta\right) + P\delta y &= 0, \\ m\rho^2\ddot{\varphi} + 2nk\left(\frac{l^2 + \frac{1}{n}\sum_{i=1}^{n} a_i^2}{v}\dot{\theta} - \frac{\gamma l}{r}y\right) &= 0, \end{aligned} \tag{6.14}$$

which, apart from the notation, is identical with the equations of a single wheel pair. Thus, the free motion of a symmetric truck with n axles and tapered wheels in the absence of elastic displacements of the axles with respect to the truck is described by exactly the same equations as the motion of a single wheel pair. In particular, the motion of such a truck is unstable for $\delta = 0$ and has a critical velocity $v_{cr} \leqq \infty$ when $\delta > 0$.

In searches for a construction that would improve the stability of a railroad truck, studies were made of models of trucks with an asymmetric arrangement of the wheel pairs with different radii of the wheels. It was found that if the radius of the front wheels is less than that of the rear wheels and the load is distributed uniformly over the wheels, such a truck may be stable. The motion is unstable if the truck runs backward. This model did not acquire any practical significance, since, given the

conditions under which railroad trucks are used, they must have the same running characteristics in both directions. For the same reason, it was not possible to exploit other forms of asymmetry that had been used to improve the stability of other wheeled vehicles (such as automobiles and aircraft) which move predominantly in one direction.

The reader can acquaint himself with the study of the traveling stability of more complicated models of railroad carriages that take into account the elasticity of the couplings between the wheel pairs, the vibrations of the framework of the carriage, etc., from the investigations cited in the bibliography.

CHAPTER VII

DYNAMICS OF NONHOLONOMIC SYSTEMS AND THE GENERAL THEORY OF ELECTRICAL MACHINES

§1. Maxwell's equations. The concept of a state in electrodynamics

The differential equations of electrodynamics, known as Maxwell's equations, consist of the following relations (in the cgse system).

1) The equations which determine the dependence of the curl of the magnetic field $\mathbf{H}$ on the density of the conduction currents $\mathbf{j}$ and the "displacement currents" $(1/4\pi)\partial\mathbf{D}/\partial t$:

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}. \tag{1.1}$$

2) An equation that expresses the law of induction of the electric field when the magnetic field varies:

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{1.2}$$

3) The equation expressing the absence of magnetic charges:

$$\operatorname{div}\mathbf{B} = 0. \tag{1.3}$$

4) The equation relating the electric displacement $\mathbf{D}$ to the charge distribution density ρ:

$$\operatorname{div}\mathbf{D} = 4\pi\rho. \tag{1.4}$$

Equations (1.1) – (1.4) play the same role in electrodynamics as Newton's laws in classical mechanics. Maxwell's equations must be complemented by relations that stem from the medium in which the electromagnetic processes take place. These additional equations, which relate the values of the fundamental vectors of the electromagnetic field in a medium at rest,[1] are the following:

$$\mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H}, \tag{1.5}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}_{\text{ext}}). \tag{1.6}$$

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[1] The results of the electrodynamics of a slowly moving medium required subsequently will be given in §5.

Here, the permittivity ϵ, the magnetic permeability μ, and the conductivity σ of the medium are assumed to be given functions of the spatial coordinates[2] that do not depend on the time. The electromotive force $\mathbf{E}_{ext}$ of nonelectromagnetic origin, which is an "external force" for the electromagnetic field, is also assumed to be a given function that depends on the coordinates and the time.

The system of equations (1.1) – (1.6) determines the variations of the electromagnetic field, the currents, and the charges in space and time.

As an additional postulate connecting the field vectors with the forces (i.e., electrodynamics and mechanics) it is usually assumed that the energy density of the electromagnetic field is

$$W = W_m + W_e = \frac{1}{8\pi}(\mathbf{BH} + \mathbf{DE}), \tag{1.7}$$

where $W_m = \mathbf{BH}/8\pi$ is the magnetic energy density and $W_e = \mathbf{DE}/8\pi$ is the electric energy density. Equation (1.7) enables one to determine the transformations of energy in which the field variations are manifested. For let us consider the variation in time of the field energy in a volume V bounded by some fixed surface S:

$$\frac{\partial W}{\partial t} = \frac{1}{8\pi}\int_V \frac{\partial}{\partial t}(\mathbf{BH} + \mathbf{DE})\,dV = \frac{1}{4\pi}\int_V \left(\mathbf{H}\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}\right)dV$$

(here we have used the fact that ϵ and μ do not depend on the time). Substituting into this equation the values of $\partial\mathbf{D}/\partial t$ and $\partial\mathbf{B}/\partial t$ from (1.1) and (1.2) and taking into account the vector identity we obtain

$$\operatorname{div}(\mathbf{a} \times \mathbf{b}) = \mathbf{b}\operatorname{curl}\mathbf{a} - \mathbf{a}\operatorname{curl}\mathbf{b}, \tag{1.8}$$

we obtain

$$\begin{aligned}\frac{\partial W}{\partial t} &= \int_V \left[-\mathbf{jE} + \frac{c}{4\pi}(\mathbf{E}\operatorname{curl}\mathbf{H} - \mathbf{H}\operatorname{curl}\mathbf{E})\right]dV \\ &= -\int_V \mathbf{jE}\,dV - \frac{c}{4\pi}\int_V \operatorname{div}(\mathbf{E} \times \mathbf{H})\,dV.\end{aligned} \tag{1.9}$$

We can transform the last integral by Gauss's theorem into an integral over the surface S:

$$\frac{c}{4\pi}\int_V \operatorname{div}(\mathbf{E} \times \mathbf{H})\,dV = \frac{c}{4\pi}\int_S (\mathbf{E} \times \mathbf{H})_n\,ds = \int_S \mathbf{U}_n\,ds,$$

[2] Such media are called linear media. More complicated media, for example, dispersing media, ferromagnetics, and ferroelectrics, are not considered here.

where the vector $\mathbf{U} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ is called the Poynting vector. Eliminating the vector $\mathbf{E}$ from the first integral of (1.9) by means of equation (1.6), we obtain finally

$$\frac{\partial W}{\partial t} = P - Q - \int_S U_n \, ds. \tag{1.10}$$

Here,

$$P = \int_V \mathbf{j}\mathbf{E}_{\text{ext}} \, dV$$

is the work done by the external electromotive forces in unit time and

$$Q = \int_V \frac{1}{\sigma} \mathbf{j}^2 \, dV$$

is the Joule heat liberated by the conduction currents in unit time in the volume V. The last term in the relation (1.10), which expresses Poynting's theorem, indicates a "loss" of electromagnetic energy of the amount $\int_S U_n ds$ ergs per second through the surface S. If the surface surrounds the entire field, (1.10) becomes the law of conservation of energy

$$\partial W/\partial t = P - Q, \tag{1.11}$$

according to which the total increase in electromagnetic energy (under the assumption of fixed material bodies) is equal to the amount by which the work of the external electromotive forces exceeds the Joule heat that is liberated.

In the case of moving bodies, additional mechanical work must be expended on moving these bodies. This will be supplied either by the external electromotive forces or at the expense of the energy of the electromagnetic field. According to (1.11), the mechanical work δA (or, the so-called work of the ponderomotive forces) of the field during the time δt is

$$\delta A = (P - Q)\delta t - \delta W. \tag{1.12}$$

We now return to the equations of electrodynamics (1.11) – (1.6), which, using (1.5), we can write in the form

$$\text{curl} \, \frac{\mathbf{B}}{\mu} = \frac{4\pi}{c} \mathbf{j} + \frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t}, \tag{1.13}$$

$$\text{curl} \, \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{1.14}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{1.15}$$

$$\operatorname{div} \epsilon \mathbf{E} = 4\pi\rho, \tag{1.16}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}_{\text{ext}}). \tag{1.17}$$

The question arises of which variables in these equations are to be regarded as the quantities that determine the state of the system. We recall that these quantities are distinguished by the fact that if their values are specified at the initial instant of time $t = 0$, one can (by means of the equations of motion) uniquely determine the values of the same variables at subsequent instants of time $t > 0$, i.e., uniquely determine the motion of the system in the future.

We shall now show that the state of the system that is formed by the electromagnetic field and the current and charges is completely determined by the vectors **E** and **B**. For, eliminating **j** from equations (1.13) and (1.17), we obtain, in conjunction with (1.14), the following system of equations:

$$\begin{aligned} \frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t} &= \operatorname{curl}\frac{\mathbf{B}}{\mu} - \frac{4\pi\sigma}{c}(\mathbf{E} + \mathbf{E}_{\text{ext}}) \\ \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= -\operatorname{curl}\mathbf{E}. \end{aligned} \tag{1.18}$$

The system of differential equations (1.18) uniquely determines the vectors **E** and **B** at subsequent times if the functions $\mathbf{E}(x, y, z)$ and $\mathbf{B}(x, y, z)$ are known at $t = 0$. At the same time, equation (1.15) is a condition that restricts (by virtue of the properties of the electromagnetic field) the class of admissible initial functions, and the remaining equations (1.16) and (1.17) serve to determine the "singularities of the electromagnetic field", which, in the general theory of the electromagnetic field, are the current density and the charge density ρ. If the condition (1.15) is satisfied at the initial instant of time, it is satisfied by virtue of equations (1.18) at all subsequent moments of time.

We shall now prove the "uniqueness theorem" stated above for a volume V on whose boundary S the vectors **B** and **E** satisfy given conditions. Let us suppose the theorem is false, i.e., suppose that there exist two different systems of solutions of equations (1.18), namely $\mathbf{E}'$, $\mathbf{B}'$, and $\mathbf{E}''$, $\mathbf{B}''$, which satisfy the same initial conditions and boundary conditions. But then the difference of these solutions $\mathbf{E}''' = \mathbf{E}' - \mathbf{E}''$, $\mathbf{B}''' = \mathbf{B}' - \mathbf{B}''$ must satisfy the system of differential equations

$$\frac{\varepsilon}{c}\frac{\partial \mathbf{E}'''}{\partial t} = \operatorname{curl}\frac{\mathbf{B}'''}{\mu} - \frac{4\pi\sigma}{c}\mathbf{E}''', \tag{1.19}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}'''}{\partial t} = -\operatorname{curl}\mathbf{E}''' \tag{1.20}$$

for null initial and boundary conditions.

We now multiply equation (1.19) by $\mathbf{E}'''$ and equation (1.20) by $\mathbf{H}'''$. Then, adding these expressions and integrating the resulting equation over the volume V, we obtain

$$\frac{1}{c}\int_V \left(\mathbf{E}''' \frac{\partial \mathbf{D}'''}{\partial t} + \mathbf{H}''' \frac{\partial \mathbf{B}'''}{\partial t}\right) dV = \int_V (\mathbf{E}''' \operatorname{curl} \mathbf{H}''' - \mathbf{H}''' \operatorname{curl} \mathbf{E}''')\, dV - \frac{4\pi}{c}\int_V \sigma (\mathbf{E}''')^2\, dV. \tag{1.21}$$

However,

$$\frac{1}{c}\int_V \left(\mathbf{E}''' \frac{\partial \mathbf{D}'''}{\partial t} + \mathbf{H}''' \frac{\partial \mathbf{B}'''}{\partial t}\right) dV = \frac{4\pi}{c}\frac{\partial}{\partial t}\int_V \frac{\mathbf{E}'''\mathbf{D}''' + \mathbf{H}'''\mathbf{B}'''}{8\pi}\, dV = \frac{4\pi}{c}\frac{\partial W'''}{\partial t},$$

where W''' is the energy of the electromagnetic field. Using (1.8) and Gauss's theorem, we obtain

$$\int_V (\mathbf{E}''' \operatorname{curl} \mathbf{H}''' - \mathbf{H}''' \operatorname{curl} \mathbf{E}''')\, dV = \int_V \operatorname{div}(\mathbf{H}''' \times \mathbf{E}''')\, dV = \int_S (\mathbf{H}''' \times \mathbf{E}''')_n\, ds = 0,$$

since the vectors $\mathbf{B}'''$ and $\mathbf{E}'''$ vanish on the boundary S. Thus, equation (1.21) implies

$$\frac{\partial W'''}{\partial t} = -\int_V \sigma (\mathbf{E}''')^2\, dV \leqslant 0,$$

i.e., the energy of the field may either decrease or (for $\mathbf{E}''' \equiv 0$) remain constant. However, since the value of the energy is initially zero and it cannot take negative values, we must have

$$W''' = \frac{1}{8\pi}\int_V \left[\varepsilon (\mathbf{E}''')^2 + \frac{1}{\mu}(\mathbf{B}''')^2\right] dV \equiv 0 \quad \text{for} \quad t > 0,$$

i.e., $\mathbf{E}''' = \mathbf{E}' - \mathbf{E}'' = 0$ and $\mathbf{B}''' = \mathbf{B}' - \mathbf{B}'' = 0$. Hence $\mathbf{E}' = \mathbf{E}''$ and $\mathbf{B}' = \mathbf{B}''$, and this proves the "uniqueness theorem".

§2. Derivation of the equations of electrodynamics from a variational principle

Instead of the vectors $\mathbf{E}$ and $\mathbf{B}$, we introduce new variables, namely, the vector potential $\mathbf{A}$ and the scalar potential φ which we associate with the original quantities by means of the equations

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} \varphi. \tag{2.1}$$

In accordance with these expressions, the state of the system is now determined by the quantities $\mathbf{A}$, $\partial\mathbf{A}/\partial t$, and φ. Equations (2.1) do not determine the vector potential $\mathbf{A}$ and the scalar potential φ uniquely. In order to eliminate this ambiguity of the variables $\mathbf{A}$ and φ, it is customary to impose on them an additional condition, for example, $\operatorname{div}\mathbf{A} = 0$ or the well-known Lorentz condition

$$\operatorname{div}\mathbf{A} + \frac{\varepsilon\mu}{c}\frac{\partial\varphi}{\partial t} = 0.$$

However, for the purposes of defining the state of the system the absence of an unambiguous correspondence in equations (2.1) is not important. Expressed in terms of the variables $\mathbf{A}$ and φ, the equations of electrodynamics (1.13) – (1.17) reduce to the following system of equations:

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A} = \frac{4\pi}{c}\mathbf{j} - \frac{\varepsilon}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right), \tag{2.2}$$

$$\operatorname{div}\varepsilon\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right) = -4\pi\rho, \tag{2.3}$$

$$\mathbf{j} = \sigma\left(-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi + \mathbf{E}_{\text{ext}}\right), \tag{2.4}$$

which contains the equation of motion (2.2) and the "coupling equations" (2.3) and (2.4), which determine the charge density ρ and the current density $\mathbf{j}$, respectively. It follows from the expression for the power of the Joule losses in a conducting medium (σ is the conductivity of the medium)

$$\int_V \frac{1}{\sigma}\mathbf{j}^2\,dV,$$

and also the power of the electromotive forces of nonelectromagnetic origin

$$\int_V \mathbf{E}_{\text{ext}}\mathbf{j}\,dV$$

that the vector $\mathbf{j}$ in electrodynamics is analogous to the velocity vector in mechanics. The charge density ρ and the current density $\mathbf{j}$ can be expressed in terms of the vector of the amount of electricity κ. The vector of the amount of electricity κ is defined at each point of space as the vector that is connected to the current density $\mathbf{j}$ and the charge density ρ by the equations

$$\mathbf{j} = \partial\kappa/\partial t, \tag{2.5}$$

$$\rho = -\operatorname{div}\kappa. \tag{2.6}$$

We note that the equation

$$\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0, \tag{2.7}$$

which is a consequence of the equations of electrodynamics (2.2) and (2.3), implies that if the relation (2.6) is satisfied at any time t and the relation (2.5) holds the relation (2.6) holds for all t. Hence, one can assume that the vector κ is determined (not uniquely) by the relation (2.5) and the initial conditions (2.6).

We shall now formulate a variational problem which leads to the equations of electrodynamics (2.2) – (2.4).

As is well known, the analogous variational problem in the dynamics of discrete systems, leading to Lagrange equations of the second kind, is an expression of the principle of stationary (or least) action. In accordance with this principle, one considers the set of trajectories of the representative point in the configuration space q_ν of the system characterized by the Lagrange function L between two positions $q_\nu(t_0)$ and $q_\nu(t_1)$. It is asserted that, as opposed to neighboring trajectories, the following extremum

$$\int_{t_0}^{t_1} (\delta L + \delta A)\, dt = 0 \tag{2.8}$$

is realized along the trajectory of the actual motion, subject to the condition that the variation in (2.8) is isochronous (i.e., the time is not varied); δA is the virtual work of the nonconservative forces; and at the ends of the interval (t_0, t_1) the variations vanish, i.e., $\delta q_\nu(t_0) = 0$ and $\delta q_\nu(t_1) = 0$. In distributed systems, the independent variables are, together with the time t, the spatial coordinates x, y, and z. The statement of the variational problem must therefore be changed somewhat. First of all, we note that the operation of variation commutes not only with the operation of differentiation with respect to the time but also with the operation of differentiation with respect to the coordinates x, y, and z. The variations of the parameters characterizing the equilibrium state of a distributed system in a certain volume V vanish not only at the ends of the time interval (t_0, t_1) but also on the boundary S of the volume V.

In the case of a system consisting of an electric field and currents and charges, the variations of its parameters are expressed in terms of $\delta\mathbf{A}$, $\delta\varphi$, and $\delta\kappa$. Since we wish to obtain not only equations of motion but also "coupling equations", the variation $\delta\kappa$ must also be regarded as independent.

After these preliminary comments, we now turn to the statement of the variational problem. Suppose that a certain volume V (Figure 7.1) contains an electromagnetic field and currents and charges characterized by $\mathbf{A}$, κ, and φ, which take definite values on the boundary of S. Together with the quantities $\mathbf{A}(x,y,z,t)$, $\varphi(x,y,z,t)$ and $\kappa(x,y,z,t)$, let us consider $\mathbf{A}'(x,y,z,t)$, and $\varphi'(x,y,z,t)$, and $\kappa'(x,y,z,t)$, which characterize different currents, charges, and fields, but such that $\mathbf{A} = \mathbf{A}'$, $\varphi = \varphi'$, $\kappa = \kappa'$ on the boundary of S. It is asserted that for the true fields, currents, and charges present in the volume V for given external electromotive forces $\mathbf{E}_{\text{ext}}(x,y,z,t)$ and for the given conditions on the boundary S of the volume

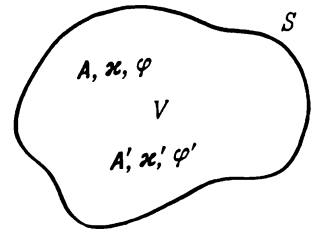

FIGURE 7.1

$$\int_{t_0}^{t_1} (\delta L_e + \delta A)\, dt = 0, \tag{2.9}$$

where the variations $\delta\mathbf{A} = \mathbf{A}' - \mathbf{A}$, $\delta\varphi = \varphi' - \varphi$, $\delta\kappa = \delta\kappa' - \delta\kappa$ are arbitrary and independent inside V and vanish at the ends of the integration interval (t_0, t_1) and also on the boundary S of the volume V. The virtual work is given by the expression

$$\delta A = \int_V \left(\mathbf{E}_{\text{ext}} - \frac{1}{\sigma}\mathbf{j}\right)\delta\varkappa\, dV, \tag{2.10}$$

and the Lagrange function L_e is

$$L_e = \int_V \left[\frac{1}{c}\mathbf{A}\frac{\partial\varkappa}{\partial t} + \varphi\nabla\varkappa + \frac{\varepsilon}{8\pi}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right)^2 - \frac{1}{8\pi\mu}(\operatorname{curl}\mathbf{A})^2\right] dV. \tag{2.11}$$

In order to prove the assertion it is sufficient to show that the equations of electrodynamics (2.2) – (2.4) are obtained when the Lagrange function (2.11) and the expression for the virtual work of the external forces and the dissipative forces (2.10) are substituted into (2.9) together with the expression for the variation of L_e in terms of the independent variations $\delta\mathbf{A}$, $\delta\varphi$, and $\delta\kappa$, the conditions at the ends of the interval (t_0, t_1) and on the boundary S of V being satisfied. To this end, we calculate the variation δL_e of the Lagrange function:

$$\delta L_e = \int_V \left[\frac{1}{c}\delta\mathbf{A}\frac{\partial\varkappa}{\partial t} + \frac{1}{c}\mathbf{A}\delta\frac{\partial\varkappa}{\partial t} + \delta\varphi\nabla\varkappa + \varphi\delta\nabla\varkappa \right.$$
$$\left. + \frac{\varepsilon}{4\pi}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right)\delta\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right) - \frac{1}{4\pi\mu}\operatorname{curl}\mathbf{A}\delta(\operatorname{curl}\mathbf{A})\right] dV.$$

Exploiting the fact that the operations of variation and differentiation with respect to the independent variables x, y, z, and t commute, we have

$$\delta L_e = \int_V \Big[\frac{1}{c}\,\delta\mathbf{A}\,\frac{\partial \varkappa}{\partial t} + \frac{\mathbf{A}}{c}\,\frac{\partial}{\partial t}\,\delta\varkappa + \delta\varphi\nabla\varkappa + \varphi\nabla(\delta\varkappa) + \frac{\varepsilon}{4\pi c}\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\frac{\partial}{\partial t}\,\delta\mathbf{A} + \frac{\varepsilon}{4\pi}\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\nabla(\delta\varphi) - \frac{1}{4\pi\mu}\,\mathrm{curl}\,\mathbf{A}\,\mathrm{curl}\,(\delta\mathbf{A})\Big]\,dV. \tag{2.12}$$

The following transformations are made with the aim of extracting the variations $\delta\mathbf{A}$, $\delta\varphi$, and $\delta\varkappa$ from under the operation of differentiation and transforming them into factors:

$$\begin{aligned}\delta L_e = \int_V \Big\{ &\frac{1}{c}\,\frac{\partial \varkappa}{\partial t}\,\delta\mathbf{A} + \frac{1}{c}\,\frac{\partial}{\partial t}(\mathbf{A}\,\delta\varkappa) - \frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t}\,\delta\varkappa + \nabla\varkappa\,\delta\varphi \\ &+ \mathrm{div}\,(\varphi\delta\varkappa) - \nabla\varphi\,\delta\varkappa + \frac{\varepsilon}{4\pi c}\,\frac{\partial}{\partial t}\Big[\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\delta\mathbf{A}\Big] \\ &- \frac{\varepsilon}{4\pi c}\,\frac{\partial}{\partial t}\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\delta\mathbf{A} + \frac{1}{4\pi}\,\mathrm{div}\Big[\varepsilon\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\delta\varphi\Big] \\ &- \frac{1}{4\pi}\,\nabla\Big[\varepsilon\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\Big]\delta\varphi - \frac{1}{4\pi}\,\mathrm{div}\Big[\delta\mathbf{A} \times \frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\Big] \\ &- \frac{1}{4\pi}\,\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\,\delta\mathbf{A}\Big\}\,dV.\end{aligned}$$

In the transformation of the last term under the integral in the expression (2.12) we have used the vector identity (1.8) taking $\mathbf{a} = \delta\mathbf{A}$ and $\mathbf{b} = \mu^{-1}\mathrm{curl}\,\mathbf{A}$.

Substituting the expressions for δL_e and δA into (2.9), one can readily see that two terms can be integrated with respect to the time. Namely,

$$\int_{t_0}^{t_1}\Big[\int_V \frac{1}{c}\,\frac{\partial}{\partial t}(\mathbf{A}\,\delta\varkappa)\,dV\Big]dt = \int_{t_0}^{t_1} d\Big[\frac{1}{c}\int_V \mathbf{A}\,\delta\varkappa\,dV\Big] = \frac{1}{c}\int_V \mathbf{A}\,\delta\varkappa\,dV\Big|_{t_0}^{t_1} = 0$$

by virtue of the fact that all the variations vanish at the ends of the interval (t_0, t_1). The same applies to the term

$$\int_{t_0}^{t_1}\Big\{\frac{\varepsilon}{4\pi c}\int_V \frac{\partial}{\partial t}\Big[\Big(\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\Big)\delta\mathbf{A}\Big]dV\Big\}\,dt.$$

In addition, one can show that the integrals with the divergence of expressions containing the variations $\delta\mathbf{A}$, $\delta\varphi$, or $\delta\varkappa$ vanish. For, applying Gauss's theorem, for example, to the integral

$$\int_V \operatorname{div}(\varphi\delta\mathbf{x})\,dV,$$

we obtain

$$\int_V \operatorname{div}(\varphi\delta\mathbf{x})\,dV = \int_S \varphi\,[\delta\mathbf{x}]_n\,dS = 0,$$

since all the variations vanish by hypothesis on the boundary of the volume of integration.

The same applies to the terms

$$\frac{1}{4\pi}\int_V \operatorname{div}\left[\varepsilon\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right)d\varphi\right]dV \quad \text{and} \quad \frac{1}{4\pi}\int_V \operatorname{div}\left[\delta\mathbf{A} \times \frac{1}{\mu}\operatorname{curl}\mathbf{A}\right]dV,$$

which vanish by virtue of the same assumption.

Grouping the remaining terms having the same factors $\delta\mathbf{A}$, $\delta\varphi$, and $\delta\kappa$, we obtain the following expression of the variational principle in the present case:

$$\int_{t_0}^{t_1} dt \int_V \left\{\left[\frac{1}{c}\frac{\partial \mathbf{x}}{\partial t} - \frac{\varepsilon}{4\pi c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right) - \frac{1}{4\pi}\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A}\right]\delta\mathbf{A}\right.$$
$$+\left[\nabla\mathbf{x} - \frac{1}{4\pi}\operatorname{div}\varepsilon\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right)\right]\delta\varphi +$$
$$\left.+\left[-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi + \mathbf{E}_{\text{ext}} - \frac{1}{\sigma}\mathbf{j}\right]\delta\mathbf{x}\right\}dV,$$

from which, taking into account the relations $\nabla\kappa = -\rho$ and $\partial\kappa/\partial t = \mathbf{j}$ and the independence of the variations $\delta\mathbf{A}$, $\delta\varphi$, and $\delta\kappa$, we obtain the system of equations of electrodynamics (2.2) – (2.4), which proves our assertion.

§3. Quasisteady approximation

Let us assume that the term $(\epsilon/c)\partial\mathbf{E}/\partial t$ in equation (1.13) can be neglected in the calculation of the magnetic field $\mathbf{B}$. In other words, suppose that the quantity $\mathbf{B}$ determined from the system of equations

$$\operatorname{curl}\frac{\mathbf{B}}{\mu} = \frac{4\pi}{c}\mathbf{j}, \tag{3.1}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{3.2}$$

is very close to $\mathbf{B}$ when it is calculated with allowance for the term $(\epsilon/c)\partial\mathbf{E}/\partial t$. In what follows, we shall call this case, which comes within

the framework of quasisteady electrodynamics, the case of the quasisteady approximation.[3] It follows from (3.1) and (3.2) that the magnetic field is now determined by the current density **j**. The electric field can be found from the equations

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3.3}$$

$$\operatorname{div} \varepsilon \mathbf{E} = 4\pi\rho. \tag{3.4}$$

As before, we go over to the quantities **A**, φ, and κ. We then obtain the following system of equations of electrodynamics in the quasisteady approximation:

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \frac{4\pi}{c}\mathbf{j}, \tag{3.5}$$

$$\operatorname{div} \varepsilon \left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right) = 4\pi \operatorname{div} \varkappa, \tag{3.6}$$

$$\frac{\partial \varkappa}{\partial t} = \sigma\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi + \mathbf{E}_{\text{ext}}\right). \tag{3.7}$$

Eliminating now $\partial\kappa/\partial t$ from (3.5) and (3.7), we obtain the equation

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \frac{4\pi\sigma}{c}\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi + \mathbf{E}_{\text{ext}}\right), \tag{3.8}$$

which, together with an additional condition (required to specify the quantities **A** and φ), for example, $\operatorname{div}\mathbf{A} = 0$, the Lorentz condition, or $\operatorname{div}\epsilon\mathbf{A} = 0$, is an equation of motion in the quasisteady approximation. It follows from (3.8) that the state of a system containing a field, currents, and charges is now determined by the quantities **A** and φ, and not by **A**, $\partial\mathbf{A}/\partial t$, and φ, as in the general case.

For what follows, it will be convenient to assume that equations (3.5) and (3.6) determine **A** and φ in terms of κ and $\partial\kappa/\partial t$ and, by the same token, regard $\mathbf{A} = \mathbf{A}(\kappa, \partial\kappa/\partial t)$ and $\varphi = \varphi(\kappa, \partial\kappa/\partial t)$ as known functions. Equation (3.7), which determines $\partial^2\kappa/\partial t^2$ in terms of κ and $\partial\kappa/\partial t$, is then the equation of motion of our system.

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[3] It should not be assumed that, since equation (3.1) holds, only processes for which $\operatorname{div}\mathbf{j} = 0$ are considered in the case of the quasisteady approximation. The point is that, although the term $(\epsilon/c)\partial\mathbf{E}/\partial t$ in equation (1.13) really can be neglected in the calculation of the magnetic field, the expression

$$\operatorname{div}\frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\varepsilon\mathbf{E} = \frac{4\pi}{c}\frac{\partial \rho}{\partial t}$$

need not be a small quantity compared with $(4\pi/c)\operatorname{div}\mathbf{j}$ (the derivative of a small quantity may be very large!).

The equations of the quasisteady approximation (3.5) – (3.7) can easily be obtained from a variational principle of the form

$$(3.9)\quad \int_{t_0}^{t_1} \left\{ \int_V \left[\delta L_e + \frac{\varepsilon}{4\pi c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi \right) \delta\mathbf{A} + \left(\mathbf{E}_{ext} - \frac{1}{\sigma} \mathbf{j} \right) \delta\boldsymbol{\varkappa} \right] dV \right\} dt = 0,$$

where L_e is given, as before, by the expression (2.11). For, calculating δL_e as was done in §2, we obtain a factor containing the term

$$-\frac{\varepsilon}{4\pi c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi \right),$$

as a result of the variation $\delta\mathbf{A}$, and this term now cancels with the analogous term in the expression (3.9). As a result, we obtain the equation of the quasisteady approximation (3.5).

However, the statement of the variational problem can be simplified. Suppose we take $\operatorname{div} \epsilon\mathbf{A} = 0$ as the additional condition used to specify the quantities $\mathbf{A}$ and φ uniquely. The equations of electrodynamics in the quasisteady approximation can then be written in the form

$$(3.10)\quad \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \frac{4\pi}{c} \frac{\partial \boldsymbol{\varkappa}}{\partial t}, \quad \operatorname{div} (\varepsilon\nabla\varphi) = 4\pi \operatorname{div} \boldsymbol{\varkappa},$$

$$\operatorname{div} \varepsilon\mathbf{A} = 0,$$

$$(3.11)\quad \frac{\partial \boldsymbol{\varkappa}}{\partial t} = \sigma \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi + \mathbf{E}_{ext} \right).$$

These equations can be obtained from the variational principle (2.9), where δA is given by (2.10), if the Lagrange function L_e is given by

$$(3.12)\quad L_e = \int_V \left[\frac{1}{c} \mathbf{A} \frac{\partial \boldsymbol{\varkappa}}{\partial t} + \nabla\boldsymbol{\varkappa}\varphi + \frac{\varepsilon}{8\pi} (\nabla\varphi)^2 - \frac{1}{8\pi\mu} (\operatorname{curl} \mathbf{A})^2 \right] dV.$$

Thus, neglect of the term $(\epsilon/c)\partial\mathbf{E}/\partial t$ in equation (1.13) is equivalent to neglecting in the Lagrange function (2.11) the term $c^{-1}\partial\mathbf{A}/\partial t$, provided $\operatorname{div} \delta\mathbf{A} = 0$.

Since we are interested in electrical machines, we shall in what follows consider systems with currents that flow in closed loops. In this case,

$$(3.13)\quad \operatorname{div} \mathbf{j} = 0$$

and as follows from the continuity equation (2.7), the charge distribution in space does not vary with the time. Assuming that there are no charges at the initial instant of time, we arrive at the following conclusion: in a system with currents that flow in closed loops with $\rho = 0$, the electrostatic field $\mathbf{E}$ is absent ($\varphi = \text{const}$) and, consequently, the electromagnetic field can be characterized by a single vector potential

A by means of the equations

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{3.14}$$

The system (3.10) and (3.11) then reduces to the two equations

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \frac{4\pi}{c}\mathbf{j}, \tag{3.15}$$

$$\mathbf{j} = \sigma\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}_{\text{ext}}\right) \tag{3.16}$$

provided $\operatorname{div} \epsilon \mathbf{A} = 0$. These equations, which are similar to (3.10) and (3.11), can be obtained from the variational principle (2.9) if L_e is given by

$$L_e = \int_V \left[\frac{1}{c}\mathbf{A}\mathbf{j} - \frac{1}{8\pi\mu}(\operatorname{curl} \mathbf{A})^2\right] dV, \tag{3.17}$$

as follows from (3.12) for $\nabla \kappa = -\rho = 0$ and $\nabla \varphi = 0$. Since the state of a system with currents that flow in closed loops can be characterized by κ and $\partial\kappa/\partial t$ in the quasisteady approximation, it is natural to take the vector κ as the independent variable and regard equation (3.15) as a definition enabling one to express $\mathbf{A}$ in terms of κ (subject to the condition $\operatorname{div} \epsilon \mathbf{A} = 0$).

In such an approach, the Lagrange function (3.17) can be transformed to a simpler form. For

$$\int_V \frac{1}{\mu}(\operatorname{curl} \mathbf{A})^2\, dV = \int_V \operatorname{div}\left[\mathbf{A} \times \frac{1}{\mu}\operatorname{curl} \mathbf{A}\right] dV + \int_V \mathbf{A} \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} \mathbf{A}\right) dV$$

$$= \int_S \left[\mathbf{A} \times \frac{1}{\mu}\operatorname{curl} \mathbf{A}\right]_n ds + \frac{4\pi}{c}\int_V \mathbf{A}\mathbf{j}\, dV.$$

The first integral vanishes if the volume of integration contains the entire field, so that the field vanishes on the boundary. Thus,[4]

$$L_e = \frac{1}{2c}\int_V \mathbf{A}\mathbf{j}\, dV, \tag{3.18}$$

where $\mathbf{A}$ is a known function of the current density $\mathbf{j}$. It is readily seen that the Lagrange function (3.18) coincides with the magnetic energy of the system.

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[4] For nonlinear media, the expression (3.18) is replaced by

$$L_e = \frac{1}{2c}\int_V \left[\int_0^{\mathbf{j}} \mathbf{A}(\mathbf{j})\, d\mathbf{j}\right] dV. \tag{3.18'}$$

For, using (3.15), we obtain

$$L_e = \frac{1}{2c}\int_V \mathbf{A}\mathbf{j}\,dV = \frac{1}{8\pi}\int \mathbf{A}\,\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\right)dV.$$

However,

$$\mathbf{A}\,\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\right) = \mathrm{div}\left[\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\times\mathbf{A}\right] + \frac{1}{\mu}(\mathrm{curl}\,\mathbf{A})^2$$

and, in addition,

$$\int_V \mathrm{div}\left[\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\times\mathbf{A}\right]dV = \int_S \left[\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A}\times\mathbf{A}\right]_n ds = 0$$

by virtue of the fact that $\mathbf{B} = \mathrm{curl}\,\mathbf{A} = 0$ on the boundary of integration. Therefore,

$$L_e = \frac{1}{8\pi}\int_V \frac{(\mathrm{curl}\,\mathbf{A})^2}{\mu}\,dV = \frac{1}{8\pi}\int_V \frac{\mathbf{B}^2}{\mu}\,dV = \frac{1}{8\pi}\int_V \mathbf{B}\mathbf{H}\,dV = W_m.$$

§4. Discrete description of electromagnetic processes in the quasisteady approximation

Equations (3.15) and (3.16) obtained in the preceding section describe electromagnetic processes in systems with currents that flow in closed loops. In the case of the quasisteady approximation, it is convenient to adopt a discrete description for the currents characterizing the state of the system. In a region V containing three-dimensional conductors we take a system of solenoidal vector functions $\mathbf{S}^r(x,y,z)$ that are complete with respect to the admissible functions of the current distribution. The current density $\mathbf{j}$ can then be represented as a converging series in $\mathbf{S}_r$:

$$\mathbf{j}(x, y, z, t) = \sum_{r=1}^{\infty} \dot{\varkappa}_r(t)\,\mathbf{S}_r(x, y, z). \tag{4.1}$$

Taking the coefficients $\dot{\kappa}_r$ of this expansion to be generalized velocities, one can regard a conductor with a current as a discrete dynamical system with a countable set of coordinates κ_r and velocities $\dot{\kappa}_r$ characterizing the system. In this case, the vector potential $\mathbf{A}$ is, in accordance with the coupling equation (3.15), a function of the generalized velocities $\dot{\kappa}_r$ and the geometrical coordinates x, y, z: $\mathbf{A} = \mathbf{A}(x, y, z, \dot{\kappa}_1, \dot{\kappa}_2, \cdots)$. At the same time, by virtue of the linearity of equation (3.15)

$$\mathbf{A}(x, y, z, \dot{\varkappa}_1, \dot{\varkappa}_2, \ldots) = \frac{1}{c}\sum_{s=1}^{\infty} \mathbf{A}_s(x, y, z)\,\dot{\varkappa}_s. \tag{4.2}$$

We introduce generalized fluxes Ψ_r by means of the relations

$$\Psi_r(\dot{\varkappa}_1, \dot{\varkappa}_2, \ldots) = \int_V \mathbf{A}\mathbf{S}_r dV = \frac{1}{c}\sum_{s=1}^{\infty} \dot{\varkappa}_s \int_V \mathbf{A}_s\mathbf{S}_s dV = \frac{1}{c}\sum_{s=1}^{\infty} L_{rs}\dot{\varkappa}_s, \tag{4.3}$$

where the coefficients

$$L_{rs} = \int_V \mathbf{A}_s(x, y, z)\,\mathbf{S}_r(x, y, z)\,dV \tag{4.4}$$

may, in analogy with linear conductors,[5] be called the generalized coefficients of selfinductance and mutual inductance of the corresponding "branches" of the current density $\mathbf{j}$ in the expansion (4.1).

Substituting (4.1) and (4.3) into equation (3.16), we find

$$\sum_{s=1}^{\infty} \dot{\varkappa}_s(t)\,\mathbf{S}_s(x, y, z) = \sigma\Big[-\frac{1}{c^2}\sum_{s=1}^{\infty} \mathbf{A}_s(x, y, z)\frac{d\dot{\varkappa}_s}{dt} + \mathbf{E}_{\text{ext}}\Big]$$

or

$$\sum_{s=1}^{\infty}\Big(\frac{1}{c^2}\mathbf{A}_s\frac{d\dot{\varkappa}_s}{dt} + \frac{1}{\sigma}\mathbf{S}_s\dot{\varkappa}_s\Big) = \mathbf{E}_{\text{ext}}.$$

We multiply both sides of this equation by $\mathbf{S}_r$ and integrate over the whole volume

$$\sum_{s=1}^{\infty}\Big[\frac{1}{c^2}\frac{d\dot{\varkappa}_s}{dt}\int_V \mathbf{A}_s\mathbf{S}_r dV + \dot{\varkappa}_s\int_V \frac{1}{\sigma}\,\mathbf{S}_s\mathbf{S}_r dV\Big] = \int_V \mathbf{E}_{\text{ext}}\mathbf{S}_r dV. \tag{4.5}$$

In analogy with the case of linear conductors, the quantities

$$R_{rs} = \int_V \frac{1}{\sigma}\mathbf{S}_r\mathbf{S}_s dV, \quad E_r = \int_V \mathbf{E}_{\text{ext}}\mathbf{S}_r\,dV \tag{4.6}$$

may be called the generalized selfresistances and mutual resistances R_{rs} and the generalized electromotive forces $\mathbf{E}_r$. In the notation of (4.4) and (4.6), the expression (4.5) is a system of a countable set of equations

$$\sum_{r=1}^{\infty}\Big(\frac{1}{c^2}L_{rs}\frac{d\dot{\varkappa}_s}{dt} + R_{rs}\dot{\varkappa}_s\Big) = E_r \quad (r = 1, 2, \ldots), \tag{4.7}$$

whose form is identical with that of the ordinary Kirchhoff equations for linear electrical circuits. Equations (4.7), which can be regarded as a generalization of Kirchhoff's equations to three-dimensional conductors,

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[5] In the case of linear conductors, equation (4.3) is identical with the usual expression for the magnetic flux through a surface bounded by a closed contour.

are the equations of motion of the distributed system in the discrete approximation.

We shall now show that the same equations (4.7) can be derived from the variational principles

$$\int_{t_0}^{t_1} (\delta L_e + \delta A)\, dt = 0, \tag{4.8}$$

by using a discrete representation of the Lagrange function L_e and the virtual work δA. For, substituting (4.1) and (4.2) into the expression for the Lagrange function (3.18), we obtain, in accordance with the notation of (4.3),

$$L_e = \frac{1}{2c^2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} L_{rs} \dot{\varkappa}_r \dot{\varkappa}_s, \tag{4.9}$$

which is a quadratic form in the generalized velocities $\dot{\varkappa}_r$. The virtual work δA of the external forces and the dissipative forces can also be expressed in terms of the variables $\varkappa_r$. Substituting (4.1) into (2.10) and using the notation of (4.6), we obtain

$$\delta A = \sum_{r=1}^{\infty} \Big(E_r - \sum_{s=1}^{\infty} R_{rs} \dot{\varkappa}_s \Big) \delta \varkappa_r. \tag{4.10}$$

We now calculate the variation of the Lagrange function

$$\delta L_e = \frac{1}{2c^2} \Big[\sum_{r,\, s=1}^{\infty} L_{rs} \delta \dot{\varkappa}_r \dot{\varkappa}_s + \sum_{r,\, s=1}^{\infty} L_{rs} \dot{\varkappa}_r \delta \dot{\varkappa}_s \Big] = \frac{1}{c^2} \sum_{r,\, s=1}^{\infty} L_{rs} \dot{\varkappa}_s \delta \varkappa_r$$

and substitute this expression together with (4.10) into (4.8):

$$\int_{t_0}^{t_1} \Big[\frac{1}{c^2} \sum_{r,\, s=1}^{\infty} L_{rs} \dot{\varkappa}_s \delta \dot{\varkappa}_r + \sum_{r=1}^{\infty} \Big(E_r - \sum_{s=1}^{\infty} R_{rs} \dot{\varkappa}_s \Big) \delta \varkappa_r \Big] dt = 0. \tag{4.11}$$

Making the usual transformation

$$\int_{t_0}^{t_1} L_{rs} \dot{\varkappa}_s \delta \dot{\varkappa}_r dt = \int_{t_0}^{t_1} d\,(L_{rs} \dot{\varkappa}_s \delta \varkappa_r) - \int_{t_0}^{t_1} \frac{d}{dt} (L_{rs} \dot{\varkappa}_s)\, \delta \varkappa_r dt$$

and remembering that the variations $\delta \varkappa_r$ vanish at the ends of the integration interval, we find

$$\int_{t_0}^{t_1} dt \sum_{r=1}^{\infty} \Big[- \sum_{s=1}^{\infty} \Big(\frac{1}{c^2} L_{rs} \frac{d\dot{\varkappa}_s}{dt} + R_{rs} \dot{\varkappa}_s \Big) + E_r \Big] \delta \varkappa_r = 0,$$

and hence, by virtue of the independence of the variations $\delta\kappa_r$,

$$\sum_{s=1}^{\infty}\left(\frac{1}{c^2}L_{rs}\frac{d\dot{\varkappa}_s}{dt}+R_{rs}\dot{\varkappa}_s\right)=E_r \quad (r=1,\,2,\,\ldots), \tag{4.12}$$

which is identical with (4.7). It is readily seen that equations (4.12), which can also be written in the form

$$\sum_{s=1}^{\infty}\left(\frac{1}{c}\frac{d\Psi_s}{dt}+R_{rs}\dot{\varkappa}_s\right)=E_r, \tag{4.13}$$

are ordinary Lagrange equations of the second kind

$$\frac{d}{dt}\left(\frac{\partial L_e}{\partial\dot{\varkappa}_r}\right)=E_r-\sum_{s=1}^{\infty}R_{rs}\dot{\varkappa}_s, \tag{4.14}$$

where κ_r are cyclic coordinates.

§5. Electrodynamics of slowly moving media. The ponderomotive forces

By slow motions we shall mean motions whose velocity v is very small compared with the velocity c of propagation of electromagnetic oscillations, i.e.,

$$v/c\ll 1. \tag{5.1}$$

If the inequality (5.1) is satisfied, all effects proportional to the square and higher powers of the ratio v/c can certainly be neglected.

As is well known, the generalization of the equations of electrodynamics to moving media reduces to a change of the coupling equations

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H}\quad\text{and}\quad \mathbf{j}=\sigma(\mathbf{E}+\mathbf{E}_{\text{ext}}),$$

which, if the condition (5.1) is satisfied, are replaced by the following equations

$$\mathbf{D}=\varepsilon\mathbf{E}+\left(\varepsilon-\frac{1}{\mu}\right)\frac{1}{c}\mathbf{v}\times\mathbf{B}, \tag{5.2}$$

$$\mathbf{B}=\mu\mathbf{H}-(\mu\varepsilon-1)\frac{1}{c}\mathbf{v}\times\mathbf{E}, \tag{5.3}$$

$$\mathbf{j}=\sigma\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}+\mathbf{E}_{\text{ext}}\right). \tag{5.4}$$

For electrodynamics in the quasisteady approximation, equation (5.2) is not in fact used and the term $(\mu\epsilon-1)c^{-1}\mathbf{v}\times\mathbf{E}$ on the right-

hand side of equation (5.3) can be neglected compared with $\mu\mathbf{H}$, because the field $\mathbf{E}$ itself is of order $1/c$ compared with the field $\mathbf{H}$. Hence, introducing the vector potential $\mathbf{A}$, we can reduce the equations of the electrodynamics of slowly moving media in the quasisteady approximation for systems with currents that flow in closed loops to two equations

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \frac{4\pi}{c} \mathbf{j}, \tag{5.5}$$

$$\mathbf{j} = \sigma\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v} \times \operatorname{curl} \mathbf{A} + \mathbf{E}_e\right). \tag{5.6}$$

It can be seen that the relation (5.5), which determines $\mathbf{A}$ in terms of $\mathbf{j}$, remains the same as in the electrodynamics of media at rest. The equation of motion (5.6) is augmented by the induced electric field $\mathbf{E}_{\text{ind}} = c^{-1}\mathbf{v} \times \mathbf{B}$. Introducing the "effective" electric field

$$\mathbf{E}^* = \mathbf{E}_{\text{ext}} + c^{-1}\mathbf{v} \times \mathbf{B}, \tag{5.7}$$

we can also reduce equation (5.6) to a form which is the same as that of the corresponding equation of the electrodynamics of media at rest.

When we consider electromagnetic phenomena in a moving conductor, we shall neglect the fact that a system of coordinates rigidly connected to the conductor is not an inertial frame. Such an approximation is completely justified for slowly moving media.

We shall now show that the equations of motion (5.6) of a system of moving three-dimensional conductors which are not connected electrically have the form of the Lagrange equations (4.14). To do this, we shall employ the discrete description introduced in the preceding section. For the mth conductor, we choose a system of vector functions $\mathbf{S}_r^m(\xi^m)$ defined in a frame ξ^m rigidly connected to the given conductor. Then the distribution of the current density $\mathbf{j}^m$ in the mth conductor can be represented as a series

$$\mathbf{j}^m = \sum_{r=1}^{m} \mathbf{S}_r^m(\xi^m)\,\dot{\varkappa}_r \tag{5.8}$$

and, consequently, the distribution of the current density $\mathbf{j}$ over all the conductors can also be expressed as an expansion in a series

$$\mathbf{j} = \sum_{k=1}^{\infty} \mathbf{S}_k(\xi^1, \xi^2, \ldots, \xi^n)\,\dot{\varkappa}_k, \tag{5.9}$$

where $\mathbf{S}_k$ is made up of the set of the $\mathbf{S}_k^m$.

Suppose that the mechanical motion of all the conductors is pre-

scribed. Then the coordinates x^m characterizing the position of all the conductors are known functions of the time.

In order to write down the equations describing the currents in the mth conductor, we associate a system of coordinates x, y, and z with this conductor. Since the mth conductor is at rest in this system of coordinates and the relative motion of the remaining conductors is manifested only in the time dependence of the vector potential $\mathbf{A}$ and the fluxes Ψ_r (through the coordinates x^m)

$$\Psi_r(x^m, \dot{\varkappa}_1, \dot{\varkappa}_2, \ldots) = \int_V \mathbf{A}(x, y, z, x^m, \dot{\varkappa}_1, \dot{\varkappa}_2, \ldots)\, \mathbf{S}_r\, dV,$$

one can use the electrodynamics of media at rest, i.e., one can repeat all the arguments of §4 that lead to equations (4.13). However, in calculating $\partial\Psi/\partial t$ one must now take into account the dependence of the flux on the time not only through the quantities $\dot{\varkappa}_r$ but also through the coordinates x^m characterizing the mutual disposition of all the conductors. The latter dependence leads to the appearance in the equation for the currents of a term describing the motion of the conductor.

In order to find the equations that describe the currents in a different conductor, we proceed in the same manner, i.e., we associate with it a system of coordinates and, ultimately, we obtain equations (4.14).

Thus, in order to find the distribution of the current density in slowly moving three-dimensional conductors that are not electrically connected, one can (in the above approach to the problem) use the electrodynamics of media at rest.

In order to derive the equations of motion of an electromechanical system, we must also consider the question of the mechanical forces of electromagnetic origin.

As follows from (1.12), the work of the ponderomotive forces on an infinitesimal displacement $\delta\mathbf{r}$ of the medium (of the three-dimensional conductor) during the time δt is

$$\delta A = \int_V \left[\left(\mathbf{E}_{\text{ext}} - \frac{1}{\sigma}\,\mathbf{j}^2\right)\delta t - \delta W_m - \delta W_e\right] dV, \tag{5.10}$$

where the first term on the right-hand side corresponds to the work of the external sources and the second term to the Joule losses, and δW_m and δW_e are the variations in the density of the magnetic and electric energy during the time δt. In a system with currents that flow in closed loops, the electric energy can be neglected compared with the magnetic energy

in the quasisteady approximation.[6] Therefore, the term δW_e in (5.10) can be neglected. Substituting into (5.10) the expression

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} + \mathbf{E}_{\text{ext}}\right),$$

we find

$$\delta A = -\int_V \left[\mathbf{E}\mathbf{j}\delta t + \frac{1}{c}\mathbf{j}(\mathbf{v} \times \mathbf{B})\,\delta t + \delta W_m\right] dV. \tag{5.11}$$

We now write $\mathbf{v} = \delta\mathbf{r}/dt$ and express the vectors $\mathbf{B}$ and $\mathbf{E}$ in terms of the vector potential $\mathbf{A}$:

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad (\operatorname{div}\varepsilon\mathbf{A} = 0). \tag{5.12}$$

Then the change of the vector potential $\mathbf{A}$ during the time δt is given by $\delta\mathbf{A} = (\partial\mathbf{A}/\partial t)\delta t$. In addition, $\mathbf{v}\delta t = (\partial\mathbf{r}/\partial t)\delta t = \delta\mathbf{r}$. The expression (5.11) can be written in the form

---

[6] For let $\lambda = 2\pi c/\omega$ be the wavelength of electromagnetic oscillations with frequency ω. These oscillations are said to be quasisteady if $l/\lambda \ll 1$, where l is a characteristic length of the system. In a system with currents that flow in closed loops, an electric field $\mathbf{E}$ is induced such that

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

We integrate this equation over an arbitrary surface S bounded by a contour whose length is of order l:

$$(*) \qquad \int_S [\operatorname{curl}\mathbf{E}]_n\, ds = -\frac{1}{c}\frac{\partial}{\partial t}\int_S B_n\, ds.$$

If all quantities vary in proportion to $e^{i\omega t}$, then

$$\int_S [\operatorname{curl}\mathbf{E}]_n\, ds = \int_L B_l\, dl \approx Bl.$$

Similarly, we can estimate the order of magnitude of the right-hand side of the equation (*)

$$\frac{1}{c}\frac{\partial}{\partial t}\int_S B_n\, ds \approx \frac{\omega}{c}Bl^2 \approx \frac{Bl^2}{\lambda}.$$

Hence, $E \approx lB/\lambda$, i.e., $E \ll B$. Hence, since

$$W_m = \frac{\mathbf{B}^2}{8\pi\mu}, \quad W_e = \frac{\varepsilon\mathbf{E}^2}{8\pi},$$

we have

$$W_e \ll W_m.$$

$$\delta A = \int_V \left[\frac{1}{c}\,\mathbf{j}\delta\mathbf{A} - \frac{1}{c}\,\mathbf{j}\,(\delta\mathbf{r} \times \operatorname{curl}\mathbf{A}) - \delta W_m\right] dV. \tag{5.13}$$

We now make the simple transformations

$$\int_V \mathbf{j}\,(\delta\mathbf{r} \times \operatorname{curl}\mathbf{A})\,dV = \int_V (\mathbf{j} \times \delta\mathbf{r})\operatorname{curl}\mathbf{A}\,dV$$

$$= \int_V \{\operatorname{div}\mathbf{A}\,(\mathbf{j} \times \delta\mathbf{r}) + \mathbf{A}\operatorname{curl}(\mathbf{j} \times \delta\mathbf{r})\}\,dV.$$

However,

$$\int_V \operatorname{div}\mathbf{A}\,(\mathbf{j} \times \delta\mathbf{r})\,dV = \int_S [\mathbf{A}\,(\mathbf{j} \times \delta\mathbf{r})]_n\,ds = 0$$

by virtue of the fact that, by hypothesis, all the fields and currents vanish on the boundary of the volume of integration. Consequently,

$$\int_V \mathbf{j}\,(\delta\mathbf{r} \times \operatorname{curl}\mathbf{A})\,dV = \int_V \mathbf{A}\operatorname{curl}(\mathbf{j} \times \delta\mathbf{r})\,dV. \tag{5.14}$$

and the relation (5.14) takes the form

$$\delta A = \delta \int_V \left(\frac{1}{c}\,\mathbf{A}\mathbf{j} - W_m\right) dV - \int_V \mathbf{A}\,[\operatorname{curl}(\mathbf{j} \times \delta\mathbf{r}) + \delta\mathbf{j}]\,dV. \tag{5.15}$$

The expression

$$\int_V \left(\frac{1}{c}\,\mathbf{A}\mathbf{j} - W_m\right) dV = \int_V \left[\frac{1}{c}\,\mathbf{A}\mathbf{j} - \frac{1}{8\pi\mu}\,(\operatorname{curl}\mathbf{A})^2\right] dV$$

is identical with (3.17) and, as was shown in §3, is the Lagrange function

$$L_e = \frac{1}{2c}\int_V \mathbf{A}\mathbf{j}\,dV$$

for the system.

It follows from the expression (5.15) that if the current density $\mathbf{j}$ is varied in such a way that

$$\delta\mathbf{j} = \operatorname{curl}(\delta\mathbf{r} \times \mathbf{j}), \tag{5.16}$$

for a displacement $\delta\mathbf{r}$, the Lagrange function L_e is a generating function for the ponderomotive forces. Let us now consider the meaning of the condition (5.16).

In the moving conductor, we take an arbitrary closed contour L that moves with the conductor and find the condition that the current J flowing through a surface S bounded by L is constant. We have

$$J = \int_S \mathbf{j}\, ds. \tag{5.17}$$

For a displacement by $\delta\mathbf{r}$, the current density at the different points of the surface S changes by $\delta\mathbf{j}$ and each element $d\mathbf{l}$ of the contour L sweeps out a surface $\delta\mathbf{s} = \delta\mathbf{r} \times d\mathbf{l}$. Thus, the condition of constancy of the current is $\delta J = 0$ or, in accordance with (5.17),

$$\delta J = \int_S \delta\mathbf{j}\, ds + \int_L \mathbf{j}\,(\delta\mathbf{r} \times d\mathbf{l}) = 0. \tag{5.18}$$

Using Stokes's theorem, we transform the second integral:

$$\int_L \mathbf{j}\,(\delta\mathbf{r} \times d\mathbf{l}) = \int_L (\mathbf{j} \times \delta\mathbf{r})\, d\mathbf{l} = \int_S \operatorname{curl}\,(\mathbf{j} \times \delta\mathbf{r})\, d\mathbf{s} = -\int_S \operatorname{curl}\,(\delta\mathbf{r} \times \mathbf{j})\, d\mathbf{s}.$$

Substituting this expression into (5.19), we find

$$\delta J = \int_S [\delta\mathbf{j} - \operatorname{curl}\,(\delta\mathbf{r} \times \mathbf{j})]\, ds = 0$$

or, by virtue of the fact that the chosen surface is arbitrary,

$$\delta\mathbf{j} = \operatorname{curl}(\delta\mathbf{r} \times \mathbf{j}),$$

which is identical with (5.16).

Thus, the condition (5.16) means differentiation for constant currents. In this case, $\delta A = \delta L$, where the left-hand side is the infinitesimal work and the right-hand side is the variation of the Lagrange function L_e. However, since the function L_e in the discrete description depends only on the parameters κ_r characterizing the currents and the coordinates x^m characterizing the mutual disposition of the three-dimensional conductors, the condition (5.16) means that the variation δL_e is made subject to $\kappa_r = \text{const}$, i.e., only the coordinates x^m are varied. Consequently, the variation δL_e has the form

$$\delta L_e = \sum_m Q_m \delta x^m.$$

This shows that L_e is a generating function for the ponderomotive forces.

§6. Lagrange-Maxwell equations for electromechanical systems

It was shown in the preceding section that, in the quasisteady approximation, the "electric" equations of motion of a system (with currents flowing in closed loops) consisting of moving three-dimensional conductors with no electric connections can be written in the form of

Lagrange equations of the second kind. If we use a discrete description (see §3), the Lagrange function L_e, which is the magnetic energy of the system in this case, has the form

$$L_e = \frac{1}{2c^2} \sum_{r,\, s=1}^{\infty} L_{rs}(x^m)\, \dot{\varkappa}_r \dot{\varkappa}_s, \tag{6.1}$$

where the generalized coefficients of selfinductance and mutual inductance L_{rs} depend on the "mechanical" coordinates x^m, which determine the mutual disposition of the three-dimensional conductors and are, in general, implicit functions of the time.

In addition, we have seen that the function L_e is a generating function for the mechanical forces of electromagnetic origin. These results enable us to state the following assertion: The union of the electric and mechanical equations of motion in a general dynamical form is effected simply by adding the Lagrange functions L_e and L_m, where L_m is the ordinary Lagrange function of a discrete mechanical system:

$$L(x, \dot{x}, \dot{\varkappa}) = L_e(x, \dot{\varkappa}) + L_m(x, \dot{x}). \tag{6.2}$$

The virtual work of the external forces and the dissipative forces is also made up of the work of the electric and mechanical forces:

$$\delta A = \sum_{r=1}^{\infty} \Big(E_r - \sum_{s=1}^{\infty} R_{rs} \dot{\varkappa}_s\Big) \delta \dot{\varkappa}_r + \sum_{k=1}^{n} X_k \delta x_k. \tag{6.3}$$

Here, we have assumed that the system has a finite number n of "mechanical" degrees of freedom.

The equations of motion of an electromechanical system can be written in the form of Lagrange equations of the second kind

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k, \tag{6.4}$$

where L is given by the sum (6.2) and the generalized forces Q_k by (6.3). It should be noted that the derivation of the equations of motion for distributed electromechanical systems is based on the assumption that the currents in each of the moving conductors flow in closed loops ($\operatorname{div} \mathbf{j} = 0$), i.e., there is no contact between the conductors and the variables $\varkappa_r$ are independent parameters.

In the special case of n linear conductors consisting of closed circuits with currents I_k, the magnetic energy of the system can be written in the form

$$W_m = \frac{1}{2c^2} \sum_{r,\, s=1}^{n} L_{rs}(x)\, I_r I_s. \tag{6.5}$$

If we wish to obtain the equations of motion of an electromechanical system containing constant magnets, we must assume that each of the latter is replaced by an equivalent solenoid in whose coil a current i_0 flows. If we assume that the field of the constant magnet does not change, the current i_0 must also remain constant. Physically, such an assumption means that we neglect the influence of external (for the magnet) fields on the state of magnetization of the magnet. The field $\mathbf{H}_0$ of a constant magnitude is due to microscopic Amperian currents, whose density $\mathbf{j}_0$ is given by $\mathbf{j}_0 = \operatorname{curl} \mathbf{H}_0$ or, if we introduce the magnetization vector $\mathbf{M}_0$, by $\mathbf{j}_0 = \operatorname{curl} \mathbf{M}_0/\mu$. The introduction of Amperian currents enables one to write all the field equations in the same form as they take when constant magnets are absent. At the same time, the electric current $\mathbf{j}$ in Maxwell's equations is augmented by the given current $\mathbf{j}_0$. One can assume that the Amperian currents $\mathbf{j}_0$, compensating each other inside the constant magnet, give rise to a certain surface current $\mathbf{i}$. Therefore, it is convenient to regard a constant magnet as an equivalent solenoid with a constant current $\mathbf{i}$. At the same time, terms which take into account the energy of the interaction of the currents of the system with the current $\mathbf{i}$ must be added to the expression for the kinetic energy of a system containing a constant magnet. The selfenergy of the magnet $Li_0^2/2$ may be neglected, since, by virtue of the assumption $i_0 = \text{const}$, it is constant and does not affect the equations of motion.

There is no need to derive the Lagrange equation of the second kind for the coordinate $q_0 = \int_{t_0}^{t} i_0 dt$ corresponding to the current i_0, since, by virtue of the assumption $i_0 = \text{const}$, the equation would be merely the condition which the emf E_0 connected to the solenoid must satisfy in order to maintain the constancy of the current i_0.

To conclude this section, let us consider some examples.

EXAMPLE 1. ELECTROMAGNETIC INSTRUMENT. The mechanism which we call an electromagnetic instrument consists of a moving coil that rotates in the magnetic field generated by another, fixed coil. The moving coil is usually fitted with a returning spring (which exerts a moment) and the winding of the coil together with the winding of the fixed coil and the external emf forms an electrical series circuit (Figure 7.2). Let L_1 and L_2 be the coefficients of self-inductance of the fixed and moving coils, respectively; R, the total resistance of the

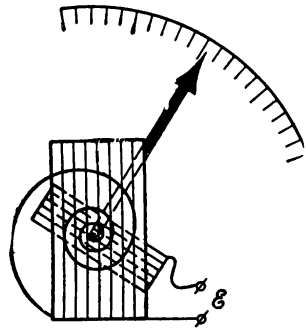

FIGURE 7.2

windings of the coils; k, the stiffness of the returning spring; h, the coefficient of viscous friction; and J, the moment of inertia of the rotating coil. The coefficient of mutual inductance $M(\varphi)$ depends on the angle φ of deflection of the moving coil. If i is the current passing through the windings of the coils, the generalized coordinates of the system are $q = \int i dt$ and φ. The Lagrange function has the form

$$L = \frac{1}{2}\{[L_1 + 2M(\varphi) + L_2]\dot{q}^2 + J\dot{\varphi}^2 - k\varphi^2\}.$$

The virtual work of the external forces and the dissipative forces is

$$\delta A = (E - R\dot{q})\delta q - h\dot{\psi}\delta\varphi.$$

Writing down the Lagrange equations of second kind, we obtain the following equations of motion of the system:

$$(L_1 + 2M(\varphi) + L_2)\ddot{q} + \left(R + 2\frac{dM}{d\varphi}\dot{\varphi}\right)\dot{q} = E,$$

$$J\ddot{\varphi} + h\dot{\varphi} + k\varphi = \frac{dM}{d\varphi}\dot{q}^2$$

or, making the substitution $\dot{q} = i$,

$$(L_1 + 2M + L_2)\frac{di}{dt} + (R + 2M_\varphi\dot{\varphi})i = E,$$

$$J\frac{d^2\varphi}{dt^2} + h\frac{d\varphi}{dt} + k\varphi = M_\varphi i^2. \tag{6.6}$$

Thus, the system has one and a half degrees of freedom. Usually, one assumes $M(\varphi) = M_0 \sin\varphi$, where φ is measured from the position in which the coils are perpendicular.

Let us consider the dynamics of the system for small angles φ. In this case, $M(\varphi) \approx M_0\varphi$ and the equations of motion (6.6) can be written in the form

$$(L_1 + L_2 + 2M_0\varphi)\frac{di}{dt} + Ri + 2M_0 i\frac{d\varphi}{dt} = E,$$

$$J\frac{d^2\varphi}{dt^2} + h\frac{d\varphi}{dt} + k\varphi = M_0 i^2. \tag{6.7}$$

In the phase space $(\varphi, d\varphi/dt, i)$, the equilibrium states (φ_0, i_0) (for $E = E_0 = \text{const}$) are situated on the plane $\dot{\varphi} = 0$ on the parabola[7]

$$k\varphi_0 = M_0 i_0^2 \qquad (Ri_0 = E_0).$$

The nature of the equilibrium state is determined by the roots of the

[7] It is for this reason that the scale of the electromagnetic instrument is quadratic.

characteristic equation

$$p^3 + Ap^2 + Bp + 1 = 0, \tag{6.8}$$

where

$$A = \frac{RJ + h(L_1 + L_2 + 2M_0\varphi_0)}{\sqrt[3]{RkJ^2(L_1 + L_2 + 2M_0\varphi_0)^2}},$$

$$B = \frac{hR + k(L_1 + L_2 + 2M_0\varphi_0) + 4M_0^2 i_0^2}{\sqrt[3]{R^2k^2J(L_1 + L_2 + 2M_0\varphi_0)}}.$$

The stability conditions of a system with the characteristic equation (6.8) reduce to the inequality

$$AB > 1,$$

which, in our case, can be expressed in the form

$$RJ(Rh + 4M_0^2 i_0^2) + h(L_1 + L_2 + 2M_0\varphi_0)[Rh + 4M_0^2 i_0^2 + k(L_1 + L_2 + 2M_0\varphi_0)] > 0.$$

Since all the quantities occurring in this inequality are positive, the equilibrium state (φ_0, i_0) is stable for all values of the physical parameters of the system.

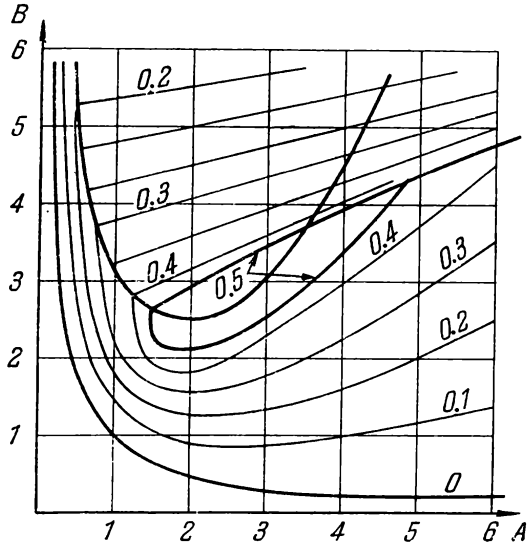

FIGURE 7.3

Using a Vyšnegradskiĭ diagram with plotted lines of equal degree of

stability (Figure 7.3), one can easily choose the optimal parameters for the operation of the electromagnetic instrument. It follows from the diagram of Figure 7.3 that the maximum possible degree of stability (equal to unity) is obtained for $A = B = 3$. However, for every actual instrument with a given class of external perturbations, it is always found that the optimal values of the parameters differ from the values for which $A = B = 3$. This is connected with the fact that, besides a high degree of stability, one also requires that the transient process in the instrument is sufficiently short lived.

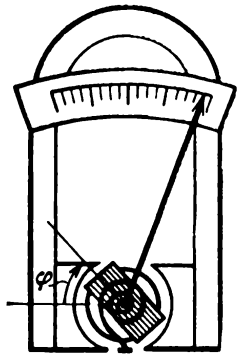

FIGURE 7.4

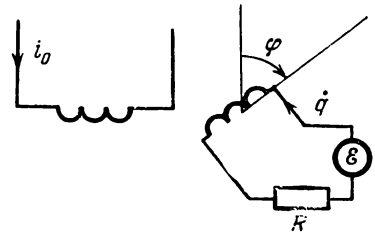

FIGURE 7.5

EXAMPLE 2. MAGNETOELECTRIC INSTRUMENT (GALVANOMETER). The instrument depicted schematically in Figure 7.4 consists of a coil which rotates in the homogeneous magnetic field of a constant magnet. The coil is fitted with a returning spring of stiffness k. If an emf E is applied to the ends of the winding of the coil, the arrow of the instrument is deflected. We shall replace the constant magnet by a solenoid with current i_0; the equivalent electric circuit of a galvanometer then takes the form shown in Figure 7.5, where φ is the deflection angle of the moving coil from the position in which its magnetic axis is perpendicular to the axis of the solenoid. Let J be the moment of inertia of the moving coil; h, the coefficient of viscous friction; L, the coefficient of self-inductance; and $M(\varphi)$, the coefficient of mutual inductance with the solenoid.

We introduce the generalized coordinates q and φ of the system and write the Lagrange function in the form

$$L = \frac{1}{2}\left[J\dot{\varphi}^2 + L\dot{q}^2 + 2M(\varphi)\, i_0\dot{q} - k\varphi^2\right].$$

The virtual work of the external forces and the dissipative forces is

$$\delta A = (E - R\dot{q})\,\delta q - h\dot{\varphi}\delta\varphi.$$

We now write down the equations of motion of the system

$$\begin{aligned} L\ddot{q} + R\dot{q} + i_0 \frac{dM}{d\varphi}\dot{\varphi} &= E, \\ J\ddot{\varphi} + h\dot{\varphi} + k\varphi - i_0 \frac{dM}{d\varphi}\dot{q} &= 0 \end{aligned} \tag{6.9}$$

or, writing $\dot{q} = i$, and assuming small angles of deflection $M(\varphi) \approx M_0\varphi$,

$$\begin{aligned} L\frac{di}{dt} + Ri + i_0 M_0 \frac{d\varphi}{dt} &= E, \\ J\frac{d^2\varphi}{dt^2} + h\frac{d\varphi}{dt} + k\varphi &= i_0 M_0 i. \end{aligned} \tag{6.10}$$

Thus, the system has one and a half degrees of freedom. If Ldi/dt in the first equation is neglected compared with the other terms, the system is described by a second-order differential equation

$$RJ\ddot{\varphi} + (Rh + i_0^2 M_0^2)\,\dot{\varphi} + Rk\varphi = i_0 M_0 E.$$

On the $(\varphi, \dot{\varphi})$ phase plane there exists a unique equilibrium state $\dot{\varphi} = 0$, $\varphi = (i_0 M_0/kR)E$ ($E = \text{const}$), which, as is easily seen, is always stable. The critical damping at which the system goes over from conditions of oscillatory motion to aperiodic motion obtains for the following relationship between the parameters:

$$R(\sqrt{kJ} - h) = i_0^2 M_0^2.$$

It follows from the second equation of (6.10) that the scale of a galvanometer is linear. Let us estimate the value of the angle $0 \leqq \varphi \leqq \varphi^*$ within which the linearity is preserved to within 10%. To this end, we use the dependence $M(\varphi) = M_0 \sin\varphi$ and, substituting it into the second equation of (6.9), we obtain

$$\varphi = i_0 M_0 \cos\varphi\, i.$$

The estimate of the limit φ^* reduces to the determination of the root of the equation $1 - \cos\varphi^* = 0.1$, the approximate value of which is

$$\varphi^* \approx 0.447\, рад \approx 25°.$$

EXAMPLE 3. ELECTRODYNAMIC LOUDSPEAKER. Figure 7.6 is a schematic representation of a dynamic loudspeaker with magnetization

effected by an iron core 1 on which the magnetization coil 2 is wound. A moving coil 3 is fitted onto the protruding end of the core (with a small gap). Its framework is connected to an elastic cone 4 whose outer edge is fixed in the holder 5. If the applied emf E varies with an acoustical frequency, the moving coil begins to vibrate under the influence of the resulting ponderomotive forces and is displaced along the core, setting the cone in motion.

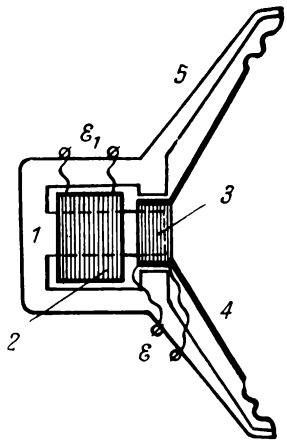

FIGURE 7.6

Let us find the equations of motion of this system. Let L, R, M, and x be, respectively, the coefficients of selfinductance, the resistance, the mass, and the coordinate measuring the displacement of the moving coil; let L_1 and R_1 be the corresponding parameters of the fixed coil; $M(x)$, the coefficient of mutual inductance of the coils; k, the stiffness of the cone; and h, the coefficient of viscous friction. Introducing the generalized coordinates x, $q = \int i dt$ and $q_1 = \int i_1 dt$, we can write the Lagrange equation in the form

$$L = \frac{1}{2}[L_1 i_1^2 + 2M(x)\, i i_1 + Li^2 + m\dot{x}^2 - k(x - x_0)^2].$$

The virtual work of the external and dissipative forces is

$$\delta A = (E - Ri)\,\delta q + (E_1 - R_1 i_1)\,\delta q_1 - h\dot{x}\,\delta x.$$

Writing down the Lagrange equations of second kind, we obtain the following equations of motion:

$$\begin{aligned} L_1 \frac{di_1}{dt} + M \frac{di}{dt} + \frac{dM}{dx}\frac{dx}{dt} i + R_1 i_1 &= E_1, \\ L \frac{di}{dt} + M \frac{di_1}{dt} + \frac{dM}{dx}\frac{dx}{dt} i_1 + Ri &= E(t) \\ m \frac{d^2x}{dt^2} + h \frac{dx}{dt} + k(x - x_0) - \frac{dM}{dx} i_1 i &= 0. \end{aligned} \tag{6.11}$$

In the first equation of (6.11), which we write in the form

$$\frac{d}{dt}(L_1 i_1 + Mi) + R_1 i_1 = E_1,$$

the second term in the brackets can be neglected compared with the first, since we usually have $Mi \ll L_1 i_1$. Assuming $E_1 = \text{const}$, we obtain $i_1 = E_1/R_1 = \text{const}$ from this equation.

Let us consider an autonomous system, setting $E(t) = 0$. For small deviations $\xi = x - x_0$, $\eta = i$ from the equilibrium position ($i = 0$, $x = x_0$), one can assume $M(x) = a(x - x_0)$ and the equations of motion (6.11) can be written in the form

$$\begin{aligned} m\ddot{\xi} + h\dot{\xi} + k\xi &= a\eta, \\ L\dot{\eta} + R\eta + a\dot{\xi} &= 0. \end{aligned} \tag{6.12}$$

It follows that the equilibrium position is always stable. If the term $L\dot{\eta}$ in the second equation can be neglected compared with the remaining terms, the system is described by second-order equations. The critical damping in the system obtains when the following relation between the parameters is satisfied:

$$R(\sqrt{km} - h) = a^2.$$

§7. Extension of the Lagrange-Maxwell equations to electromechanical systems with currents that do not flow in closed loops

In the preceding section, we have shown that the equations of motion of an electromechanical system with currents that flow in closed loops can, in the quasisteady approximation, be written in the form of Lagrange equations with a Lagrange function that is a sum of the "electric" and "mechanical" Lagrange functions. In the general case of an electromagnetic system with currents that do not flow in closed loops such a method of deriving the equations of motion, even in the quasisteady approximation, is not possible; however, one can find conditions under which the same method of deriving the equations of motion can be used in such systems.

Suppose that the region V occupied by the electromagnetic field of an electromechanical system with currents that do not flow in closed loops can be split into two nonoverlapping regions V_1 and V_2 in which the following conditions are satisfied.

a) In the region V_1, the electrical energy W_e of the field is negligibly small compared with the magnetic energy W_m of the field and, at any point of V_1,

$$\operatorname{div} \mathbf{j} = 0. \tag{7.1}$$

b) In the region V_2, the magnetic energy W_m of the field is negligibly small compared with the electric energy W_e of the field and there is no conduction current:

$$\mathbf{j} = 0. \tag{7.2}$$

We shall show that under these conditions the equations of motion

of the system in the quasisteady approximation can be written in the form of Lagrange-Maxwell equations.

In accordance with (3.12), the "electric" part L_e of the Lagrange function $L = L_e + L_m$ (L_m is the "mechanical" part of the Lagrange function expressed in the usual manner) in the quasisteady approximation has the form

$$L_e = \int_V \left[\frac{1}{c}\mathbf{A}\frac{\partial \varkappa}{\partial t} + \nabla \varkappa \varphi + \frac{\varepsilon}{8\pi}(\nabla\varphi)^2 - \frac{1}{8\pi\mu}(\operatorname{curl}\mathbf{A})^2\right] dV. \tag{7.3}$$

In accordance with the conditions formulated at the beginning of this section, we divide the region of integration V in (7.3) into the two regions V_1 and V_2. In accordance with (7.1), the continuity equation (2.8) shows that the charge distribution in the region V_1 does not change with the time. If there are no charges in the region at the initial instant of time, the relation $\rho = -\nabla\varkappa$ shows that $\nabla\varkappa = 0$. In addition, if $\rho = 0$, the electric field $\mathbf{E}$ will also be absent and, therefore, the relation $\Delta\varphi = 0$ is always satisfied in the region V_1.

In the region V_2, we have $\partial\varkappa/\partial t = \mathbf{j} = 0$. Since $\operatorname{curl}\mathbf{B} = 4\pi\mathbf{j}/c$ in the quasisteady approximation, the condition (7.2) yields $\operatorname{curl}\mathbf{B} = 0$. Taking into account the equation $\operatorname{div}\mathbf{B} = 0$, we obtain $\mathbf{B} = \operatorname{curl}\mathbf{A} = 0$ in the region V_2. Thus, the expression (7.3) can be represented in the form

$$L_e = \int_{V_1}\left[\frac{1}{c}\mathbf{A}\frac{\partial\varkappa}{\partial t} - \frac{1}{8\pi\mu}(\operatorname{curl}\mathbf{A})^2\right]dV + \int_{V_2}\left[\nabla\varkappa\varphi + \frac{\varepsilon}{8\pi}(\nabla\varphi)^2\right]dV.$$

The integral over the region V_1 is equal to the expression (3.17) and, as was shown in §3, is the magnetic energy:

$$\int_{V_1}\left[\frac{1}{c}\mathbf{A}\frac{\partial\varkappa}{\partial t} - \frac{1}{8\pi\mu}(\operatorname{curl}\mathbf{A})^2\right]dV = \frac{1}{8\pi}\int_{V_1}\mathbf{BH}\,dV = W_m. \tag{7.4}$$

In accordance with our assumptions, the expression (7.4) is the total magnetic energy of the system. The integral over the region V_2, taken with the opposite sign, gives us the electric energy of the system. For, in accordance with our earlier remark and the assumptions we have made, the field $\mathbf{E}$ in the region V_2 is a potential field, i.e.,

$$\mathbf{E} = -\nabla\varphi.$$

Using this relation, we find

$$\int_{V_2}\left[\varphi\nabla\varkappa + \frac{\varepsilon}{8\pi}(\nabla\varphi)^2\right]dV = -\int_{V_2}\varphi\rho\,dV + \frac{1}{8\pi}\int_{V_2}\mathbf{DE}\,dV.$$

We now transform the first integral on the right-hand side of the equation

$$\int_{V_2} \varphi\rho\, dV = \frac{1}{4\pi}\int_{V_2} \varphi \operatorname{div} \mathbf{D}\, dV = \frac{1}{4\pi}\int_{V_2} \operatorname{div}(\varphi\mathbf{D})\, dV - \frac{1}{4\pi}\int_{V_2} \mathbf{D}\nabla\varphi\, dV$$
$$= \frac{1}{4\pi}\int_{V_2} \mathbf{DE}\, dV + \frac{1}{4\pi}\int_{S_2} \varphi\mathbf{D}\, d\mathbf{s}.$$

However, the integral over the surface S_2 bounding the volume V_2 vanishes because, by assumption, the electric field $\mathbf{E}$ outside the volume V_2 is negligibly small. Thus, we obtain finally

$$\int_{V_2} \left[\varphi\nabla\varkappa + \frac{\varepsilon}{8\pi}(\nabla\varphi)^2\right] dV = -\frac{1}{4\pi}\int_{V_2} \mathbf{DE}\, dV + \frac{1}{8\pi}\int_{V_2} \mathbf{DE}\, dV$$
$$= -\frac{1}{8\pi}\int_{V_2} \mathbf{DE}\, dV = -W_e, \tag{7.5}$$

which is the expresion for the electric energy of the system. Consequently, the Lagrange function L_e can be written in the form

$$L_e = W_m - W_e, \tag{7.6}$$

where the magnetic energy W_m plays the role of a kinetic energy and the electric energy W_e plays the role of a potential energy. In the discrete description of the electromagnetic processes, the expression (7.6) has, in accordance with (4.9), the form

$$L_e = \frac{1}{2c^2}\sum_{r,\,s=1}^{\infty} L_{rs}(x_1, \ldots, x_n)\dot{\varkappa}_r\dot{\varkappa}_s - W_e(\varkappa_1, \varkappa_2, \ldots; x_1, \ldots, x_n), \tag{7.7}$$

where the number n of mechanical generalized coordinates is assumed to be finite.

In the same way as in §4, we can find the "electric" equations of motion of the electromechanical system from the variational principle

$$\int_{t_0}^{t_1} (\delta L + \delta A)\, dt = 0, \tag{7.8}$$

where $L = L_e + L_m$; the function L_e is given by (7.7), and the virtual work δA of the external and dissipative forces by (6.3). As a result, we obtain the Lagrange-Maxwell equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varkappa}_r} - \frac{\partial L}{\partial \varkappa_r} = Q_r \quad (r = 1, 2, \ldots), \tag{7.9}$$

which, with allowance for (6.3) and (7.7), take the form

$$\frac{1}{c^2} L_{rs} \frac{d\dot{\varkappa}_s}{dt} + \frac{1}{c^2} \frac{\partial L_{rs}}{\partial x_i} \dot{\varkappa}_i \dot{\varkappa}_s + R_{rs} \dot{\varkappa}_s + \frac{\partial W_e}{\partial \varkappa_r} = E_r \quad (r = 1, 2, \ldots). \tag{7.10}$$

[8]

We shall now show that the "mechanical" Lagrange-Maxwell equations can also be written in the form of Lagrange equations of the second kind

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = Q_i \quad (i = 1, 2, \ldots n). \tag{7.11}$$

To this end, we shall show that the Lagrange function L is a generating function for the ponderomotive forces. In §5, we considered the question of the mechanical forces of electromagnetic origin and proved that in the case $L_e = W_m$ (i.e., when there is no electric energy W_e)

$$Q_i^{\text{mech}} = \left(\frac{\partial W_m}{\partial x_i} \right)_{\dot{\varkappa}_s = \text{const}}. \tag{7.12}$$

Thus, in the absence of electric energy W_e, the function L_e is indeed a generating function of the ponderomotive forces.

Let us now consider the case $W_e \neq 0$. Since the electric energy in the given system is given by the integral (7.5) taken over the region B_2, where there is no conduction current ($\mathbf{j} = 0$), the virtual work δA of the ponderomotive forces, given by the expression (5.1):

$$\delta A = \int_{V_2} \left[\left(\mathbf{E}_{\text{ext}} \mathbf{j} - \frac{1}{\sigma} \mathbf{j}^2 \right) \delta t - \delta W_m - \delta W_e \right] dV,$$

simplifies in our case to

$$\delta A = - \int_{V_2} \delta W_e dV = - \delta W_e.$$

Performing the variation, we obtain

$$\delta A = - \left(\frac{\partial W_e}{\partial x_i} \right)_{\varkappa_s = \text{const}} \delta x_i - \left(\frac{\partial W_e}{\partial \varkappa_r} \right)_{x_k = \text{const}} \delta \varkappa_r. \tag{7.13}$$

From this it follows directly that if the electric energy W_e of the system is varied subject to the condition $\varkappa_s = \text{const}$ (i.e., $\delta \varkappa_r = 0$), we obtain the expression

$$\delta A = - \left(\frac{\partial W_e}{\partial x_i} \right)_{\varkappa_s = \text{const}} \delta x_i,$$

[8] Summation over repeated indices is to be understood.

which yields the generalized forces

$$Q_i^{\text{mech}} = -\left(\frac{\partial W}{\partial x_i}\right)_{x_s = \text{const}}. \tag{7.14}$$

Thus, if electric energy W_e is present, the Lagrange function $L = L_e + L_m$ is a generating function for the ponderomotive forces. Consequently, the "mechanical" Lagrange-Maxwell equations (7.11) do indeed describe the behavior of an electromechanical system with currents that do not flow in closed loops under the given assumptions.

Before we consider examples, we should like to point out that, in the case of linear conductors and capacitors, the electric energy stored in a capacitor is

$$W_e = q^2/2c. \tag{7.15}$$

If this capacitor is connected in a circuit in which a current I flows, the continuity equation (2.7)

$$I = dq/dt, \tag{7.16}$$

shows that the charge q plays the role of a generalized coordinate and the current I that of a generalized velocity. Then W_m is the kinetic energy of the system and W_e the potential energy; the Lagrange function L_e in this case can be written as usual in the form

$$L_e = W_m - W_e = \frac{1}{2c^2} L_{rs} I_r I_s - \frac{q_k^2}{2c_k}. \tag{7.17}$$

EXAMPLE 4. THREE-COMPONENT FILTER (FIGURE 7.7). We take the charges on the capacitors as the generalized coordinates q_1, q_2, and q_3. The generalized forces are then the currents $i_1 = dq_1/dt$, $i_2 = dq_2/dt$, and $i_3 = dq_3/dt$. The magnetic energy is

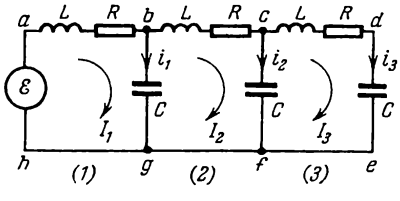

FIGURE 7.7

$$W_m = \frac{L}{2}[(i_1 + i_2 + i_3)^2 + (i_2 + i_3)^2 + i_3^2].$$

The electric energy is

$$W_e = \frac{1}{2c}(q_1^2 + q_2^2 + q_3^2).$$

The virtual work of the external E and dissipative forces across the resistances R is given by the expressions

$$\delta A = \sum_{k=1}^{3} Q_k \delta q_k = E(\delta q_1 + \delta q_2 + \delta q_3)$$
$$- R(i_1 + i_2 + i_3)(\delta q_1 + \delta q_2 + \delta q_3) - R(i_2 + i_3)(\delta q_2 + \delta q_3) - R i_3 \delta q_3$$
$$= [E - R(i_1 + i_2 + i_3)]\,\delta q_1 + [E - R(i_1 + 2i_2 + 2i_3)]\,\delta q_2$$
$$+ [E - R(i_1 + 2i_2 + 3i_3)]\,\delta q_3.$$

We now write down the Lagrange-Maxwell equations. Using the Lagrange function

$$L = \frac{L}{2}[(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 + (\dot{q}_2 + \dot{q}_3)^2 + \dot{q}_3^2] - \frac{1}{2c}(q_1^2 + q_2^2 + q_3^2),$$

we find

$$L(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3) + R(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) + \frac{1}{c} q_1 = E, \tag{7.18}$$

$$L(\ddot{q}_1 + 2\ddot{q}_2 + 2\ddot{q}_3) + R(\dot{q}_1 + 2\dot{q}_2 + 2\dot{q}_3) + \frac{1}{c} q_2 = E, \tag{7.19}$$

$$L(\ddot{q}_1 + 2\ddot{q}_2 + 3\ddot{q}_3) + R(\dot{q}_1 + 2\dot{q}_2 + 3\dot{q}_3) + \frac{1}{c} q_3 = E. \tag{7.20}$$

It is easily seen that equations (7.18), (7.19), and (7.20) are Kirchhoff equations expressed in voltages for the circuits *abgh*, *abcfgh*, and *abcdefgh*, respectively. To prove this, we introduce currents I_1, I_2, and I_3 in the cells (1), (2), and (3) in Figure 7.7 connected to the currents $i_1 = \dot{q}_1$, $i_2 = \dot{q}_2$, and $i_3 = \dot{q}_3$ by the equations

$$I_1 = \dot{q}_1 + \dot{q}_2 + \dot{q}_3, \quad I_2 = \dot{q}_2 + \dot{q}_3, \quad I_3 = \dot{q}_3. \tag{7.21}$$

The Kirchhoff equation for the circuit *abgh* can then be written in the form

$$L\frac{dI_1}{dt} + RI_1 + \frac{1}{c}\int (I_1 - I_2)\,dt = E.$$

Using the relation (7.21), we find that this equation is identical with (7.18). Similarly, one can show that the Kirchhoff equation for the circuit *abcfgh*

$$L\left(\frac{dI_1}{dt} + \frac{dI_2}{dt}\right) + R(I_1 + I_2) + \frac{1}{c}\int (I_2 - I_3)\,dt = E$$

is, after the change of variables (7.1), identical with equation (7.19), etc.

EXAMPLE 5. CAPACITOR MICROPHONE. The dynamical model of a capacitor microphone is shown in Figure 7.8. The sound waves give rise to vibrations of the membrane, a moving capacitor plate (m is

the mass and k the stiffness of the membrane). Because of the displacement of the plate, the capacitance of the capacitor varies and an alternating current arises in the circuit consisting of the capacitor $C(x)$, the resistance R, and the source of the constant emf E_0. The signal is detected at the input of an amplifier with resistance R. Let x be the coordinate of the displacement of the moving plate on which the external force $F(t)$ acts. As generalized coordinates, we shall take x and q, where q is the charge on the capacitor. The Lagrange function has the form

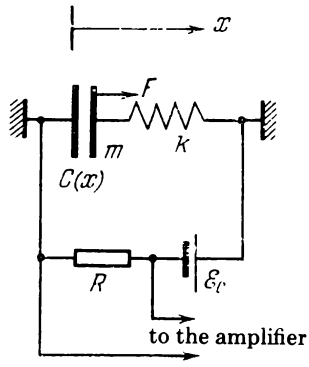

FIGURE 7.8

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}(x - x_1)^2 - \frac{1}{2C(x)}q^2$$

(x_1 is the coordinate of the slab when the spring is relaxed).

We can find the generalized forces from the expression for the virtual work of the external and dissipative forces

$$\delta A = (E_0 - R\dot{q})\,\delta q + (F - h\dot{x})\,\delta x,$$

where h is the coefficient of the viscous friction associated with the displacement of the moving plate of the capacitor. The Lagrange equations of second kind are

$$m\ddot{x} + h\dot{x} + k(x - x_1) + \frac{q^2}{2}\frac{d}{dx}\left(\frac{1}{C(x)}\right) = F,$$

$$R\dot{q} + \frac{q}{C(x)} = E_0.$$

For a plane-parallel capacitor, $C(x) = a/x$; hence,

$$m\ddot{x} + h\dot{x} + k(x - x_1) + \frac{q^2}{2a} = F,$$

$$R\dot{q} + \frac{qx}{a} = E_0. \tag{7.22}$$

We introduce the new quantities

$$x_n = aR\sqrt{\frac{k}{m}}\,x, \quad q_n = aq\sqrt{2kR\sqrt{\frac{k}{m}}}, \quad t_n = \sqrt{\frac{m}{k}}\,t,$$

$$\delta = \frac{r}{\sqrt{km}}, \quad F_n = \frac{1}{kaR}\sqrt{\frac{m}{k}}\,F, \quad E_{0n} = \frac{1}{akR}\sqrt{\frac{m}{2R}\sqrt{\frac{m}{k}}}\,E_0,$$

and equations (7.22) can then be written in the form (we omit the suffixes n)

$$\begin{aligned} \ddot{x} + \delta\dot{x} + x - x_1 + q^2 &= F, \\ \dot{q} + qx &= E_0. \end{aligned} \tag{7.23}$$

In the absence of external perturbations ($F = 0$), the equilibrium position (x_0, q_0) of the system can be determined from (7.23) by setting $\dot{x} = \ddot{x} = \dot{q} = 0$:

$$x_0 - x_1 + q_0^2 = 0, \quad q_0 x_0 = E_0. \tag{7.24}$$

Hence, we can find an equation for the equilibrum position x_0:

$$x_0^3 - x_1 x_0^2 + E_0^2 = 0, \tag{7.25}$$

which has positive roots only if

$$4x_1 > 27E_0^2. \tag{7.26}$$

In order to study the stability of these equilibrium states, we find the linearized equation for small deviations ξ and η from the equilibrium state (x_0, q_0). Setting $x = x_0 + \xi$ and $q = q_0 + \eta$ in (7.23) and using (7.24), we find

$$\begin{aligned} \ddot{\xi} + \delta\dot{\xi} + \xi + 2q_0\eta &= 0, \\ \dot{\eta} + x_0\eta + q_0\xi &= 0. \end{aligned}$$

The characteristic equation has the form

$$p^3 + (\delta + x_0)\, p^2 + (\delta x_0 + 1)\, p + x_0 - 2q_0^2 = 0. \tag{7.27}$$

The stability conditions of the equilibrium state (x_0, q_0) reduce to

$$x_0 > 2q_0^2.$$

In accordance with the first equation of (7.24), this inequality can be written in the form

$$x_0 > 2x_1/3.$$

It follows that, since the two equilibrium states are situated to the left and the right of the point $2x_1/3$, the former is unstable and the latter is stable. Since the characteristic equation (7.27) is of third order, we can, as in Example 1 of §6, use the diagram of Figure 7.3 with all the results that follow from it to investigate the nature of the transient processes in the given example.

§8. Models of electrical machines described by the Lagrange-Maxwell equations

In the present section, we shall consider more complicated examples of electromechanical systems described by the Lagrange-Maxwell equations.

An electrical machine is an electromechanical system whose mechanical part is usually a holonomic system with a finite number of degrees of freedom. If such a machine contains a finite number of thin (quasi-linear) conductors, in each of which the electrical current can be characterized by a single independent variable, we have a discrete electromechanical system. As an example of such systems, one may consider the ordinary induction machines, namely, synchronous salient and nonsalient pole machines and also asynchronous machines with a short-circuited coil on the armature.

If an electrical machine contains three-dimensional or surface conductors in which the current distribution changes during the motion we have a distributed electromechanical system.[9] As examples of continuous systems, we may take Barlow's wheel or an asynchronous machine with a solid rotor.

EXAMPLE 6. ELECTRICAL MACHINE WITHOUT COMMUTATOR. Let us consider a machine that operates under conditions when the iron is not saturated. The function L_e is a quadratic function of the currents of the rotor and the stator. Suppose the stator has an N-phase coil and the rotor an M-phase coil. Let $\dot{q}_a$ $(a = 1, 2, \cdots, N)$ and $\dot{q}_\nu$ $(\nu = 1, 2, \cdots, M)$ be the currents in the phases of the stator and the rotor respectively, and let $\mathcal{E}_a$ and $\mathcal{E}_\nu$ be the corresponding emf's. Let R_{ab} and $R_{\mu\nu}$ be self-resistances and mutual resistances of the phases and L_{ab} and $L_{\mu\nu}$ the coefficients of selfinductance and mutual inductance of the coils of the stator and rotor, respectively; finally, let $L_{\nu a}$ be the coefficients of mutual inductance of the coils of the rotor and stator. In the case when the rotor and stator do not have salient poles and the air gap between the rotor and stator is uniform, only the coefficients $L_{\nu a} = L_{\nu a}(\varphi)$ depend on the angle φ, and the quantities L_{ab} and $L_{\mu\nu}$ are constants. The Lagrange function has the form

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[9] Sine, in general, the Lagrange-Maxwell equations cannot be applied to distributed electrical machines containing sliding contacts, we shall not consider such machines in this section.

$$L = L_e + \tfrac{1}{2} J\dot{\varphi}^2,$$

where J is the moment of inertia of the rotor and L_e is[10]

$$L_e = \tfrac{1}{2} L_{\mu\nu}\dot{q}_\mu\dot{q}_\nu + L_{\nu a}(\varphi)\,\dot{q}_\nu\,\dot{q}_a + \tfrac{1}{2} L_{ab}\dot{q}_a\dot{q}_b.$$

The generalized forces can be determined from the expression for the virtual work

$$\delta A = (\mathscr{E}_a - R_{ab}\dot{q}_b)\,\delta q_a + (\mathscr{E}_\nu - R_{\mu\nu}\dot{q}_\mu)\delta q_\nu + (K_\varphi - h\dot{\varphi})\delta\varphi,$$

where K_φ is the moment of the external mechanical forces and h is the coefficient of viscous friction.

The Lagrange-Maxwell equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = \mathscr{E}_a - R_{ab}\dot{q}_b, \tag{8.1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\nu} = \mathscr{E}_\nu - R_{\mu\nu}\dot{q}_\mu, \tag{8.2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = K_\varphi - h\dot{\varphi}. \tag{8.3}$$

We have

$$\frac{\partial L}{\partial \dot{q}_a} = L_{\nu a}(\varphi)\,\dot{q}_\nu + L_{ab}\dot{q}_b,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = L_{\nu a}(\varphi)\,\ddot{q}_\nu + \frac{\partial L_{\nu a}}{\partial \varphi}\dot{\varphi}\dot{q}_\nu + L_{ab}\ddot{q}_b.$$

Similarly,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\nu} = L_{\mu\nu}\ddot{q}_\mu + L_{\nu a}(\varphi)\,\ddot{q}_a + \frac{\partial L_{\nu a}}{\partial \varphi}\dot{\varphi}\dot{q}_a$$

and, finally,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = J\ddot{\varphi}, \quad \frac{\partial L}{\partial \varphi} = \frac{\partial L_{\nu a}}{\partial \varphi}\,\dot{q}_\nu\dot{q}_a.$$

Substituting these expressions into (8.1)–(8.3), we obtain the following equations of motion for a machine without commutator:

$$L_{ab}\ddot{q}_b + R_{ab}\dot{q}_b + L_{\nu a}(\varphi)\,\ddot{q}_\nu + \frac{\partial L_{\nu a}}{\partial \varphi}\dot{\varphi}\dot{q}_\nu = \mathscr{E}_a, \tag{8.4}$$

$$L_{\mu\nu}\ddot{q}_\mu + R_{\mu\nu}\dot{q}_\mu + L_{\nu a}(\varphi)\,\ddot{q}_a + \frac{\partial L_{\nu a}}{\partial \varphi}\dot{\varphi}\dot{q}_a = \mathscr{E}_\nu, \tag{8.5}$$

[10] We recall that summation is always understood over repeated indices. The indices a and b take the values from 1 to N and the indices μ and ν the values from 1 to M.

$$J\ddot{\varphi} + h\dot{\varphi} - \frac{\partial L_{va}}{\partial \varphi}\dot{q}_v\dot{q}_a = K_\varphi. \tag{8.6}$$

In the case when there is a single coil on the stator and also on the rotor, we obtain the following equations, in which the subscript 1 refers to the stator and the subscript 2 to the rotor:

$$L_1\ddot{q}_1 + R_1\dot{q}_1 + M(\varphi)\ddot{q}_2 + \frac{\partial M}{\partial \varphi}\dot{\varphi}\dot{q}_2 = \mathscr{E}_1, \tag{8.7}$$

$$L_2\ddot{q}_2 + R_2\dot{q}_2 + M(\varphi)\ddot{q}_1 + \frac{\partial M}{\partial \varphi}\dot{\varphi}\dot{q}_1 = \mathscr{E}_2, \tag{8.8}$$

$$J\ddot{\varphi} + h\dot{\varphi} - \frac{\partial M}{\partial \varphi}\dot{q}_1\dot{q}_2 = K_\varphi. \tag{8.9}$$

EXAMPLE 7. PARENT MODEL OF THE GENERAL THEORY OF NONSALIENT POLE ELECTRICAL MACHINES.[11] Our model of a nonsalient pole electrical machine with a cylindrical drum armature is a system consisting of two hollow cylinders of radius a and b with thin conducting walls. The external cylinder (the stator) is fixed and the internal cylinder (the rotor) can rotate about their common axis O (Figure 7.9). Since the active conductors of the coils of a real electrical machine are parallel to the axis of rotation Oz, we shall assume that the model of Figure 7.9 possesses an appropriate anisotropy of the conductivity, i.e., the currents of the stator j_s and the rotor j_r flow along the generators of the cylinders; in other words, they only have z components. Leaving aside the boundary effects at the ends of the cylinders, we shall assume that the latter are infinite and that the parameters of the medium are constant. We shall use the discrete description of the currents. Assuming that the total current flowing along each of the cylinders vanishes, one can write

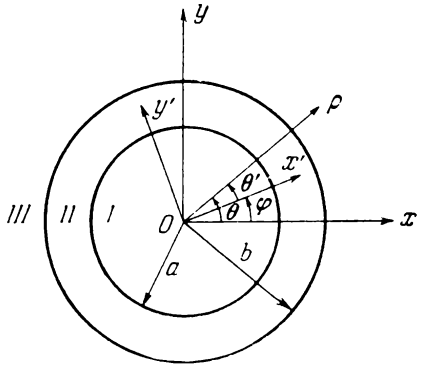

FIGURE 7.9

$$j'_r = \frac{1}{a}(\dot{x}'_{1k}\cos k\theta' + \dot{x}'_{2k}\sin k\theta'),$$

$$j_s = \frac{1}{b}(\dot{x}_{3k}\cos k\theta + \dot{x}_{4k}\sin k\theta).$$

[11] This model is called a parent model because, by imposing appropriate connections, one can obtain from it models of different specific nonsalient pole machines with a definite type of coil which are fitted with a commutator or contact ring.

Here, the angle θ is measured in a fixed system of coordinates and the angle θ' in a rotating system of coordinates connected with the rotor. Consequently, $j_s = j_s(\theta)$ and $j'_r = j'_r(\theta) = j_r(\theta - \varphi)$. The quantities

$$\dot{\varkappa}'_{1k}, \quad \dot{\varkappa}'_{2k}, \quad \dot{\varkappa}'_{3k} \text{ and } \dot{\varkappa}'_{4k}$$

are generalized velocities.

In order to calculate the Lagrange function

$$L_e = \frac{1}{2c^2} L_{rs}\dot{\varkappa}_r\dot{\varkappa}_s$$

we must find the quantity

$$L_{rs} = \int_V \mathbf{A}_s\mathbf{S}_r dV, \tag{8.10}$$

where the vector functions $\mathbf{S}_r$ are known

$$\mathbf{S}_k^{(r)} = \mathbf{z}^0 \frac{1}{a} \begin{matrix}\sin \\ \cos\end{matrix} k\theta', \quad \mathbf{S}_k^{(s)} = \mathbf{z}_0 \frac{1}{b} \begin{matrix}\sin \\ \cos\end{matrix} k\theta,$$

and the vector potential $\mathbf{A} = c^{-1}\mathbf{A}_s\dot{\varkappa}_s$, determined by the currents j_r and j_s can be found by the following arguments. The vector potential $\mathbf{A}$ must satisfy the homogeneous Laplace equation in the whole of the space (with the exception of the walls of the cylinders), i.e., in the regions I, II, and III (see Figure 7.9). Since the vector potential has only a z component in the case under consideration, the problem is essentially two-dimensional and $A_z = A(\rho, \theta, t)$. Therefore, the equation $\Delta_z A = 0$ has the form

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2 A}{\partial\theta^2} = 0$$

or

$$\rho^2\frac{\partial^2 A}{\partial\rho^2} + \rho\frac{\partial A}{\partial\rho} + \frac{\partial^2 A}{\partial\theta^2} = 0.$$

We seek the solution of this equation in the form

$$A = A_1(\rho)A_2(\theta)$$

and, substituting this equation into the foregoing equation, we obtain

$$A_2\left(\rho^2\frac{d^2A_1}{d\rho^2} + \rho\frac{dA_1}{d\rho}\right) + A_1\frac{d^2A_2}{d\theta^2} = 0$$

or

$$\frac{1}{A_1}\left(\rho^2\frac{d^2A_1}{d\rho^2} + \rho\frac{dA_1}{d\rho}\right) = -\frac{1}{A_2}\frac{d^2A_2}{d\theta^2} = n^2.$$

Hence,

$$\frac{d^2A_2}{d\theta^2} + n^2A_2 = 0, \quad \rho^2\frac{d^2A_1}{d\rho^2} + \rho\frac{dA_1}{d\rho} + n^2A_1 = 0.$$

For $n = 0$, we have $A_0 = (C_0\theta + D_0)(E_0 \ln \rho + F_0)$. For $n \neq 0$, we obtain, restricting ourselves to real positive numbers $n > 0$,

$$A_n = (E_n\rho^n + F_n\rho^{-n})(C_n \cos n\theta + D_n \sin n\theta),$$

where the coefficients C_0, D_0, E_0, F_0, E_n, F_n, C_n, and D_n are arbitrary constants. By virtue of the fact that A must be periodic in θ (with period 2π), we must take $C_0 = 0$ and n can only be an integer, i.e., $n = 1, 2, \cdots$. In addition, A is finite at the origin for $\rho = 0$ and at infinity for $\rho = \infty$. Hence, in the regions I, II, and III (Figure 7.9), we have

$$A_{\mathrm{I}} = D_0 + \sum_{k=1}^{\infty} \rho^k (C_k \cos k\theta + D_k \sin k\theta),$$

$$A_{\mathrm{II}} = E_0' \ln \rho + F_0' + \sum_{k=1}^{\infty} (E_k'\rho^k + F_k'\rho^{-k})(C_k' \cos k\theta + D_k' \sin k\theta),$$

$$A_{\mathrm{III}} = D_0' + \sum_{k=1}^{\infty} \rho^{-k} (C_k'' \cos k\theta + D_k'' \sin k\theta).$$

Using the boundary conditions

$$(A_{\mathrm{I}})_{\rho=a} = (A_{\mathrm{II}})_{\rho=a}, \quad (A_{\mathrm{II}})_{\rho=b} = (A_{\mathrm{III}})_{\rho=b},$$

we find

$$D_0 = E_0' \ln a + F_0', \; C_k a^k = (E_k' a^k + F_k' a^{-k}) C_k', \; C_k'' b^{-k} = (E_k' b^k + F_k' b^{-k}) C_k',$$
$$D_0'' = E_0' \ln b + F_0', \; D_k a^k = (E_k' a^k + F_k' a^{-k}) D_k', \; D_k'' b^{-k} = (E_k' b^k + F_k' b^{-k}) D_k'.$$

In addition, the following boundary conditions must be satisfied:

$$\left(\frac{\partial A_{\mathrm{I}}}{\partial \rho}\right)_{\rho=a} - \left(\frac{\partial A_{\mathrm{II}}}{\partial \rho}\right)_{\rho=a} = \frac{4\pi\mu}{c} j_r, \quad \left(\frac{\partial A_{\mathrm{II}}}{\partial \rho}\right)_{\rho=b} - \left(\frac{\partial A_{\mathrm{III}}}{\partial \rho}\right)_{\rho=b} = \frac{4\pi\mu}{c} j_s. \tag{8.11}$$

Hence, $E_0' = 0$ and, ignoring the irrelevant additive constant in the expression for the potential and introducing the new coefficients $\alpha_k = E_k' C_k'$, $\beta_k = F_k' C_k'$, $\gamma_k = E_k' D_k'$, and $\delta_k = F_k' D_k'$, we obtain the following expressions:

$$A_{\mathrm{I}} = \sum_{k=1}^{\infty} \rho^k [(\alpha_k + \beta_k a^{-2k}) \cos k\theta + (\gamma_k + \delta_k a^{-2k}) \sin k\theta],$$

$$A_{\mathrm{II}} = \sum_{k=1}^{\infty} [(\alpha_k \rho^k + \beta_k \rho^{-k}) \cos k\theta + (\gamma_k \rho^k + \delta_k \rho^{-k}) \sin k\theta], \tag{8.12}$$

$$A_{\mathrm{III}} = \sum_{k=1}^{\infty} \rho^{-k} [(\alpha_k b^{2k} + \beta_k) \cos k\theta + (\gamma_k b^{2k} + \delta_k) \sin k\theta].$$

We substitute these expressions into (8.10), whose right-hand sides must contain

$$j_r = \frac{1}{a}\sum_{k=1}^{\infty}[\dot{\varkappa}'_{k1}\cos k(\theta-\varphi)+\dot{\varkappa}_{k2}\sin k(\theta-\varphi)],$$

(8.13)

$$j_s = \frac{1}{b}\sum_{k=1}^{\infty}[\dot{\varkappa}_{k3}\cos k\theta+\dot{\varkappa}_{k4}\sin k\theta].$$

Equating the terms having the same $\cos k\theta$ and $\sin k\theta$ as factors, we find

$$\alpha_k = \frac{2\pi\mu b^{-k}}{ck}\dot{\varkappa}_{k3},\quad \beta_k = \frac{2\pi\mu a^k}{ck}(\dot{\varkappa}'_{k1}\cos k\varphi - \dot{\varkappa}'_{k2}\sin k\varphi),$$

$$\gamma_k = \frac{2\pi\mu b^{-k}}{ck}\dot{\varkappa}_{k4},\quad \delta_k = \frac{2\pi\mu a^k}{ck}(\dot{\varkappa}'_{k1}\sin k\varphi + \dot{\varkappa}'_{k2}\cos k\varphi).$$

Having found the expression (8.12) for the vector potential, we can easily determine from (8.10) the values of the coefficients of selfinductance and mutual inductance L_{rs} per unit length of the stator and rotor. However, in order to find the Lagrange function L_e it is simpler to use the following expression directly:

$$L_e = \frac{1}{2c}\int_V \mathbf{A}\mathbf{j}\,dv. \tag{8.14}$$

The integral on the right-hand side breaks up into two integrals over the cylinders $\rho = a$ and $\rho = b$, where

$$(A_I)_{\rho=a} = (A_{II})_{\rho=a} = \frac{2\pi\mu}{c}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\left[\left(\frac{a}{b}\right)^k\dot{\varkappa}_{k3}+\dot{\varkappa}'_{k1}\cos k\varphi - \dot{\varkappa}'_{k2}\sin k\varphi\right]\cos k\theta + \left[\left(\frac{a}{b}\right)^k\dot{\varkappa}_{k4}+\dot{\varkappa}'_{k1}\sin k\varphi+\dot{\varkappa}'_{k2}\cos k\varphi\right]\sin k\theta\right\},$$

$$(A_{II})_{\rho=b} = (A_{III})_{\rho=b} = \frac{2\pi\mu}{c}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\left[\dot{\varkappa}_{k3}+\left(\frac{a}{b}\right)^k(\dot{\varkappa}'_{k1}\cos k\varphi - \dot{\varkappa}'_{k2}\sin k\varphi)\right]\cos k\theta + \left[\dot{\varkappa}_{k4}+\left(\frac{a}{b}\right)^k(\dot{\varkappa}'_{k1}\sin k\varphi+\dot{\varkappa}'_{k2}\cos k\varphi)\right]\sin k\theta\right\}.$$

Substituting these expressions together with (8.13) and (8.14), we obtain

$$L_e = \frac{\pi^2\mu}{c^2}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\left[\left(\frac{a}{b}\right)^k\dot{\varkappa}_{k3}+\dot{\varkappa}'_{k1}\cos k\varphi - \dot{\varkappa}'_{k2}\sin k\varphi\right](\dot{\varkappa}'_{k1}\cos k\varphi - \dot{\varkappa}'_{k2}\sin k\varphi) + \left[\left(\frac{a}{b}\right)^k\dot{\varkappa}_{k4}+\dot{\varkappa}'_{k1}\sin k\varphi+\dot{\varkappa}'_{k2}\cos k\varphi\right](\dot{\varkappa}'_{k1}\sin k\varphi+\dot{\varkappa}'_{k2}\cos k\varphi) + \left[\dot{\varkappa}_{k3}+\left(\frac{a}{b}\right)^k(\dot{\varkappa}'_{k1}\cos k\varphi - \dot{\varkappa}'_{k2}\sin k\varphi)\right]\dot{\varkappa}_{k3} + \left[\dot{\varkappa}_{k4}+\left(\frac{a}{b}\right)^k(\dot{\varkappa}'_{k1}\sin k\varphi+\dot{\varkappa}'_{k2}\cos k\varphi)\right]\dot{\varkappa}_{k4}\right\}$$

or, introducing the coefficients

$$L_k = 2\left(\frac{\pi}{c}\right)^2 \frac{\mu}{k}, \quad M_k = 2\left(\frac{\pi}{c}\right)^2 \frac{\mu}{k}\left(\frac{a}{b}\right)^k,$$

we obtain

$$L_e = \frac{1}{2}\sum_{k=1}^{\infty}\Big\{L_k\,(\dot{\varkappa}_{k1}'^2 + \dot{\varkappa}_{k2}'^2 + \dot{\varkappa}_{k3}^2 + \dot{\varkappa}_{k4}^2) + 2M_k\,[(\dot{\varkappa}_{k1}'\cos k\varphi - \dot{\varkappa}_{k2}'\sin k\varphi)\,\dot{\varkappa}_{k3} + (\dot{\varkappa}_{k1}'\sin k\varphi + \dot{\varkappa}_{k2}'\cos k\varphi)\,\dot{\varkappa}_{k4}]\Big\}.$$

The total Lagrange function is

$$L = L_e + \tfrac{1}{2} J\dot{\varphi}^2,$$

where J is the moment of inertia of the rotor. In accordance with formulas (4.6), the resistances of the rotor and stator per unit length are

$$R_r = \pi/a\sigma_r, \qquad R_s = \pi/b\sigma_s,$$

where σ_r and σ_s are constants. The generalized forces can be found from the expression for the virtual work

$$\delta A = \sum_{k=1}^{\infty}[(\mathscr{E}_{1k}' - R_r\dot{\varkappa}_{k1}')\,\delta\varkappa_{k1}' + (\mathscr{E}_{2k}' - R_r\dot{\varkappa}_{k2}')\,\delta\varkappa_{k2}' + (\mathscr{E}_{3k} - R_s\dot{\varkappa}_{k3})\,\delta\varkappa_{k3} + (\mathscr{E}_{4k} - R_s\dot{\varkappa}_{k4})\,\delta\varkappa_{k4}] + (K - h\dot{\varphi})\,\delta\varphi = \sum_{\mu=1}^{\infty} Q_\mu \delta q_\mu.$$

Writing down the Lagrange-Maxwell equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = Q_\mu,$$

we obtain the following equations of motion of the parent model of a nonsalient pole electrical machine

$$L_k\ddot{\varkappa}_{k1,2}' + M_k\left[\pm\ddot{\varkappa}_{k3}\begin{matrix}\cos\\\sin\end{matrix} k\varphi + \ddot{\varkappa}_{k4}\begin{matrix}\sin\\\cos\end{matrix} k\varphi + k\dot{\varphi}\left(-\dot{\varkappa}_{k3}\begin{matrix}\sin\\\cos\end{matrix} k\varphi \pm \dot{\varkappa}_{k4}\begin{matrix}\cos\\\sin\end{matrix} k\varphi\right)\right] = \mathscr{E}_{k1,2}' - R_r\dot{\varkappa}_{k1,2}' \quad (k = 1, 2, \ldots), \tag{8.15}$$

$$L_k\ddot{\varkappa}_{k3,4} + M_k\left[\ddot{\varkappa}_{k1}'\begin{matrix}\cos\\\sin\end{matrix} k\varphi \mp \ddot{\varkappa}_{k2}'\begin{matrix}\sin\\\cos\end{matrix} k\varphi + k\dot{\varphi}\left(\mp\dot{\varkappa}_{k1}'\begin{matrix}\sin\\\cos\end{matrix} k\varphi - \dot{\varkappa}_{k2}'\begin{matrix}\cos\\\sin\end{matrix} k\varphi\right)\right] = \mathscr{E}_{k3,4} - R_s\dot{\varkappa}_{k3,4} \quad (k = 1, 2, \ldots) \tag{8.16}$$

and, for the coordinate φ,

$$J\ddot{\varphi} - \sum_{k=1}^{\infty} k M_k [- (\dot{x}'_k \sin k\varphi + \dot{x}'_k \cos k\varphi) \dot{x}_k + (\dot{x}'_k \cos k\varphi - \dot{x}'_k \sin k\varphi) \dot{x}_k] = K - h\dot{\varphi}. \tag{8.17}$$

§9. Examples of electromechanical systems with nonholonomic constraints realized by sliding contacts

All the electromechanical systems considered in the preceding sections are holonomic systems, and their equations of motion can be written in the form of Lagrange equations of the second kind.

However, it is well known that the application of Lagrange equations of the second kind to systems containing three-dimensional conductors with sliding contacts leads to completely incorrect results.

Historically, it would seem that the first electromechanical system whose equations could not be written in the form of Lagrange equations of the second kind was Barlow's wheel. The formal Lagrange equations derived for Barlow's wheel could not explain its rotation in a magnetic field and the induction of an electromotive force when the wheel rotates.

At the time, several obscure derivations of the correct equations of motion for Barlow's wheel were put forward, and these were the subject for some time of a lively discussion. It was conjectured that a similar situation obtains in the case of Barlow's wheel as arises in the derivation of the equations of motion of a disc rolling on a rough plane.[12] However, until very recently these surmises were not developed to the stage at which one could write down the equations of the nonholonomic constraints realized by sliding contacts so that the apparatus of the analytical mechanics of nonholonomic systems could be exploited.

In the present section, we consider examples of electromechanical systems with sliding contacts and establish the form of the resulting constraints.

EXAMPLE 8. MODEL OF A COMMUTATOR MACHINE WITH UNIFORM COILS OF THE ROTOR AND STATOR. In the preceding section, we considered the parent model of the general theory of electrical machines (Example 7) consisting of two thin-walled cylinders which have a definite anisotropy of the conductivity. The current in the walls of the cylinders is an

[12] As we have seen, the equations of motion of a disc cannot be expressed in the form of Lagrange equations of the second kind because of the presence of the nonholonomic constraints.

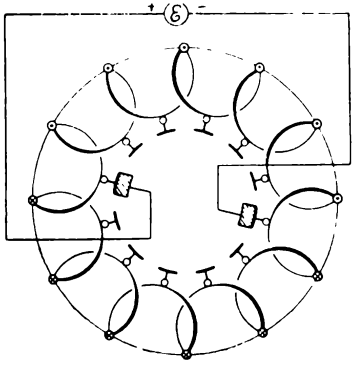

FIGURE 7.10

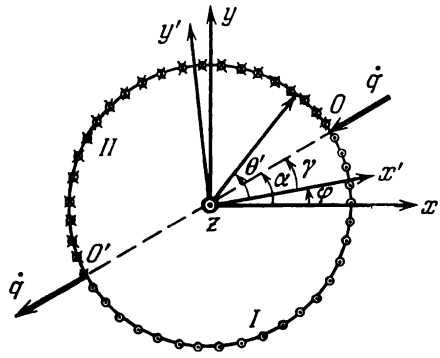

FIGURE 7.11

idealization of the currents in a dense wire coil of an actual machine. Therefore, the distribution of this current is completely determined by the distribution of the conductors over the rotor (or stator) and the way in which they are connected (with contact rings, commutators, etc.).

Let us consider a commutator machine with a drum rotor which has a uniform coil. The model of such a machine can be obtained from the parent machine if, for example, the conductors of the rotor are connected in a loop (or wave) coil with a tap to the plates of a commutator fitted with brushes. For the sake of better visualization, we have depicted the coil of a ring rotor schematically in Figure 7.10. In principle, it does not differ from the coils of a drum rotor.[13] Let the current $\dot{q}$ be passed through the brushes to the coil of the rotor at the points O and O' (Figure 7.11). In accordance with Figures 7.10 and 7.11, these points divide all the conductors into two groups: in one of them (say, I) the current is directed in the $+z$ direction and in the other in the $-z$ direction. The current of the rotor can be expressed in the form

$$j'_r(\theta') = \frac{1}{a}\dot{q}f'(\gamma, \theta'),$$

where the "distribution function of the coil" $f'(\gamma, \theta')$ depends on the parameter γ, the angle characterizing the position of the lead contacts O and O' in a system of coordinates $x'y'$ rigidly connected to the rotor. In our case,

[13] It is well known that a loop coil of a drum rotor (armature) can be represented as a variant of the coil of a ring armature for which the nonactive conductors arranged on the internal surface of the ring are taken onto the outer surface of the armature.

$$f'(\gamma, \theta') = \begin{cases} +1, & \text{if } \gamma - \pi < \theta' < \gamma, \\ -1, & \text{if } \gamma < \theta' < \gamma + \pi. \end{cases}$$

If the commutator brushes O and O' are fixed in space, some of the conductors pass from group I to group II when the rotor revolves, and the same number of conductors pass from group II to group I. Thus, the distribution of the current over the surface of the rotor in the fixed coordinate system xy does not change.

If the position of the brush O in the fixed coordinate system is characterized by the constant angle α, the angle γ is related to the angle θ of revolution of the rotor by the equation

$$\gamma = \alpha - \varphi.$$

Then, in the case of a uniform coil,

$$\begin{aligned} f'(\gamma, \theta') &= f(\theta' - \gamma) \\ &= f(\theta' + \varphi - \alpha). \end{aligned}$$

Hence, the distribution function depends on the angle of revolution of the rotor. Suppose $\alpha = 0$ (the brushes are arranged horizontally); the distribution function $f(\theta' + \varphi)$ has the form shown in Figure 7.12. Expanding the function $f(\theta' + \varphi)$ in a Fourier series, we obtain

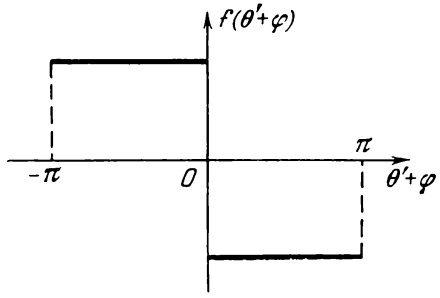

FIGURE 7.12

$$\begin{aligned} f(\theta' + \varphi) &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)(\theta' + \varphi)}{2k-1} \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} [\sin(2k-1)\varphi \cos(2k-1)\theta' \\ &\qquad + \cos(2k-1)\varphi \sin(2k-1)\theta']. \end{aligned}$$

Thus, the current density can be represented in the form

$$\begin{aligned} j_r'(\theta') = \frac{4}{\pi a} \dot{q} \sum_{k=1}^{\infty} \frac{1}{2k-1} &[\sin(2k-1)\varphi \cos(2k-1)\theta' \\ &+ \cos(2k-1)\varphi \sin(2k-1)\theta']. \end{aligned}$$

Comparing this with the expression

$$j_r'(\theta') = \frac{1}{a} \sum_{k=1}^{\infty} (\dot{x}_{k_1}' \cos k\theta' + \dot{x}_{k_2}' \sin k\theta'),$$

we obtain the relations

$$\dot{\varkappa}'_{1\,2k} = \dot{\varkappa}'_{2\,2k} \qquad (k = 1, 2, \ldots), \tag{9.1}$$

$$\dot{\varkappa}'_{1\,2k-1} = \frac{4\dot{q}}{(2k-1)\pi} \sin(2k-1)\varphi \quad (k = 1, 2, \ldots), \tag{9.2}$$

$$\dot{\varkappa}'_{2\,2k-1} = \frac{4\dot{q}}{(2k-1)\pi} \cos(2k-1)\varphi \quad (k = 1, 2, \ldots). \tag{9.3}$$

In the general case $\varphi = \varphi(t)$, equations (9.2) and (9.3) are nonintegrable. Consequently, the commutator device generates nonholonomic constraints in the general case.

EXAMPLE 9. BARLOW'S WHEEL. This name is usually given to the electromechanical system depicted in Figure 7.13. A thin metallic disc (or wheel with spokes), which can rotate about an axis perpendicular to its plane is placed in a magnetic field.[14] The disc is connected to an electrical circuit containing an emf $\mathscr{E}$ by means of contacts at the points O and O' (the contact at O' is a sliding contact).

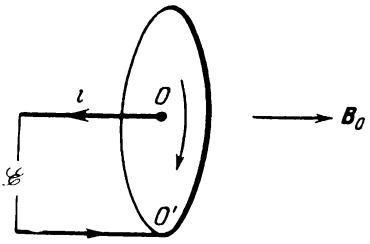

FIGURE 7.13

We shall consider the problem of Barlow's wheel under the following simplifying assumptions: 1) The magnetic field is constant and homogeneous and the direction of the vector of the magnetic induction B_0 is perpendicular to the plane of the disc. 2) The angular velocity of the disc is not too large and the equations of quasisteady electrodynamics of slowly moving media can be used to study the electromagnetic field and the currents in the disc. 3) The field $\mathbf{B}'$ of the currents in the disc is negligibly small compared with the field $\mathbf{B}_0$.

In order to find the distribution of the current density $\mathbf{j}$ in the rotating disc, we write down the equations of the quasisteady electrodynamics of slowly moving media

$$\begin{aligned} &\operatorname{curl} \frac{\mathbf{B}}{\mu} = \frac{4\pi}{c}\mathbf{j}, \quad \operatorname{div}\mathbf{B} = 0, \\ &\operatorname{curl}\mathbf{E}^* = \frac{1}{c}\operatorname{curl}(\mathbf{v}\times\mathbf{B}) - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ &\mathbf{j} = \sigma(\mathbf{E}^* + \mathbf{E}_{\text{ext}}). \end{aligned} \tag{9.4}$$

[14] This field may be generated by a constant magnet or a solenoid. In the case when the coil of the solenoid is connected in series with an electrical circuit containing an emf ϵ, we have a "Barlow's wheel with selfexcitation".

Here, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$ is the total magnetic field: $\mathbf{E}^* = \mathbf{E} + \mathbf{v}/c \times \mathbf{B}$ is the "effective" electric field; and $\mathbf{v}$ is the velocity of the given point of the disc. Under the assumptions we have made, the system of equations (9.4) can be greatly simplified. For $\mathbf{B}'$ can be neglected everywhere since $\mathbf{B}' \ll \mathbf{B}_0$. Then the magnetic field in the system is characterized by the constant vector $\mathbf{B}_0$. In this connection, the first two equations of the system (9.4) are superfluous and should be ignored. Since $\mathbf{B}_0$ is constant, the term $\partial \mathbf{B}_0/\partial t$ drops out of the third equation. In addition, since $\mathbf{v} \times \mathbf{B}_0 = (\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}_0 = \omega B_0 \mathbf{r}$, where $\mathbf{r}$ is the radius vector from the center of the disc to some point of the disc, we obtain $\operatorname{curl}(\mathbf{v} \times \mathbf{B}_0) = 0$. Thus, the electric field $\mathbf{E}^*$ in the system can be found from the equation[15]

$$\operatorname{curl} \mathbf{E}^* = 0. \tag{9.5}$$

The current density $\mathbf{j}$ in the disc can then be found from the last equation of the system (9.4), where, for the sake of simplicity, we can assume $\mathbf{E}_{\text{ext}} = 0$.

It follows from equation (9.5) that $\mathbf{E}^* = \operatorname{grad} U$. Thus, the problem reduces to finding the solution of the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{9.6}$$

subject to the following boundary conditions (Figure 7.14):

$$\sigma \int_0^{2\pi} \left[\frac{\partial U}{\partial r}\right]_{\substack{z=0 \\ r \to 0}} r \, d\theta = \dot{q}, \tag{9.7}$$

$$\left[\frac{\partial U}{\partial r}\right]_{\substack{z=0 \\ r=a}} = \frac{\dot{q}}{a\sigma} \delta(\theta + \varphi). \tag{9.8}$$

These conditions reflect the presence on the disc of point contacts, through which the current $\dot{q}$ flows. In equations (9.7) and (9.8), the conductivity σ is assumed constant and the symbol $\delta(x)$ stands for the delta-function.

We shall now show that the determination of the distribution of the current density in the disc reduces to the solution of a two-dimensional problem. We note first that the current density in the disc is determined by the values which the vector $\mathbf{E}^*$ (and consequently U) takes on the plane

---

[15] The form of the equation indicates that, under the assumptions made, the distribution of the current density $\mathbf{j}$ in the disc does not depend on the angular velocity of the disc.

$z = 0$ (i.e., on the plane of the disc). However, of the three components of the vector $\mathbf{E}^*\{E_x(x,y,z),\ E_y(x,y,z),\ E_z(x,y,z)\}$ we always have $E_z(x,y,0) = 0$ at the points of the plane $z = 0$. For it follows from equation (9.5) that

$$\begin{aligned} \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} &= 0, \\ \frac{\partial E_z}{\partial x'} - \frac{\partial E_x}{\partial z'} &= 0, \end{aligned} \tag{9.9}$$

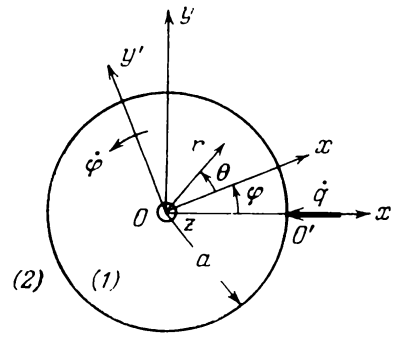

FIGURE 7.14

where x', y', and z' are axes rigidly fixed to the wheel (see Figure 7.14). However, by virtue of the symmetry of the field with respect to the plane $z = 0$, we have

$$\left[\frac{\partial E_z}{\partial y'}\right]_{z=0} = \left[\frac{\partial E_z}{\partial x'}\right]_{z=0} = 0$$

i.e., the function $E_z(x,y,0)$ is a constant and, since the system contains no charges, the constant is equal to zero. Therefore, in the plane $z = 0$ we arrive at the two-dimensional problem of determining $\mathbf{E}^*\{E_x(x',y'),\ E_y(x',y')\}$, i.e., of finding the solution of the equation

$$\frac{\partial^2 U}{\partial x'^2} + \frac{\partial^2 U}{\partial y'^2} = 0 \tag{9.6'}$$

subject to the boundary conditions

$$\sigma \int_0^{2\pi} \left(\frac{\partial U}{\partial r}\right)_{r\to 0} r\, d\theta = \dot{q}, \tag{9.7'}$$

$$\left(\frac{\partial U}{\partial r}\right)_{r=a} = \frac{\dot{q}}{a\sigma}\,\delta(\theta + \varphi). \tag{9.8'}$$

Further, let U_1 be the potential on the disc and U_2 the potential outside the disc. Obviously,

$$(U_1)_{r=a} = (U_2)_{r=a}. \tag{9.10}$$

The solution of equation (9.6′) can be written in the form

$$U_1 = A \ln r + B + \sum_{n=1}^{\infty} (C_n \cos n\theta + C_n' \sin n\theta)\, r^n,$$

$$U_2 = \sum_{n=1}^{\infty} (D_n \cos n\theta + D_n' \sin n\theta)\, r^{-n}.$$

Hence,

$$\frac{\partial U_1}{\partial r} = \frac{A}{r} + \sum_{n=1}^{\infty} n\,(C_n \cos n\theta + C_n' \sin n\theta)\, r^{n-1},$$

$$\frac{\partial U_1}{\partial \theta} = \sum_{n=1}^{\infty} n\,(-C_n \sin n\theta + C_n' \cos n\theta)\, r^n. \tag{9.11}$$

The current density in the disc is $\mathbf{j} = \sigma \mathbf{E}^* = \sigma \operatorname{grad} U_1$. Using (9.11), we find

$$\mathbf{j} = \sigma \left[\frac{A}{r} \mathbf{r}^0 + \sum_{n=1}^{\infty} C_n (\mathbf{r}^0 \cos n\theta - \boldsymbol{\theta}^0 \sin n\theta)\, n r^{n-1} + \sum_{n=1}^{\infty} C_n' (\mathbf{r}^0 \sin n\theta + \boldsymbol{\theta}^0 \cos n\theta)\, n r^{n-1} \right], \tag{9.12}$$

where $\mathbf{r}^0$ and $\boldsymbol{\theta}^0$ are, respectively, radial and transverse unit vectors.

The expression (9.12) is an expansion of the current density $\mathbf{j}$ in a series with respect to an orthogonal system of vector functions

$$\mathbf{S}_0 = \frac{1}{r} \mathbf{r}^0,$$

$$\underset{1}{\mathbf{S}_n} = (\mathbf{r}^0 \cos n\theta - \boldsymbol{\theta}^0 \sin n\,\theta)\, r^{n-1},$$

$$\underset{2}{\mathbf{S}_n} = (\mathbf{r}^0 \sin n\,\theta + \boldsymbol{\theta}^0 \cos n\,\theta)\, r^{n-1},$$

the coefficients

$$\sigma A = \dot{\varkappa}_0, \quad \sigma n C_n = \underset{1}{\dot{\varkappa}_n}, \quad \sigma n C_n' = \underset{2}{\dot{\varkappa}_n}$$

playing the role of generalized velocities. Thus,

$$\mathbf{j} = \dot{\varkappa}_0 \mathbf{S}_0 + \sum_{n=1}^{\infty} (\underset{1}{\dot{\varkappa}_n} \underset{1}{\mathbf{S}_n} + \underset{2}{\dot{\varkappa}_n} \underset{2}{\mathbf{S}_n}). \tag{9.13}$$

In order to find the constraints that arise because of the sliding contact, we shall use the boundary conditions (9.7′) and (9.8′) and the fitting conditions (9.10).

By virtue of the relation (9.7′), we have $A = \dot{q}/2\pi\sigma$. It follows from (9.10) that

$$B = -\frac{\dot{q}}{2\pi\sigma} \ln a, \quad a^{2n} C_n = D_n, \quad a^{2n} C_n' = D_n'.$$

Since

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos nx,$$

equation (9.8′) can be written in the form

$$\frac{\dot{q}}{2\pi\sigma a} + \sum_{n=1}^{\infty} (C_n \cos n\theta + C_n' \sin n\theta)\, n a^{n-1}$$

$$= \frac{\dot{q}}{a\sigma}\left[\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos n(\theta + \varphi)\right].$$

Hence,

$$n a^{n-1} C_n = \frac{\dot{q}}{\pi a \sigma} \cos n\varphi, \quad n a^{n-1} C_n' = -\frac{\dot{q}}{\pi a \sigma} \sin n\varphi.$$

Consequently,

$$\dot{\varkappa}_0 = \frac{1}{2\pi}\dot{q}, \tag{9.14}$$

$$\dot{\varkappa}_{1n} a^n \pi - \dot{q} \cos n\varphi = 0, \tag{9.15}$$

$$\dot{\varkappa}_{2n} a^n \pi + \dot{q} \sin n\varphi = 0. \tag{9.16}$$

Thus, the parameters

$$\varphi, q, \varkappa_0, \varkappa_{1n}, \varkappa_{2n},$$

which can be regarded as the generalized coordinates of our system, are subject to the constraints (9.14) – (9.16), which are differential equations. The constraint (9.14) is integrable. However, equations (9.15) and (9.16) cannot be integrated. These two constraints are therefore nonholonomic.

Equations (9.15) and (9.16) represent a countable set of nonholonomic constraints; as a result, the system, which is essentially distributed, has two degrees of freedom.

Even from these examples, it can be seen that, in studying an electrical machine with a three-dimensional rotor fitted with sliding contacts, one can regard the current density $\mathbf{j}$ as a function of a finite number of independent variables $\dot{q}_1, \cdots, \dot{q}_n$, where n is the number of "electrical" degrees of freedom. In the case of a discrete description, this is reflected by the fact that the corresponding variables $\dot{\varkappa}_r$ are related to the currents $\dot{q}_j$ by equations of the form

$$\dot{\varkappa}_r = \sum_{j=1}^{n} k_{rj}(\varphi)\, \dot{q}_j, \tag{9.17}$$

where φ is the angle of revolution of the rotor. If the system has m mechanical degrees of freedom, the number of variables φ is also increased accordingly. At the same time, equations (9.17) take the form

$$\dot{\varkappa}_r = \sum_{j=1}^{n} k_{rj}(\varphi_1, \varphi_2, \ldots, \varphi_m)\,\dot{q}_j \quad (r = 1, 2, \ldots), \tag{9.18}$$

i.e., they are the equations of nonholonomic constraints. A feature of these equations is the fact that the coefficients k_{rj} on the right-hand sides do not depend on the generalized coordinates, whose derivatives appear on the left-hand sides.

§10. Gaponov's equations of motion of nonholonomic electromechanical systems

In the preceding section, we have shown that nonholonomic constraints which have the form of equations (9.18) are realized in a system containing three-dimensional conductors with sliding contacts. The equations of motion of dynamical systems with such constraints in the case of a finite number of degrees of freedom were first derived by Čaplygin. Thus, electromechanical systems with sliding contacts belong to the class of nonholonomic dynamical systems of Čaplygin. However, because of the fact that Čaplygin's equations of motion contain the coefficients of the nonholonomic constraints, it is difficult to use these equations in the case of a system with a countable set of constraints. This is because Čaplygin's equations contain the original Lagrange function L and also the transformed function $\tilde{L}$, obtained from L by eliminating the variables $\dot{\varkappa}_r$ by means of the equations of the nonholonomic constraints (9.18). As a result, the equations of motion of electrical machines derived in this manner contain a countable set of coefficients of nonholonomic constraints.

In his investigations [**166**, **168**, **169**], Gaponov proposed equations of motion which have the advantage that, for applications to electromechanical systems and, particularly, systems containing three-dimensional conductors with sliding contacts, they only contain a single function L^*, whose "electrical" part may be written down without knowledge of the function L and the equations of the nonholonomic constraints.

We now turn to the derivation of Gaponov's equations. Let $q_1, q_2, \cdots$ be the generalized coordinates of the system and let $L(q_1, q_2, \cdots, q_m; \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_m, \dot{q}_{m+1}, \cdots)$ be its Lagrange function, which does not explicitly contain the coordinates with indices greater than some number $m > 1$. We shall also assume that the system is subjected to a countable set of nonholonomic constraints of the form

$$\dot{q}_{m+l} = \sum_{j=1}^{m} A_{lj}(q_1, q_2, \ldots, q_m)\,\dot{q}_j \qquad (l = 1, 2, \ldots). \tag{10.1}$$

In these equations, we replace the quantities q_i, the arguments of the coefficients A_{lj}, by the independent variables u_{ij} $(i, j = 1, 2, \cdots, m)$. As a result of this substitution, the coefficients A_{lj} are transformed into the functions $A_{lj}(u_{1j}, u_{2j}, \cdots, u_{mj})$. Obviously, the number of additional parameters introduced (which we shall henceforth call "constraint variables") cannot exceed m^2. The system of equations (10.1) now takes the form

$$\dot{q}_{m+l} = \sum_{j=1}^{m} A_{lj}(u_{1j}, u_{2j}, \ldots, u_{mj})\,\dot{q}_j \quad (l = 1, 2, \ldots). \tag{10.2}$$

Let L be the Lagrange function written down without allowance for the nonholonomic constraints (10.1). Let L^* be the function obtained from L by eliminating the generalized velocities $\dot{q}_{m+1}, \dot{q}_{m+2}, \cdots$ by means of equations (10.2). As a result of this transformation, we obtain the following equation

$$\begin{aligned} &L^*(q_1, q_2, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m; u_{11}, \ldots, u_{mm}) \\ &\quad = L\left(q_1, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m, \sum_{j=1}^{m} A_{1j}\dot{q}_j, \ldots, \sum_{j=1}^{m} A_{mj}\dot{q}_j\right). \end{aligned} \tag{10.3}$$

We shall take as our starting point d'Alembert's principle, according to which

$$\sum_{\alpha=1}^{\infty} \left\{ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} \right\} \delta q_\alpha = 0. \tag{10.4}$$

It follows from the relations (10.2) that

$$\delta q_{m+l} = \sum_{j=1}^{m} A_{lj}(u_{1j}, \ldots, u_{mj})\,\delta q_j \qquad (l = 1, 2, \ldots).$$

Substituting these expressions for δq_{m+l} into equation (10.4) and then equating to zero the coefficients of the independent variations, we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \sum_{l=1}^{\infty} A_{lj}\,\frac{d}{dt}\frac{\partial L}{\partial q_{m+l}} = 0. \tag{10.5}$$

If we now introduce the function $\widetilde{L}$ discussed at the beginning of this section into these equations we obtain Čaplygin's equations. In order to obtain Gaponov's equations, we must replace the function L every-

where in equation (10.5) by the function L^*. Differentiating the relation (10.3), we obtain

$$\frac{\partial L^*}{\partial q_j} = \frac{\partial L}{\partial q_j}, \quad \frac{\partial L^*}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \sum_{l=1}^{\infty} A_{lj} \frac{\partial L}{\partial \dot{q}_{m+l}}, \tag{10.6}$$

$$\frac{\partial L^*}{\partial u_{ij}} = \dot{q}_j \sum_{l=1}^{\infty} \frac{\partial L}{\partial \dot{q}_{m+l}} \frac{\partial A_{lj}}{\partial u_{ij}} \quad (i, j = 1, 2, \ldots, m). \tag{10.7}$$

Hence,

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{l=1}^{\infty} A_{lj} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{m+l}} + \sum_{l=1}^{\infty}\sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_{m+l}} \frac{\partial A_{lj}}{\partial u_{ij}} \dot{u}_{ij}. \tag{10.8}$$

The last sum on the right-hand side of this equation can, in accordance with (10.7), be transformed to the following form:

$$\sum_{l=1}^{\infty}\sum_{i=1}^{m} \frac{\partial L}{\partial q_{m+l}} \frac{\partial A_{lj}}{\partial u_{ij}} \dot{u}_{ij} = \sum_{i=1}^{m} \dot{u}_{ij} \left(\sum_{l=1}^{\infty} \frac{\partial L}{\partial \dot{q}_{m+l}} \frac{\partial A_{li}}{\partial u_{ij}} \right) = \sum_{i=1}^{m} \frac{\dot{u}_{ij}}{q_j} \frac{\partial L^*}{\partial u_{ij}}.$$

Substituting this into (10.8), we find

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{l=1}^{\infty} A_{lj} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{m+l}} = \frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_j} - \frac{1}{\dot{q}_j} \sum_{i=1}^{m} \frac{\partial L^*}{\partial u_{ij}} \dot{u}_{ij}.$$

Now, substituting this expression together with the first relation of (10.6) into equations (10.5), we obtain equations which contain only $L^*(q, \dot{q}, u)$:

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_j} - \frac{\partial L^*}{\partial q_j} - \frac{1}{\dot{q}_j} \sum_{i=1}^{m} \frac{\partial L^*}{\partial u_{ij}} \dot{u}_{ij} = 0 \quad (j = 1, 2, \ldots, m). \tag{10.9}$$

Carrying out the differentiation in these equations and then making the substitution $u_{ij} = q_i$ $(i = 1, 2, \cdots, m)$, we obtain the final equations of motion containing $q_1, q_2, \cdots, q_m, \dot{q}_1, \cdots, \dot{q}_m$. Equations (10.9) with a slightly different notation were derived in 1953 by Gaponov.

We should mention that if no holonomic constraint is imposed on the coordinate q_α, equation (10.9) for the index $j = \alpha$ has the usual form of Lagrange equations of the second kind, since, in this case, the parameters $u_{i\alpha}$ corresponding to the coordinate q_α do not occur in the function L^*.

As an illustration of the use of equations (10.9), let us find the equations of free rolling (and pivoting) of a homogeneous sphere on a rough horizontal plane. We shall first use Gaponov's equations and then, for comparison, obtain the same equations by the methods of Čaplygin

and Appell.

In accordance with the notation of Figure 7.15, the Cartesian coordinates x and y of the projection of the center of the sphere onto the horizontal plane can be taken as the generalized coordinates of the sphere together with the Eulerian angles t, φ, and ψ.

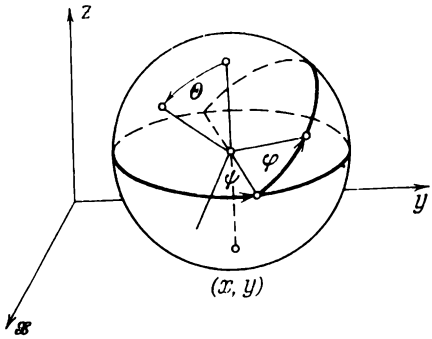

FIGURE 7.15

We first write down the Lagrange function L without allowance for the nonholonomic constraints

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta). \tag{10.10}$$

Here, J is the moment of inertia of the sphere about an axis passing through its center. The condition of rolling of the sphere without sliding leads to the two nonintegrable equations

$$\begin{aligned} \dot{x} &= a\dot{\theta}\sin\psi - a\dot{\varphi}\sin\theta\cos\psi, \\ \dot{y} &= -(a\dot{\theta}\cos\psi - a\dot{\varphi}\sin\theta\sin\psi). \end{aligned} \tag{10.11}$$

Let the coordinates θ, φ, and ψ be the independent parameters.

a) *Gaponov's equations.* In the equations of the nonholonomic constraints (10.11), we introduce the constraint variables ψ_θ, ψ_φ, and θ_φ so that

$$\begin{aligned} \dot{x} &= a\dot{\theta}\sin\psi_\theta - a\dot{\varphi}\sin\theta_\varphi\cos\psi_\varphi, \\ \dot{y} &= -a\dot{\theta}\cos\psi_\theta - a\dot{\varphi}\sin\theta_\varphi\sin\psi_\varphi. \end{aligned} \tag{10.12}$$

This notation (rather than u_{ij}) is convenient in that in order to derive the equations of motion it will be sufficient to drop the subscripts in the final Gaponov equations and thus return to the variables θ, φ, and ψ. Substituting (10.12) into (10.10), we obtain the function L^* in the form

$$\begin{aligned} L^* = \frac{ma^2}{2}[\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta_\varphi + 2\dot{\varphi}\dot{\theta}\sin\theta_\varphi\sin(\psi_\varphi - \psi_\theta)] \\ + \frac{1}{2}J(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta). \end{aligned}$$

We now write down equations (10.9), in which we have $q_1 = \theta$, $q_2 = \psi$, $q_3 = \varphi$, $u_{11} = \psi_\theta$, $u_{13} = \theta_\varphi$, $u_{23} = \psi_\varphi$, and $u_{ij} = 0$ for the remaining values

of i and j. We now make the necessary calculations.

For the coordinate θ:

$$\frac{\partial L^*}{\partial \dot{\theta}} = ma^2 [\dot{\theta} + \dot{\varphi} \sin \theta_\varphi \sin (\psi_\varphi - \psi_\theta)] + J\dot{\theta},$$

$$\frac{\partial L^*}{\partial \theta} = -J\dot{\varphi}\dot{\psi} \sin \theta, \quad \frac{\partial L^*}{\partial \psi_\theta} = -ma^2 \dot{\varphi}\dot{\theta} \sin \theta_\varphi \cos (\psi_\varphi - \psi_\theta).$$

Hence, dropping the subscripts, we obtain

$$(J + ma^2)\, \ddot{\theta} + J\dot{\varphi}\dot{\psi} \sin \theta + ma^2 \dot{\varphi}\dot{\psi} \sin \theta = 0 \tag{10.13}$$

or

$$\ddot{\theta} + \dot{\varphi}\dot{\psi} \sin \theta = 0.$$

For the coordinate ψ:

$$\frac{\partial L^*}{\partial \dot{\psi}} = J (\dot{\psi} + \dot{\varphi} \cos \theta), \quad \frac{\partial L^*}{\partial \psi} = 0.$$

Hence

$$\frac{d}{dt} J (\dot{\psi} + \dot{\varphi} \cos \theta) = 0$$

or

$$\psi + \varphi \cos \theta = \text{const}, \tag{10.14}$$

i.e., we obtain an integral of the motion: the conservation of the vertical projection of the angular momentum.

For the coordinate φ:

$$\frac{\partial L^*}{\partial \dot{\varphi}} = ma^2 [\dot{\varphi} \sin^2 \theta_\varphi + \dot{\theta} \sin \theta_\varphi \sin (\psi_\varphi - \psi_\theta)] + J (\dot{\varphi} + \dot{\psi} \cos \theta),$$

$$\frac{\partial L^*}{\partial \varphi} = 0,$$

$$\frac{\partial L^*}{\partial \theta_\varphi} = ma^2 [\dot{\varphi}^2 \sin \theta_\varphi \cos \theta_\varphi + 2\dot{\varphi}\dot{\theta} \cos \theta_\varphi \sin (\psi_\varphi - \psi_\theta)],$$

$$\frac{\partial L^*}{\partial \psi_\varphi} = ma^2 \dot{\varphi}\, \dot{\theta} \sin \theta_\varphi \cos (\psi_\varphi - \psi_\theta).$$

Hence, dropping the subscripts, we obtain

$$(J + ma^2 \sin^2 \theta)\, \ddot{\varphi} + J\ddot{\psi} \cos \theta + ma^2 \dot{\varphi}\dot{\theta} \sin \theta \cos \theta - (J + ma^2)\, \dot{\theta} \sin \theta = 0. \tag{10.15}$$

b) *Derivation of the equations by Čaplygin's method.* In accordance with the notation adopted in Čaplygin's investigation of the problem

of the sphere, we have

$$q_1 = x,\ q_2 = y,\ q_3 = \theta,\ q_4 = \psi,\ q_5 = \varphi,$$
$$B_3^{(1)} = a\sin\psi,\ B_4^{(1)} = 0,\ B_5^{(1)} = -a\sin\theta\cos\psi,$$
$$B_3^{(2)} = -a\sin\psi,\ B_4^{(2)} = 0,\ B_5^{(2)} = -a\sin\theta\sin\psi.$$

The function $\widetilde{L}$ obtained from L by eliminating $\dot{x}$ and $\dot{y}$ by means of the equations of the nonholonomic constraints is

$$\widetilde{L} = \tfrac{1}{2}(J + ma^2)\dot{\theta}^2 + \tfrac{1}{2}(J + ma^2\sin^2\theta)\dot{\varphi}^2 + \tfrac{1}{2}J\dot{\psi}^2 + J\dot{\varphi}\dot{\psi}\cos\theta.$$

Čaplygin's equations are

$$\frac{d}{dt}\frac{\partial\widetilde{L}}{\partial\dot{q}_s} - \frac{\partial\widetilde{L}}{\partial q_s} + \sum_{i=1}^{2}\frac{\partial L}{\partial\dot{q}_i}\left[\sum_{k=3}^{5}\left(\frac{\partial B_k^{(i)}}{\partial q_s} - \frac{\partial B_s^{(i)}}{\partial q_k}\right)\dot{q}_k\right] = 0 \quad (s = 3, 4, 5).$$

We now make the necessary calculations.

For the coordinate $q_3 = \theta$:

$$\frac{\partial\widetilde{L}}{\partial\dot{\theta}} = (J + ma^2)\,\dot{\theta},\ \frac{\partial\widetilde{L}}{\partial\theta} = ma^2\dot{\varphi}\sin\theta\cos\theta - J\dot{\varphi}\dot{\psi}\sin\theta,$$
$$\sum_{i=1}^{2}\frac{\partial L}{\partial\dot{q}_i}\left\{\sum_{k=3}^{5}\left(\frac{\partial B_k^{(i)}}{\partial q_3} - \frac{\partial B_3^{(i)}}{\partial q_k}\right)\dot{q}_k\right\} = ma^2\dot{\varphi}\,(\psi + \dot{\varphi}\cos\theta)\sin\theta.$$

Hence

$$(J + ma^2)\,\ddot{\theta} + J\dot{\varphi}\dot{\psi}\sin\theta + ma^2\dot{\varphi}\dot{\psi}\sin\theta = 0$$

or

$$\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta = 0. \tag{10.13′}$$

For the coordinate $q_4 = \psi$:

$$\frac{\partial\widetilde{L}}{\partial\psi} = J\,(\psi + \dot{\varphi}\cos\theta),\ \frac{\partial\widetilde{L}}{\partial\psi} = 0,$$
$$\sum_{i=1}^{2}\frac{\partial L}{\partial\dot{q}_i}\left\{\sum_{k=3}^{5}\left(\frac{\partial B_k^{(i)}}{\partial q_4} - \frac{\partial B_4^{(i)}}{\partial q_k}\right)\dot{q}_k\right\} = ma^2\,(\dot{\theta}\dot{\varphi}\sin\theta - \dot{\theta}\dot{\varphi}\sin\theta) = 0.$$

Hence

$$\frac{d}{dt}(\psi + \dot{\varphi}\cos\theta) = 0$$

or

$$\dot{\varphi} + \dot{\psi}\cos\theta = \text{const}. \tag{10.14′}$$

For the coordinate $q_5 = \varphi$:

$$\frac{\partial \widetilde{L}}{\partial \dot{\varphi}} = (J + ma^2 \sin^2\theta)\,\dot{\varphi} + J\dot{\psi}\cos\theta, \quad \frac{\partial \widetilde{L}}{\partial \varphi} = 0,$$

$$\sum_{i=1}^{2} \frac{\partial L}{\partial \dot{q}_i} \left\{ \sum_{k=3}^{5} \left(\frac{\partial B_k^{(i)}}{\partial q_5} - \frac{\partial B_5^{(i)}}{\partial q_k} \right) \dot{q}_k \right\} = ma^2 (-\dot{\theta}\dot{\psi}\sin\theta - \dot{\varphi}\dot{\theta}\cos\theta).$$

Hence

$$(J + ma^2 \sin^2\theta)\,\ddot{\varphi} + J\ddot{\psi}\cos\theta + ma^2\,\dot{\theta}\dot{\varphi}\sin\theta\cos\theta - (J + ma^2)\,\ddot{\theta}\dot{\psi}\sin\theta = 0. \tag{10.15'}$$

c) *Derivation of the equations by Appell's method.* We first find the expression for the acceleration energy S:

$$S = \frac{m}{2}(\ddot{x}^2 + \ddot{y}^2) + \frac{1}{2} J (\omega_x^2 + \omega_y^2 + \dot{\omega}_z^2)$$

$$= \frac{1}{2}(J + ma^2)(\ddot{\theta}^2 + 2\ddot{\theta}\dot{\varphi}\dot{\psi}\sin\theta - 2\dot{\varphi}\dot{\theta}\dot{\psi}\sin\theta)$$

$$+ \frac{ma^2}{2}(\dot{\varphi}^2\sin^2\theta + 2\dot{\varphi}\dot{\varphi}\dot{\theta}\sin\theta\cos\theta)$$

$$+ \frac{1}{2} J (\ddot{\varphi}^2 + \ddot{\psi}^2 + 2\ddot{\psi}\ddot{\varphi}\cos\theta - 2\ddot{\psi}\dot{\varphi}\dot{\theta}\sin\theta).$$

Appell's equations for the free motion of the system can be written in the form

$$\frac{\partial S}{\partial \ddot{q}_k} = 0 \qquad (k = 1, 2, 3;\ q_1 = \theta,\ q_2 = \psi,\ q_3 = \varphi).$$

Carrying out the differentiation of the function S, we obtain in our case the following equations of motion:

$$\frac{\partial S}{\partial \ddot{\theta}} = (J + ma^2)(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta) = 0, \tag{10.13''}$$

$$\frac{\partial S}{\partial \ddot{\psi}} = J(\ddot{\psi} + \dot{\varphi}\cos\theta - \dot{\varphi}\theta\sin\theta) = 0$$

or

$$\psi + \dot{\varphi}\cos\theta = \text{const},$$

$$\frac{\partial S}{\partial \ddot{\varphi}} = -(J + ma^2)\,\dot{\theta}\dot{\psi}\sin\theta + ma^2(\ddot{\varphi}\sin^2\theta + \dot{\varphi}\dot{\theta}\sin\theta\cos\theta) + J\ddot{\psi}\cos\theta + \varphi \tag{10.14''}$$

or

$$(J + ma^2\sin\theta)\,\ddot{\varphi} + J\ddot{\psi}\cos\theta + ma^2\,\dot{\varphi}\dot{\theta}\sin\theta\cos\theta - (J + ma^2)\,\dot{\theta}\dot{\psi}\sin\theta = 0. \tag{10.15''}$$

It can be seen immediately that equations (10.9) contain only the function L^* and do not contain the coefficients of the equations of the

nonholonomic constraints. Therefore, in order to write down the equations of motion, one need only know the expression $L^*(q, \dot{q}, u)$. For applications to electromechanical systems, for which, obviously,

$$L^* = L^*_{\text{mech}} + L^*_{\text{el}},$$

this leads to a considerable simplification because the expression for the function L^*_{el} can be written down directly without prior knowledge of the expression for L_{el} and the equations of the nonholonomic constraints.

First of all we note that in the constraint equations (9.18) the coefficients k_{rj} are, in general, determined by the parameters of the system and, in particular, depend on the coordinates γ_{ij} of the sliding contacts through which the current $\dot{q}_j$ is supplied to the given three-dimensional conductor (γ_{ij} are measured in a system of coordinates rigidly connected to the given three-dimensional conductor). In electrical machines, the quantities γ_{ij} for the rotor can be expressed in terms of the angle φ_i of revolution of the rotor[16] and the coordinate β_{ij} of the brush (β_{ij} is measured in a fixed coordinate system)

$$\gamma_{ij} = \beta_{ij} - \varphi_i \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n), \tag{10.16}$$

where m is the number of rotors of the electrical machines making up the given electromechanical system and n is the number of circuits or "electrical" degrees of freedom of the system. The quantities γ_{ij}, together with the φ_i, can be regarded as the coordinates of the system. We now use the relation (10.16) to return from the φ_j to the γ_{ij} in the coefficients of the constraint (9.18), and, replacing these coordinates by the constraint variables u_{ij} in accordance with the general rule, we obtain the function $L^*(\varphi_i, \dot{\varphi}_i, u_{ij}, \dot{q}_j)$, and the equations of motion can be written in the form (10.9). In what follows, we shall, in order to avoid superfluous notation, regard the quantities γ_{ij} themselves as the constraint variables. Then L^* is a function of the γ_{ij}:

$$L^* = L^*(\varphi, \dot{\varphi}, \dot{q}, \gamma) = L^*_{\text{el}}(\dot{q}, \gamma) + L^*_{\text{mech}}(\varphi, \dot{\varphi}). \tag{10.17}$$

Assuming that nonconservative and external forces act in the system, we obtain, in accordance with (10.9), the following equations of motion of the electromechanical system (the coordinates q_j are cyclic):

---

[16] It is for this reason that the coefficients k_{ij} depend on the angles φ_i, as was noted in the expression (9.18).

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_j} - \frac{1}{\dot{q}_j}\sum_{i=1}^{m_j}\frac{\partial L^*}{\partial \gamma_{ij}}\dot{\gamma}_{ij} = \mathscr{E}_j - \sum_{k=1}^{n} R_{kj}\dot{q}_k,$$

(10.18)

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{\varphi}_i} - \frac{\partial L^*}{\partial \varphi_i} = P_i,$$

where m_i is the number of brushes encountered in a passage once round the ith circuit.

We shall now show that the expression for the function $L^*_{el}(\dot{q}, \gamma)$ can be written down directly without prior knowledge of the expression for L_{el} and the equations of the nonholonomic constraints.

We shall take as our starting point the expression (3.18′) for the Lagrange function:

$$L_{el} = \int_V \left[\int_0^{\mathbf{j}} \mathbf{A}(\mathbf{j})\, d\mathbf{j}\right] dv, \tag{10.19}$$

which we transform by using the representation for the current density in three-dimensional conductors in terms of the currents $\dot{q}_j$ and the distribution function $\mathbf{S}_j(\gamma_{ij})$ of the currents associated with the jth circuit,

$$\mathbf{j} = \sum_{j=1}^{n} \dot{q}_j \mathbf{S}_j(\gamma_{il}). \tag{10.20}$$

In accordance with (10.19) and (10.20), we obtain

$$dL_{el} = \int_V [\mathbf{A}(\mathbf{j})\, d\mathbf{j}]\, dv,$$

$$d\mathbf{j} = \sum_{j=1}^{n} \mathbf{S}_j(\gamma_{il})\, d\dot{q}_j + \sum_{i=1}^{m}\sum_{j=1}^{n} \dot{q}_l \frac{\partial \mathbf{S}_j}{\partial \gamma_{ij}} d\gamma_{ij}.$$

Substituting this expression into the preceding expression, we obtain the differential of the transformed function L^*_{el}:

$$dL^*_{el} = \sum_{j=1}^{n} \Phi_j d\dot{q}_j + \sum_{i=1}^{m}\sum_{j=1}^{n} \dot{q}_l \Psi_{ij} d\gamma_{ij}, \tag{10.21}$$

where

$$\Phi_l(\dot{q}, \gamma) = \int_V \mathbf{A}(\dot{q}, \gamma)\, \mathbf{S}_l(\gamma_{ij})\, dv,$$

$$\Psi_{il}(\dot{q}, \gamma) = \int_V \mathbf{A}(\dot{q}, \gamma) \frac{\partial \mathbf{S}_j}{\partial \gamma_{il}}\, dv.$$

In order to obtain an expression for L_{el}^*, we integrate dL_{el}^* in the space $(\dot{q}, \gamma)$ from the origin to the point $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, \gamma_{11}, \cdots, \gamma_{mn}$ along a path composed of segments taken along the coordinate lines (Figure 7.16) so that we first integrate along the coordinates γ_{ij} with $\dot{q}_1 = \cdots = \dot{q}_n = 0$ and then continue along $\dot{q}_1$, $\dot{q}_2$, etc. to $\dot{q}_n$. It is easily seen that we obtain zero as a result of the integration along γ_{ij} (because of the factor $\dot{q}_j$ occurring in the double sum of the expression (10.21)), and that the integration on the remaining section of the path yields the following expression for the function L_{el}^*:

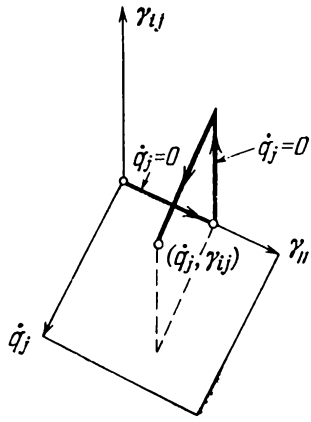

FIGURE 7.16

$$L_{el}^*(\dot{q}, \gamma) = \sum_{j=1}^{n} \int_0^{\dot{q}_j} \Phi_j(\dot{q}_1, \ldots, \dot{q}_j, 0, \ldots, 0, \ldots, \gamma_{ij}, \ldots)\, d\dot{q}_j, \tag{10.22}$$

where the integrand Φ_j is the flux of magnetic induction associated with the jth circuit.

In view of applications to commutator machines with fixed brushes, it is more convenient in the theory of electrical machines to use the angular coefficients of the brushes β_{ij} instead of the quantities γ_{ij}.

Introducing the quantities β_{ij}, which are connected to the γ_{ij} by the equations (10.16), into the function L_{el}^*, we obtain a new function $\overline{L}_{el}(\dot{q}, \beta)$. Thus, the complete function $\overline{L} = \overline{L}_{el} + L_{mech}$ can be expressed in terms of the function L^* as follows:

(10.23)

in which we can assume that the substitution $\varphi_{ij} = \beta_{ij} - \varphi_i$ has already been made in the integrand of (10.22). It follows from (10.23) that

$$\frac{\partial L^*}{\partial \dot{q}_j} = \frac{\partial \overline{L}}{\partial \dot{q}_j}, \quad \frac{\partial L^*}{\partial \gamma_{ij}} = \frac{\partial \overline{L}}{\partial \beta_{ij}}, \quad \frac{\partial L^*}{\partial \dot{\varphi}_i} = \frac{\partial \overline{L}}{\partial \dot{\varphi}_i},$$

$$\frac{\partial L^*}{\partial \varphi_i} - \sum_{j=1}^{n_i} \frac{\partial L^*}{\partial \gamma_{ij}} = \frac{\partial \overline{L}}{\partial \varphi_i}$$

or

$$\frac{\partial L^*}{\partial \varphi_i} = \frac{\partial \overline{L}}{\partial \varphi_i} + \sum_{j=1}^{n_i} \frac{\partial L}{\partial \beta_{ij}},$$

where n_i is the number of brushes that touch the commutator of the ith rotor. Substituting these relations into (10.18), we can transform the equations of motion to the form

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{q}_j} - \frac{1}{\dot{q}_j}\sum_{i=1}^{m_j}\frac{\partial \overline{L}}{\partial \beta_{ij}}(\dot{\beta}_{ij} - \dot{\varphi}_i) = \mathscr{E}_j - \sum_{k=1}^{n} R_{kj}\dot{q}_k \quad (j = 1, 2, \ldots, n),$$

(10.24)

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{\varphi}_i} - \frac{\partial \overline{L}}{\partial \varphi_i} - \sum_{j=1}^{n_i}\frac{\partial \overline{L}}{\partial \beta_{ij}} = P_i \quad (i = 1, 2, \ldots, m),$$

in which we must set $\dot{\beta}_{ij} = 0$ for commutator machines with fixed brushes.

Finally, we note that a uniform winding of the rotor can be regarded, relative to a pair of brushes of the jth circuit,[17] as a coil for which the direction of the magnetic axis is characterized by the single angle α_{ij}. The magnitude of the angle α_{ij} is obviously related to the angles β'_{ij} and β''_{ij} of the corresponding pair of brushes by a simple dependence, namely

$$\alpha_{ij} = \frac{1}{2}(\beta'_{ij} + \beta''_{ij}) + \frac{\pi}{2}.$$

In this case, $\overline{L}$ is a function of the α_{ij} calculated in accordance with the general rules from the formula

$$\overline{L}(\varphi, \dot{\varphi}, \dot{q}, \alpha) = \sum_{j=1}^{n}\int_0^{\dot{q}_j}\Phi_j(\varphi, \dot{q}_1, \ldots, \dot{q}_j, 0, \ldots, 0, \alpha)\,dq_j + L_{\text{mech}}, \tag{10.25}$$

and the equations of motion (10.24) take the following final form

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{q}_j} + \frac{1}{\dot{q}_j}\sum_{i=1}^{m_j}\frac{\partial \overline{L}}{\partial \alpha_{ij}}\dot{\varphi}_i = \mathscr{E}_j - \sum_{k=1}^{n} R_{kj}\dot{q}_k \quad (j = 1, 2, \ldots, n),$$

(10.26)

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{\varphi}_i} - \frac{\partial \overline{L}}{\partial \varphi_i} - \sum_{j=1}^{n_i}\frac{\partial \overline{L}}{\partial \alpha_{ij}} = P_i \quad (i = 1, 2, \ldots, m).$$

Here, n is the number of "electrical" degrees of freedom (the number of electrical circuits); m is the number of mechanical degrees of freedom (the number of rotors of the electrical machines making up the electromechanical system); n_i is the number of equivalent coils on the ith rotor

[17] If there is an odd number of brushes on the commutator, each brush should be regarded as split into two (or more) parts and these parts should be considered as belonging to a corresponding circuit with current $\dot{q}_j$.

(the number of circuits associated with the ith rotor); and m_j is the number of rotors with which the jth circuit is connected.

Equations (10.24) are the fundamental equations of the dynamical theory of electrical machines, and, in the form (10.26), they are also the equations of the theory of commutator electrical machines with fixed brushes. In fact, equations (10.24) can be applied not only to commutator machines but also to electrical machines without commutators, since $\dot{\beta} = \dot{\varphi}$ for the latter and equations (10.24) are identical with the ordinary Lagrange equations.

To conclude this section, we shall consider some examples of the application of equations (10.26) to commutator electrical machines of various type.

EXAMPLE 10. SERIES EXCITATION GENERATOR. Figure 7.17 is a schematic representation of a series generator with load resistance R. In accordance with (10.25), the function $\overline{L}$ for this machine is

$$\overline{L} = \int_0^{\dot{q}} \Phi(\dot{q}, \alpha)\, d\dot{q} + \frac{1}{2} J\dot{\varphi}^2,$$

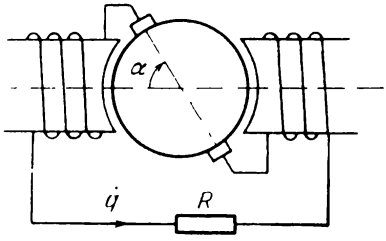

FIGURE 7.17

where $\Phi(\dot{q}, \alpha)$ is the flux of magnetic induction associated with the current $\dot{q}$ (this function is a characteristic of the machine and its dependence on $\dot{q}$ and α can be determined experimentally) and J is the moment of inertia of the rotor. The equation of motion (10.26) is

$$\begin{gathered} \frac{d}{dt}\Phi(\dot{q}, \alpha) + \frac{\dot{\varphi}}{\dot{q}} \int_0^{\dot{q}} \frac{\partial \Phi(\dot{q}, \alpha)}{\partial \alpha}\, d\dot{q} + R\dot{q} = 0, \\ J\ddot{\varphi} - \int_0^{\dot{q}} \frac{\partial \Phi(\dot{q}, \alpha)}{\partial \alpha}\, d\dot{q} = P_\varphi. \end{gathered} \tag{10.27}$$

If the characteristic of the generator for small currents is approximated by a linear dependence $\Phi = L(\alpha)\dot{q}$, the equations (10.27) become

$$\begin{gathered} L(\alpha)\ddot{q} + R\dot{q} + \frac{1}{2}\frac{\partial L(\alpha)}{\partial \alpha}\dot{q}\dot{\varphi} = 0, \\ J\ddot{\varphi} - \frac{1}{2}\frac{\partial L(\alpha)}{\partial \alpha}\dot{q}^2 = P_\varphi. \end{gathered} \tag{10.28}$$

The conditions of selfexcitation of the generator can be easily obtained

from the first equation of (10.28):

$$R + \frac{1}{2}\frac{\partial L(\alpha)}{\partial \alpha}\dot{\varphi} < 0.$$

For steady-state conditions $\dot{\varphi}_0$ and $\dot{q}_0$, equations (10.27) yield

$$\frac{\dot{\varphi}_0}{\dot{q}_0}\int_0^{\dot{q}_0}\frac{\partial \Phi(\dot{q}, \alpha)}{\partial \alpha}d\dot{q} + R\dot{q}_0 = 0, \quad -\int_0^{\dot{q}_0}\frac{\partial \Phi(\dot{q}, \alpha)}{\partial \alpha}d\dot{q} = P_\varphi.$$

The first of these equations gives the dependence of the steady-state current $\dot{q}_0$ on the angular velocity $\dot{\varphi}_0$ of rotation of the rotor and the value of the load resistance, and the second equation determines the value of the mechanical moment P_φ which must be applied to the drum of the generator to achieve the velocity $\dot{\varphi}_0$ and the current $\dot{q}_0$.

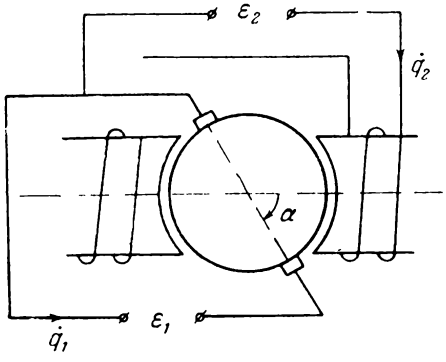

FIGURE 7.18

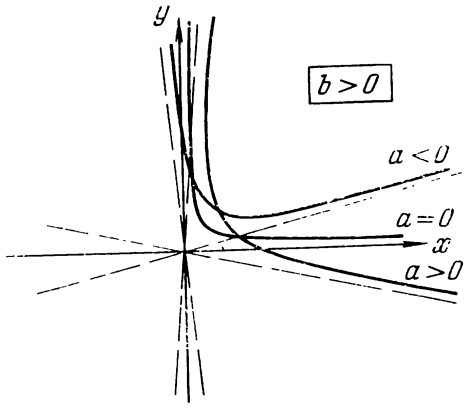

FIGURE 7.19

EXAMPLE 11. COMMUTATOR MOTOR WITH INDEPENDENT EXCITATION. In accordance with the notation of Figure 7.18, the function $\overline{L}$ of the system has the form

$$\overline{L} = \frac{1}{2}L_1(\alpha)\dot{q}_1^2 + M(\alpha)\dot{q}_1\dot{q}_2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2}J\dot{\varphi}^2.$$

Here, we have assumed that the chosen rotor has a uniform winding and, in addition, there is no hysteresis or nonlinearity when the armature and poles are magnetized.

We write down equations (10.26):

$$\begin{aligned} L_1\ddot{q}_1 + M\ddot{q}_2 + \left(\frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{q}_1 + \frac{\partial M}{\partial \alpha}\dot{q}_2\right)\dot{\varphi} + R_1\dot{q}_1 &= \mathcal{E}_1, \\ L_2\ddot{q}_2 + M\ddot{q}_1 + R_2\dot{q}_2 &= \mathcal{E}_2, \\ J\ddot{\varphi} - \frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{q}_1^2 - \frac{\partial M}{\partial \alpha}\dot{q}_1\dot{q}_2 &= P_\varphi. \end{aligned} \tag{10.29}$$

Introducing the notation

$$\dot{q}_1 = x,\ \dot{q}_2 = y,\ \dot{\varphi} = z,\ \frac{1}{2}\frac{\partial L_1}{\partial \alpha} = a,\ \frac{\partial M}{\partial \alpha} = b,\ \frac{1}{L_1L_2 - M^2} = c > 0,$$

we can rewrite this system of nonlinear differential equations in the form

$$\dot{x} = c\,(MR_2y - R_1L_2x - aL_2xz - bL_2yz + L_2\mathscr{E}_1 - M\mathscr{E}_2),$$
$$\dot{y} = c\,(MR_1x - L_1R_2y + aMxz + bMyz - M\mathscr{E}_1 + L_1\mathscr{E}_2),$$
$$\dot{z} = \frac{1}{J}(ax^2 + bxy + P_\varphi).$$

Let us study the running of the motor in the case $\mathscr{E}_1 = \text{const}$, $\mathscr{E}_2 = \text{const}$ and $P_\varphi = \text{const}$. We define

$$L_2\mathscr{E}_1 - M\mathscr{E}_2 = A,\ L_1\mathscr{E}_2 - M\mathscr{E}_1 = B.$$

In the phase space (x, y, z), the position of the critical points is determined by the system of nonlinear equations:

$$MR_2y - R_1L_2x - L_2z\,(ax + by) + A = 0, \tag{10.30}$$

$$MR_1x - R_2L_1y + Mz\,(ax + by) + B = 0, \tag{10.31}$$

$$x\,(ax + by) + P_\varphi = 0. \tag{10.32}$$

Eliminating z from the first two equations, we find

$$MR_1\,(L_2 - L_1)\,x - R_2\,(L_1L_2 - M^2)\,y + MA + L_2B = 0. \tag{10.33}$$

Equations (10.32) and (10.33) determine the coordinates x_0 and y_0 of the critical points. Since (10.32) is the equation of a hyperbola ($a \gtrless 0$, $b > 0$, $P_\varphi > 0$), whose position on the (x, y) plane is shown in Figure 7.19, the straight line (10.33) can intersect this hyperbola in the first quadrant of the (x, y) plane for $L_2 > L_1$ and $M > 0$ only at a single point. In the phase space (x, y, z) there can therefore, in this case, exist only a single critical point (x_0, y_0, z_0), reflecting the existence of a steady-state regime.

For the steady-state regime,

$$x_0 = \dot{q}_{10} = \frac{\mathscr{E}_1 - \dot{\varphi}_0\frac{\partial M}{\partial \alpha}\frac{\mathscr{E}_2}{R_2}}{\frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{\varphi}_0 + R_1}, \qquad y_0 = \dot{q}_{20} = \frac{\mathscr{E}_2}{R_2}.$$

The electromechanical moment developed by the motor is given by

$$P_{em} = -P_\varphi = \frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{q}_{10}^2 + \frac{\partial M}{\partial \alpha}\dot{q}_{10}\dot{q}_{20}.$$

In these expressions, the quantity $\dot{\varphi}_0(\partial M/\partial\alpha)\dot{q}_{20}$ is the counter emf that arises in the winding of the rotor when the latter revolves.

EXAMPLE 12. REPULSION MOTOR (FIGURE 7.20). This electrical machine differs in the dynamical respect from an ordinary motor of independent excitation only by the fact that the winding of its armature is shorted through the brushes. Therefore, the equations of motion of a repulsion motor can be obtained from (10.29) by setting $\mathscr{E}_2 = \mathscr{E}$ and $\mathscr{E}_1 = 0$:

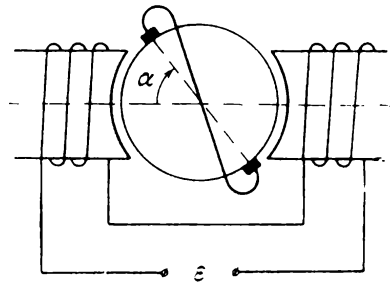

FIGURE 7.20

$$
\begin{aligned}
L_1\ddot{q}_1 + M\ddot{q}_2 + \left(\frac{1}{2}\frac{\partial L_1}{\partial\alpha}\dot{q}_1 + \frac{\partial M}{\partial\alpha}\dot{q}_2\right)\dot{\varphi} + R_1\dot{q}_1 &= 0,\\
L_2\ddot{q}_2 + M\ddot{q}_1 + R_2\dot{q}_2 &= \mathscr{E},\\
J\ddot{\varphi} - \frac{1}{2}\frac{\partial L_1}{\partial\alpha}\dot{q}_1^2 - \frac{\partial M}{\partial\alpha}\dot{q}_1\dot{q}_2 &= P_\varphi.
\end{aligned}
\tag{10.34}
$$

A repulsion motor is driven by an emf that depends sinusoidally on the time: $\mathscr{E} = \mathscr{E}_0 e^{i\omega t}$. Suppose that the period of the emf is small compared with the time constant of the armature and the winding. In order to investigate the transient conditions in this case, the "electrical" equations of the system (10.34) can be integrated independently of the "mechanical" equation, the quantity $\dot{\varphi}$ being regarded as a constant. The equation for φ, in which the currents $\dot{q}_1$ and $\dot{q}_2$ are then known functions of the velocity $\dot{\varphi}$ and the time t, can then be integrated. Since, by hypothesis, $\dot{\varphi}$ is a slowly varying function of the time, an approximate integral of the latter equation can be obtained by replacing the terms $\dot{q}_1^2$ and $\dot{q}_1\dot{q}_2$ by their average values over a period.

If these assumptions are made, we find the following expressions for the currents $\dot{q}_1$ and $\dot{q}_2$ for the steady-state regime:

$$
\dot{q}_1 = \frac{\mathscr{E}_0}{\Delta}\left(\dot{\varphi}\frac{\partial M}{\partial\alpha} + i\omega M\right)e^{i\omega t}, \quad \dot{q}_2 = -\frac{\mathscr{E}_0}{\Delta}\left(\frac{1}{2}\dot{\varphi}\frac{\partial L_1}{\partial\alpha} + R_1 + i\omega L_1\right)e^{i\omega t},
$$

where

$$
\begin{aligned}
\Delta = \omega^2(L_1L_2 - M^2) - R_2\left(\frac{1}{2}\dot{\varphi}\frac{\partial L_1}{\partial\alpha} + R_1\right)\\
+ i\omega\left[\left(M\frac{\partial M}{\partial\alpha} - \frac{1}{2}L_2\frac{\partial L_1}{\partial\alpha}\right)\dot{\varphi} - R_2L_1 - R_1L_2\right].
\end{aligned}
$$

The electromechanical moment $\dot{P}_\varphi$ can be found by averaging the left-hand side of the third equation of (10.34) over a period $T = 2\pi/\omega$ after the expressions found for $\dot{q}_1$ and $\dot{q}_2$ have been substituted into the left-hand side.

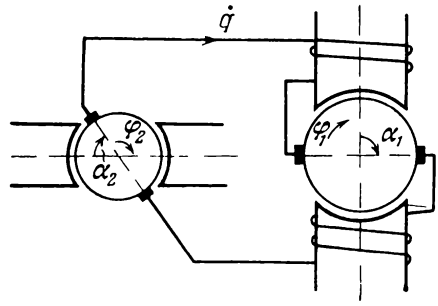

FIGURE 7.21

EXAMPLE 13. CONNECTION IN SERIES OF TWO SERIES MACHINES. Let us write down the dynamical equations for the processes in the system shown in Figure 7.21.

In the linear idealization, the Lagrange function $\overline{L}$ is

$$\overline{L} = \frac{1}{2}[L_1(\alpha_1) + L_2(\alpha_2)]\,\dot{q}^2 + \frac{1}{2}J_1\dot{\varphi}_1^2 + \frac{1}{2}J_2\dot{\varphi}_2^2.$$

The equations of motion are

$$(L_1 + L_2)\ddot{q} + \frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{\varphi}_1\dot{q} + \frac{1}{2}\frac{\partial L_2}{\partial \alpha}\dot{\varphi}_2\dot{q} + R\dot{q} = 0,$$

$$J_1\ddot{\varphi}_1 - \frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{q}^2 = K_1 - h_1\dot{\varphi}_1,$$

$$J_2\ddot{\varphi}_2 - \frac{1}{2}\frac{\partial L_2}{\partial \alpha}\dot{q}^2 = K_2 - h_2\dot{\varphi}_2.$$

Here, J_1 and J_2 are the moments of inertia of the rotors; K_1 and K_2 are the moments of the external mechanical forces; and h_1 and h_2 are the coefficients of viscous friction. Under steady-state conditions

$$\dot{q}\left(\frac{\partial L_1}{\partial \alpha}\dot{\varphi}_1 + \frac{\partial L_2}{\partial \alpha}\dot{\varphi}_2 + 2R\right) = 0,$$

$$h_1\dot{\varphi}_1 - \frac{1}{2}\frac{\partial L_1}{\partial \alpha}\dot{q}^2 = K_1, \tag{10.35}$$

$$h_2\dot{\varphi}_2 - \frac{1}{2}\frac{\partial L_2}{\partial \alpha}\dot{q}^2 = K_2.$$

This system of equations has two solutions: In the first case ($\dot{q} = 0$, $\dot{\varphi}_1 = K_1/h_1$, $\dot{\varphi}_2 = K_2/h_2$), there is no electrical connection in the system; in the other, more interesting case ($\dot{q} \neq 0$), the values of the currents and the angular velocities are given by the expressions

$$\dot{\varphi}_1 = \frac{\left(K_1\frac{\partial L_2}{\partial \alpha} - K_2\frac{\partial L_1}{\partial \alpha}\right)\frac{\partial L_2}{\partial \alpha} - 2Rh_2\frac{\partial L_1}{\partial \alpha}}{h_1\left(\frac{\partial L_2}{\partial \alpha}\right)^2 + h_2\left(\frac{\partial L_1}{\partial \alpha}\right)^2},$$

$$\dot{\Phi}_2 = \frac{\left(K_2 \frac{\partial L_1}{\partial \alpha} - K_1 \frac{\partial L_2}{\partial \alpha}\right) \frac{\partial L_1}{\partial \alpha} - 2Rh_1 \frac{\partial L_2}{\partial \alpha}}{h_1 \left(\frac{\partial L_2}{\partial \alpha}\right)^2 + h_2 \left(\frac{\partial L_1}{\partial \alpha}\right)^2},$$

$$\dot{q}^2 = -2 \frac{h_1 K_2 \frac{\partial L_2}{\partial \alpha} + h_2 K_1 \frac{\partial L_1}{\partial \alpha} + 2Rh_1 h_2}{h_1 \left(\frac{\partial L_2}{\partial \alpha}\right)^2 + h_2 \left(\frac{\partial L_1}{\partial \alpha}\right)^2},$$

from which it follows that the second case is realized only if the "condition of current excitation" is satisfied:

$$h_1 K_2 \frac{\partial L_2}{\partial \alpha} + h_2 K_1 \frac{\partial L_1}{\partial \alpha} + 2Rh_1 h_2 < 0.$$

It follows from equations (10.35) that, for $\ddot{q} \neq 0$, the ratio of the electromechanical moments on the drums of the first and the second machines remains constant irrespective of the value of the current $\dot{q}$, the resistance R, and the rate of revolution of the rotors of the machines. This means that the system can operate as an unusual "electrical amplifier" of an electromechanical moment.

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